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Octagon Quadrangle Systems nesting 4-kite-designs having equi-indices

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Abstract

An *octagon quadrangle* is the graph consisting of an 8-cycle $(x_1, ..., x_8)$ with two additional chords: the edges $\{x_1, x_4\}$ and $\{x_5, x_8\}$. An *octagon quadrangle system* of order v and index λ $[OQS]$ is a pair $(X, \mathcal{B})$, where X is a finite set of v vertices and $\mathcal{B}$ is a collection of edge disjoint octagon quadrangles (called *blocks*) which partition the edge set of λK_v defined on X. A 4-*kite* is the graph having five vertices x_1, x_2, x_3, x_4, y and consisting of an 4-cycle $(x_1, x_2, ..., x_4)$ and an additional edge $\{x_1, y\}$. A 4-*kite design* of order n and index μ is a pair $\mathcal{K} = (Y, \mathcal{H})$, where Y is a finite set of n vertices and $\mathcal{H}$ is a collection of edge disjoint 4-*kite* which partition the edge set of μK_n defined on Y. An *Octagon Kite System* $[OKS]$ of order v and indices (λ, μ) is an $OQS(v)$ of index λ in which it is possible to divide every block in two 4-*kites* so that an 4-*kite design* of order v and index μ is defined.

In this paper we determine the spectrum for $OKS(v)$ nesting 4-kite-designs of *equi-indices* (2,3).

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1 Introduction

Let $\lambda \cdot K_v$ be the complete multigraph defined on a vertex set X. Let G be a subgraph of $\lambda \cdot K_v$. A *G-decomposition* of $\lambda \cdot K_v$ is a pair $\Sigma = (X, \mathcal{B})$, where $\mathcal{B}$ is a partition of the edge set of $\lambda \cdot K_v$ into subsets all of which yield subgraphs that are isomorphic to G. A G-decomposition is also called a *G-design* of order v and index λ; the classes of the partition $\mathcal{B}$ are said to be the *blocks* of Σ. Thus, $\mathcal{B}$ is a collection of graphs all isomorphic to G such that every pair of distinct elements of X is contained in λ blocks of Σ.

A 4-*kite* is a graph $G = C_4 + e$ (Fig.1), formed by a cycle $C_4 = (x, y_1, y_2, y_3)$, where the vertices are written in cyclic order, with an additional edge $\{x, z\}$. In what follows, we will denote such a graph by $[(y_1, y_2, y_3), (x), z]$. We will say that x is the *centre* of the kite, z the *terminal* point, y_1, y_3 the *lateral* points and y_2 the *median* point. A $(C_4 + e)$-design will also be called a 4-*kite-design*. It is known that a 4-kite-design of order v exists if and only if: $v \equiv 0$ or 1 mod 5, $v \geq 10$.

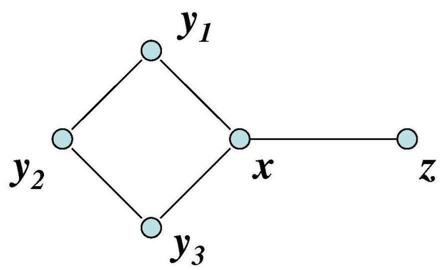

Figure 1. 4-kite

A λ-fold *m-cycle system* of order v is a pair $\Sigma = (X, \mathcal{C})$, where X is a finite set of v elements, called *vertices*, and C is a collection of edge disjoint m-cycles which partitions the edge set of λK_v, (the complete multigraph defined on X, where every pair of vertices is joined by λ edges). In this case, $|\mathcal{C}| = \lambda v(v-1)/2m$. The integer number λ is also called the *index* of the system. When $\lambda = 1$, we will simply say that Σ is an *m-cycle system.* The spectrum for λ-fold *m-cycle systems* for $\lambda \geq 2$ is still an open problem.

The graph given in Fig.2 is called an *octagon quadrangle* and will be denoted by $[(x_1), x_2, x_3, (x_4), (x_5), x_6, x_7, (x_8)]$. An octagon quadrangle system $[OQS]$ is a G-design, where G is an octagon quadrangle. $OQS(v)$s have been studied by the authors in [1][2][3][5].

