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Some Aspects of Mathematical and Physical Approaches for Topological Quantum Computation

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Abstract

A paradigm to build a quantum computer, based on topological invariants is highlighted. The identities in the ensemble of knots, links and braids originally discovered in relation to topological quantum field theory are shown: how they define Artin braid group – the mathematical basis of topological quantum computation (TQC). Vector spaces of TQC correspond to associated strings of particle interactions, and TQC operates its calculations on braided strings of special physical quasiparticles – anyons – with non-Abelian statistics. The physical platform of TQC is to use the topological quantum numbers of such small groups of anyons as qubits and to perform operations on these qubits by exchanging the anyons, both within the groups that form the qubits and, for multi-qubit gates, between groups. By braiding two or more anyons, they acquire up a topological phase or Berry phase similar to that found in the Aharonov-Bohm effect. Topological matter such as fractional quantum Hall systems and novel discovered topological insulators open the way to form system of anyons – Majorana fermions – with the unique property of encoding and processing quantum information in a naturally fault-tolerant way. In the topological insulators, due to its fundamental attribute of topological surface state occurrence of the bound, Majorana fermions are generated at its heterocontact with superconductors. One of the key operations of TQC – braiding of non-Abelian anyons: it is illustrated how it can be implemented in one-dimensional topological isolator wire networks.

Keywords: topological computation, braided strings, anyons, topological phase, braiding, wire networks.

1 Introduction

In the last 10-15 years it has been realized that the most fundamental attributes of quantum mechanics can be configured for quantum computing and communication. Formally, a quantum computation is performed through a set of transformations, called quantum gates [1]. Like quantum operators a quantum gate applies unitary transformation U to a set of quantum bits (qubits) in a quantum state $|\psi>$. At the end of the calculation, a measurement is performed on the qubits (which are in the state $|\psi'>= U|\psi>$). During last decade many paths have been paved to build sets of universal quantum gates, which allows one to perform any arbitrary calculation without inventing a new gate each time. The implementation of a set of universal gates is therefore of crucial importance. It can be shown that it is possible to construct such a set with gates that act only on one or two qubits at a time.

The great promise of quantum computers (QC) has to be balanced against the great difficulty of actually building them. The main problem is quantum decoherence, the inevitable continuous dephasing of a quantum state due to its interaction with the environment. Small improvements to current strategies are not sufficient to overcome this problem and radically new ideas are required. A new paradigm is to build a quantum computer which will be topologically immune to quantum decoherence and such platform was recently proposed and was called topological quantum computation (TQC) [1]. The idea is to use the topological quantum numbers of small groups of special physical quasiparticles – anyons – as qubits and to perform operations on these qubits by exchanging the anyons, both within the groups that form the qubits and, for multi-qubit gates, between groups. Anyons are unusual special type quasiparticles unlike the electrons and protons and having the desired topological properties. The importance of such a paradigm is that it allows one to make direct contact with the circuit model of quantum computation and it enables algorithmic questions to be tackled independently of the details of experimental implementation, at least initially. TQC suppose to employ many-body physical systems with the unique property of encoding and processing quantum information in a naturally fault-tolerant way.

Research on topological quantum computation has become a highly interdisciplinary field, with frontiers in physics, mathematics, and computer science [2]. Last years significant theoretical and experimental developments have been realized, which include major experimental and theoretical advances in fractional quantum Hall systems that support the existence of non-Abelian anyons – the physical building block of topological quantum computation – as well as the predication and experimental discovery of novel solid-state phase – topological insulators (TI).

The aim of the paper is to review together the latest developments in a TQC and TI with the goal of underlining the synergy between computer and material sciences approaches.

2 Mathematical Considerations on Topological Computation

Knots, links and braids are the mathematical basis of TQC. A representation of a knot was defined to be a closed polygonal curve in space and links are a combination of knots that are intertwined leading to braids. Thus the representations of braids which are defined to be a set of n polygonal curves stretching from $z = 0$ plane (in $R3$) to the $z = 1$ plane where the k_{th} curve stretches from (1/2, k/n, 0) to (1/2, k/n, 1) and the z value is strictly increasing and the curves do not intersect [3]. Braids clearly have some algebraic properties. There is a clear identity braid, which is just formed by connecting the start and end points with straight lines. We can imagine "adding" two braids with the same number of strands. This addition will be "associative" a(bc) = (ab)c. Similarly, we could imagine by exactly reversing the way we did the braiding that we could add two braids which could be manipulated to obtain the identity (an inverse braid). Finally, if we add many braids together it is clear it will still remain a braid. Thus a group – Artin braid group – is obtained for which we can establish about how it may be generated and what equalities are required for combinations of those generators so that we can determine if two braids are equivalent [3].

Let us define k as the exchange of the k^{th} curve with $(k+1)^{st}$ curve where the k^{th} curve passes over the $(k+1)^{st}$ one (Fig 1). It is easy to observe the following set of identities:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for} \quad |i-j| < 3; \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \tag{1}$$

The first of equations (1) indicates that two disjoint exchanges are commutative and the second can be seen in Fig.1. The abovementioned conditions are all which are required to define general braid group. In TQC, vector spaces correspond to associated strings of particle interactions being interrelated by recouping transformations that generalize the usual QC mapping. A full representation of the Artin braid group on each space is defined in terms of the local interchange phase gates and the recoupling transformations [1].

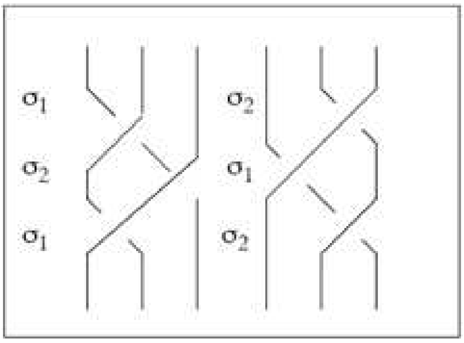


Figure 1. Two braids continuously deformed into each other without cutting any of the strands illustrating the second equality (1).

These gates and transformations have to satisfy a number of identities in order to produce a well-defined representation of the braid group. These identities were discovered originally in relation to topological quantum field theory.

At first sight, a TQC does not seem much like a computer at all. It works its calculations on braided strings – but not physical strings in the conventional sense. Rather, they are what physicists refer to as world lines, representations of particles as they move through time and space. However TQC are based on non-Abelian statistics and a special type of particles – anions – is required, which can appear in physical systems as the result of many-body interactions.

3 Topological Computation with Anyons

In conventional computing the bits of zeroes and ones are created by switching an electric current on and off in a CMOS transistor. Ordinary quantum computation uses simple quantum two level systems (e.g. electron or nuclear spins, atomic hyperfine states, etc.) as quantum bits ('qubits') with one-and two-qubit unitary operations serving as universal quantum gates. Physical basis of TQC is more subtle and uses the anions – excitations in a two-dimensional electronic system that behave a lot like the particles and antiparticles of high-energy physics. They are able to carry charges that are fractions of the fundamental charge of the electron and its spin can take on any real value (Fig2).

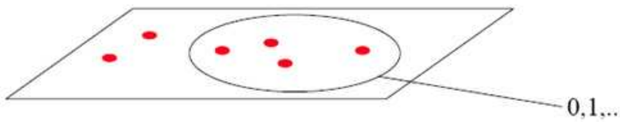


Figure 2. Groups of particles have quantum numbers

This is of course related to their statistics and the fact that they are neither fermions nor bosons. There are no physical processes that can create or destroy isolated anyons, that is important if we intend to use them in a quantum computer. If the anyons could spontaneously appear or disappear any quantum operation using them would fail. They also have antiparticles, with which they can interact to combine or annihilate. Anyons can also combine with other anyons that are not their antiparticles. Fundamental attribute of a group of anyons is its quantum topological number. By braiding two anyons, they acquire up a topological phase similar to that occurring in the Aharonov-Bohm effect – that is the phase given to a charge, which the particle accumulates when it travels around a solenoid. Just like Aharonov-Bohm effect phase the topological phase only depends on how many times the anyons wrap around each other and not the path they follow. In the one dimensional representation of the braid group, we obtain $\sigma_j = e^{i\theta j}$ for identical anyons, where θ_j is the topological phase added by the σ_j operation [2].