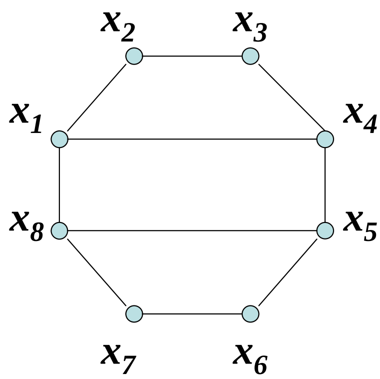

Figure 2. Octagon Quadrangle

Observe that, if we consider an *octagon quadrangle* $Q = [(x_1), x_2, x_3, (x_4), (x_5), x_6, x_7, (x_8)]$, it is possible to partition it into the two 4-kites $K_1(Q) = [(x_1, x_2, x_3), (x_4), x_5]$, $K_2(Q) = [(x_5, x_6, x_7), (x_8), x_1]$.

In this paper we study OQSs which can be partitioned into two $(C_4{+}e)$-designs (see Fig.3).

2 Definitions

Let $\Sigma = (X, \mathcal{B})$ be an OQS of order v and index λ. We say that Σ is 4-*kite nesting*, if for every octagon quadrangle $Q \in \mathcal{B}$ there exists at least a 4-kite $K(Q) \in \{K_1(Q), K_2(Q)\}$ such that the collection $\mathcal{K}$ of all these 4-kites $K(Q)$ form a 4-kite-design of order μ. This kite system is said to be *nested* in Σ. We will call it an *octagon 4-kite system* of order v and indices (λ, μ), briefly also OKS or $OKS_{\lambda,\mu}$, $OKS_{\lambda,\mu}(v)$.

If Ω is the family of all the 4-kites $\{K_1(Q), K_2(Q)\}$ contained in the octagon quadrangles $Q \in \mathcal{B}$, we observe that also the family $\mathcal{K}^c =$

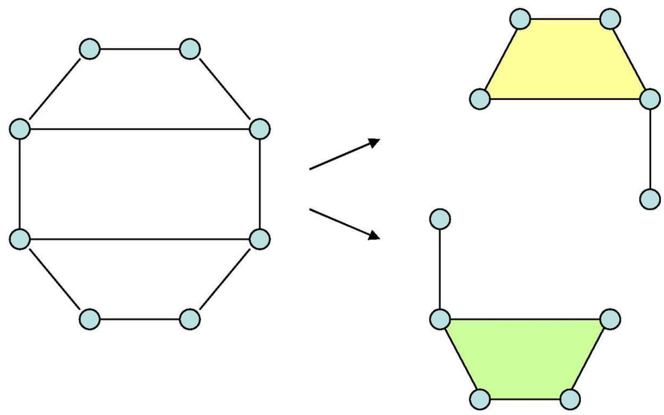

Figure 3. Decomposition of an OQ into two 4-kites

$= \Omega - \mathcal{K}$ forms a 4-kite-design of index $\mu' = \lambda - \mu$. If, for every octagon quadrangle $Q \in \mathcal{B}$, both families of 4-kites $\Omega_1 = \{K_1(Q) : Q \in \mathcal{B}\}$, $\Omega_2 = \{K_2(Q) : Q \in \mathcal{B}\}$ form a 4-kite design of index μ, we will say that the *OQS* is an *octagon bi-kite system.*

3 The new concept of G-Designs with equi-indices

In this section we give a new concept, which considers the possibility for a G-design to have more indices. We consider the case of *two* indices and order $v = 4h + 1$, because these will be considered in what follows, but the definition can be extended for the case with k indices, $k \geq 2$, and order every admissible v.

Definition of G-design with two equi-indices

Let G be a graph and let $v = 4h + 1$ an integer.
A G-design of equi-indices λ, μ is a pair $\Sigma = (X, \mathcal{B})$ where $X = Z_v$ and

$\mathcal{B}$ is a collection of graphs, all isomorphic to G, called blocks and defined in a subset of Z_v, such that for every pair of distinct element $x, y \in Z_v$:

1) *if the distance (difference) between x, y is equal to 1,2,...,h, then the pair x, y is contained in exactly λ blocks of Σ;*

2) *if the distance (difference) between x, y is equal to h+1,h+2,...,2h, then the pair x, y is contained in exactly μ blocks of Σ.*

This definition generalizes the well known concept of G-design and it can be done in many different ways and conditions. For them we keep the usual terminology.

Example 1

Let $v = 13$. In Z_{13} the set of all the possible differences is $\Delta = \{1, 2, 3, 4, 5, 6\}$. Partition Δ into the following two classes: $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$. It is possible to define a K_3-design (*Steiner Triple System*) of order $v = 13$ and *equi-indices* (λ, μ)=(1,2), as follows:

$$\forall \{x, y\} \subseteq Z_{13}, \;\; x \neq y, \;\; |x - y| = 1, 2, 3 \Longrightarrow \lambda = 1,$$

$$\forall \{x, y\} \subseteq Z_{13}, \;\; x \neq y, \;\; |x - y| = 4, 5, 6 \Longrightarrow \mu = 2,$$

where $\lambda = 1$ and $\mu = 2$ mean respectively that the pair x, y is contained in exactly one or two blocks of the system.

The blocks

$$\{i, i + 1, i + 5\}, \{i, i + 2, i + 7\}, \{i, i + 3, i + 7\},$$

for every $i \in Z_{13}$,

define such a system.

Example 2

Let $v = 9$. In Z_9 the set of all the possible differences is $\Delta = \{1, 2, 3, 4\}$. Partition Δ into the following two classes: $A = \{1, 2\}$, $B = \{3, 4\}$. It is possible to define a $(K_4 - e)$-design of order $v = 9$ and *equi-indices* (λ, μ)=(3,2), as follows:

$$\forall \{x, y\} \subseteq Z_9, x \neq y, |x - y| = 1, 2 \Longrightarrow \lambda = 3,$$
$$\forall \{x, y\} \subseteq Z_{13}, x \neq y, |x - y| = 3, 4 \Longrightarrow \mu = 2,$$

where $\lambda = 3$ and $\mu = 2$ mean respectively that the pair x, y is contained in exactly three or two blocks of the system.
If we indicate by $[x, y, (z, t)]$ a block of a $(K_4 - e)$-design, where (z, t) are the only two vertices non adjacent in $(K_4 - e)$, then the blocks

$$\{i, i + 2, (i + 1, i + 4)\}, \{i, i + 4, (i + 1, i + 6)\},$$

for every $i \in Z_9$,

define such a system.

4 Strongly Balanced 4-kite-Designs

It is known that a G-design Σ is said to be *balanced* if the *degree* of each vertex $x \in X$ is a constant: in other words, the number of blocks of Σ containing x is a constant.

In [4] the authors have introduced the following concept.
Let G be a graph and let $A_1, A_2, ..., A_h$ be the orbits of the automorphism group of G on its vertex-set. Let $\Sigma = (X, \mathcal{B})$ be a G-design.

We define the degree $d_{A_i(x)}$ of a vertex $x \in X$ as the number of blocks of Σ containing x as an element of A_i.

We say that:

$\Sigma = (X, \mathcal{B})$ is a strongly balanced G-design if, for every $i = 1, 2, ..., h$, there exists a constant C_i such that

$$d_{A_i}(x) = C_i,$$

for every $x \in X$.

It follows that:

Theorem 4.1. *A strongly balanced G-design is a balanced G-design.*

Proof Clearly, if $\Sigma = (X, \mathcal{B})$ is a *balanced* G-design, then for each vertex $x \in X$ the relation $d(x) = \sum_{i=1}^{h} d_{A_i}(x)$ holds. Hence, there exists $k \in N$ such that $d(x) = k$, for every $x \in X$. □

We say that a G-design is *simply balanced* if it is balanced, but not strongly balanced.

Theorem 4.2. *If Σ=$(X, \mathcal{B})$ in a balanced OQS of order v and index λ, then Σ is strongly balanced.*

Proof If G is an octagon quadrangle $[(x_1), x_2, x_3, (x_4), (x_5), x_6, x_7, (x_8)]$, the automorphism group of its vertices has the two orbits $A_1 = \{x_1, x_4, x_5, x_8\}$, $A_2 = \{x_2, x_3, x_6, x_7\}$. For every vertex $x \in X$, denote by C_x the number of blocks of Σ containing x as a central vertex, i.e. in a position of degree three, and by M_x the number of blocks of Σ containing x as a median vertex, i.e. in a position of degree two. Since Σ is balanced, it follows that:

$$C_x + M_x = d(x) = \frac{8 \cdot |\mathcal{B}|}{v} = 4 \cdot \lambda \cdot (v-1);$$

$$4 \cdot C_x = \frac{8 \cdot |\mathcal{B}|}{v} = 4 \cdot \lambda \cdot (v-1).$$

Hence:

$$C_x = M_x = 2 \cdot \lambda \cdot (v-1),$$

and the statement is so proved. □

If Ω is a balanced 4-kite-design, it is possible that Ω is not strongly balanced. In Fig.4 there is a *simply balanced* 4-kite- design of order 11. We can see that all the vertices x have degree $d(x) = 5$, but the vertex 2 has *median-degree* equal to *three*, while the vertex 3 has *median-degree* equal to *zero*.

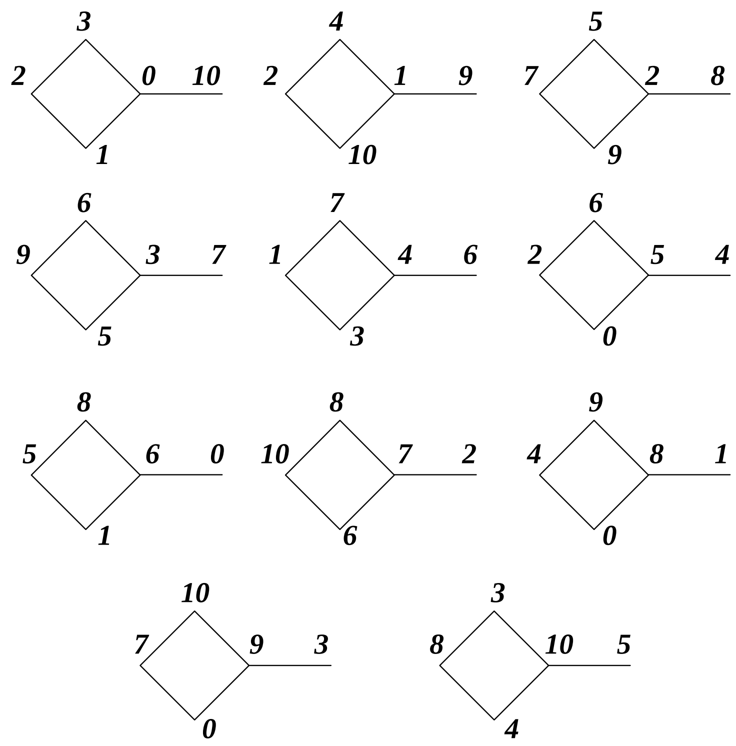

Figure 4. Simply balanced 4-kite-design of order 11

5 Necessary existence conditions

In this section we prove some necessary conditions for the existence of 4-kite-designs.

Theorem 5.1. *Let* $\Omega = (X, \mathcal{B})$ *be an* $OKS_{\lambda,\mu}(v)$*. Then*

i) $\lambda = 2 \cdot \mu$*;*

ii) (λ, μ)*=(2,1) or (4,2) or (6,3) or (8,4) implies* $v \equiv$ *0,1 mod 5,* $v \geq 8$*;*

iii) (λ, μ)*=(10,5) implies* $v \equiv$ *0,1 mod 2,* $v \geq 8$*.*

Proof Let $\Omega = (X, \mathcal{B})$ be and let $\Sigma = (X, \mathcal{K})$ the 4-kite system of index μ, nested in it. It follows that:

$|\mathcal{B}| = |\mathcal{K}|$,

$|\mathcal{B}| = v(v-1) \cdot \lambda/20$,

$|\mathcal{K}| = v(v-1) \cdot \mu/10$.

Hence: *i)* $\lambda = 2 \cdot \mu$. For *ii)* and *iii)*, it suffices to consider that:

$|\mathcal{K}| = v(v-1) \cdot \mu/10$. □

Theorem 5.2. *Let* $\Sigma = (Z_v, \mathcal{B})$ *be a* 4*-kite-design of equi-indices* (λ, μ)*, with* v *odd. Then*

i) $|\mathcal{B}|$=$(\lambda + \mu) \cdot v \cdot (v-1)/20 \in N$*;*

ii) (λ, μ)=$(2, 3)$ *implies* $v \equiv$ *1 mod 4,* $v \geq 5$*.*

Proof *i*) Let $\Sigma = (Z_v, \mathcal{B})$ be a 4-kite-design of equi-indices (λ, μ), with v odd. It follows:

$$|\mathcal{B}| = (\frac{\lambda}{2} \cdot \binom{v}{2} + \frac{\mu}{2} \cdot \binom{v}{2})/5 = \frac{\lambda + \mu}{10} \cdot \binom{v}{2},$$

which must be a positive integer. *ii*) Directly from *i*). □

The *ii*) of Theorem 5.2 will be used in the next section.

6 Main Existence Theorems

In this section we prove the conclusive Theorems of this paper. In what follows, if $B = [(a), b, c, (d), (\alpha), \beta, \gamma, (\delta)]$ in a block of a system Σ defined in Z_v, then the *translates* of B are all the blocks of type $B_j = [(a+j), b+j, c+j, (d+j), (\alpha+j), \beta+j, \gamma+j, (\delta+j)]$, for every $j \in Z_v$. B is called a base block of Σ.

Theorem 6.1. *There exists an OKS of order v and equi-indices $(2,3)$, with v odd, if and only if :*

$$v \equiv 1 \ mod \ 4, \ \ v \geq 9.$$

Proof $\Rightarrow$ Let $\Sigma = (Z_v, \mathcal{B})$ be an OQS of order v and equi-indices $(2,3)$, with v odd. Since every block contains eight vertices, from Theorem 5.2.*ii*), it follows

$$v \equiv 1 \ mod \ 4, \ \ v \geq 9.$$

$\Leftarrow$ Let $v = 4h+1, \ \ h \in N, \ \ h \geq 2.$
Consider the following octagon quadrangles:

$$B_1 = [(0), h, 3h+1, (1), (2h+1), 3h, h+1, (2)],$$

$$B_2 = [(0), 1, 3h+1, (2), (2h+1), 3h-1, h+1, (3)],$$

$$B_3 = [(0), 2, 3h+1, (3), (2h+1), 3h-2, h+1, (4)],$$

....

$$B_i = [(0), i-1, 3h+1, (i), (2h+1), 3h-(i-1), h+1, (i+1)],$$

....

$$B_{h-1} = [(0), h-2, 3h+1, (h-1), (2h+1), 3h-(h-2), h+1, (h)],$$

$$B_h = [(0), h-1, 3h+1, (h), (2h+1), h+1, 3h+2, (1)].$$

Consider the system $\Sigma = (X, \mathcal{B})$, defined in $X = Z_v$, having $B_1, ..., B_i, ..., B_h$ as *base blocks*. This means that $B_1, B_2, ..., B_i, ..., B_h$ belong to $\mathcal{B}$ and also all the *translates*.

It is possible to verify that Σ is an *OQS* of order $v = 4h + 1$ and index $\lambda = 5$. Further, if we divide every block $Q = [(x_1), x_2, x_3, (x_4), (x_5), x_6, x_7, (x_8)]$, into the two 4-kites

$$K_1(Q) = [(x_1, x_2, x_3), (x_4), x_5],$$

$$K_2(Q) = [(x_5, x_6, x_7), (x_8), x_1],$$

we can verify that the collection of all the *upper* 4-kites form a 4-kite-design $\Sigma_1 = (Z_v, \mathcal{B}_1)$ of equi-indices $(\lambda = 2, \mu = 3)$, while the collection of all the *lower* 4-kites form a 4-kite-design $\Sigma_2 = (Z_v, \mathcal{B}_2)$ of equi-indices $(\lambda = 3, \mu = 2)$.

Observe that:

i) in Σ_1 all the pairs $x, y \in Z_v$ associated with the index $\lambda = 2$ have difference $|x-y|$ belonging to $A = \{1, 2, ..., h\}$, while the pairs associated with $\mu = 3$ have difference belonging to $B = \{h+1, h+2, ..., 2h\}$;

ii) in Σ_2 all the pairs $x, y \in Z_v$ associated with the index $\lambda = 3$ have difference $|x-y|$ belonging to A, while the pairs associated with $\mu = 2$ have difference belonging to B.

This proves that Σ is an OKS of order $v = 4h + 1$, $h \geq 2$, where the two 4-kite-designs nested in it have equi-indices $(2, 3)$ and $(3, 2)$ respectively. □

Theorem 6.1 permits to prove the following Theorems:

Theorem 6.2. *There exists a 4-kite-design of order v and equi-indices* $(2, 3)$*, with v odd, if and only if* :

$$v \equiv 1 \ mod \ 4, \ \ v \geq 5.$$

Proof The statement follows directly from Theorem 6.1 and considering that the design Σ_5, defined in Z_5 and having for blocks all the translates of the *base* 4-kite:

$$[(2, 1, 3), (0), 4],$$

is a 4-kite-design of order 5 and equi-indices (2,3). □

Theorem 6.3. *For every* $v \equiv 1 \ mod \ 4, \ \ v \geq 5$, *there exists a strongly balanced 4-kite-design of order v.*

Proof See Theorems 6.1 and 6.2. □

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