Alternatively, we could extend to multidimensional representation, which allows us to have non-abelian anyons as well, which are more suitable for quantum computing than abelian anyons. We now must consider how anyons can combine and split. Each model of anyons will have different fusion rules. The fusion rules determine the total charge, c, when a and b combine. These are written as $a \times b = \Sigma_c N_{cab}$, where N_{cab} is a nonnegative integer and the sum is over the complete set of labels of the composite. The composition rules are symmetric ($a \times b = b \times a$) so the possible charges do not depend on which side the anyon came from. Note that if N_{cab} is zero, the charge c cannot be formed, while if it is one there is a unique way of obtaining c, and N_{cab} can also be greater than one. Thus N_{cab} represents the number of distinguishable ways that a charge c can obtain. The distinguishable ways, that a and b can combine to form c, then represent an orthonormal basis for a Hilbert space, which is called a fusion space.

The next idea that is introduced is the R matrix, which is the braid operator, and the F matrix, which is the fusion operator. The following step is to use the topological quantum numbers of small groups of anyons as qubits and to perform operations on these qubits exchanging the anyons, both within the groups that form the qubits and, for multi-qubit gates, between groups. Thus the summary of TQC basis includes:

- *Uses 2D systems which have quasiparticles with NonAbelian Statistics.*
- *Quantum Information is encoded in nonlocal topological degrees*

of freedom that do not couple to any local quantity.

- *States can be manipulated by dragging (braiding) quasiparticles around each other.*
- *The operations (gates) performed on the qubits depend only on the topology of the braids.*

4 Topological states of Condensed Matter

More than three decades ago there was established that spin-orbit interaction (SOI) has an important pattern on band structure of solid state matter. Among different qualitative features induced by SOI the band inversion of electronic spectrum near the Fermi level has been discovered in different types of semimetals and narrow-gap semiconductors. In the context of low dimensional structure investigations the band spectrum inversion was shown to generate new type of interface gapless states with linear spectrum at the heterocontact boundaries [4]. Last years investigations [5] have reopened the interest to materials with inverted band spectra. Due to new type of the symmetry break like that characteristic for the quantum Hall effects, the electronic states were shown to have topological nature and materials have been named topological insulators (TI). In TI a new state of matter appears, distinguished from a regular band insulator by a nontrivial time-reversal topological invariant, which characterizes its band structure and leads to new physics and phenomena.

The most robust observable consequence of a nontrivial topological character of these materials is the presence of gapless helical edge states (interface states of inverted heterocontacts), protected by time-reversal symmetry and robust to perturbations that do not break this symmetry (Fig.3). Like the quantum Hall state the "bulk" of the electron gas of TI is an insulator, but along its surface, the states are gapless with a spectrum described by a Dirac cone like graphene and characterized by prohibiting backscattering. Topological nature of boundary states of TI leads to a fundamental attributes each momentum of the state along the surface has only a single spin state at the Fermi level, and the spin direction rotates as the momentum moves around the Fermi surface ensuring a non-trivial Berry's phase. These two defining properties of TI – the spin-momentum locking of surface states and π Berry's phase – are crucial for the generation of the anyons.

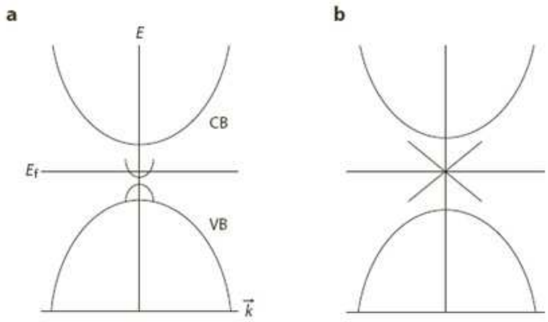


Figure 3. The electronic band structures of topological insulators with a robust metallic state at the surface and insulating properties in the bulk.

At the same time, the spectrum and characteristics of topological surface states, depending on geometrical configuration, can be manipulated by different factors: electrical and magnetic fields, strain and deformation etc. For this reason TI are being explored with a view towards applications, as a potential platform for TQC [5].

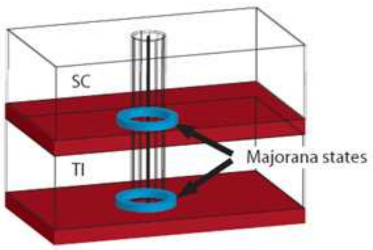
5 Majorana Fermions of Topological Insulator – a new physical approach for TQC

In condensed matter physics, Majorana fermions can arise due to a paired condensate that allows a pair of fermionic quasiparticles to "disappear" into the condensate. They have been predicted in a number of physical systems. Majorana zero modes must always come in pairs (for instance, a 1D superconductor has two ends [2]), and a well separated pair defines a degenerate two level system – a qubit, whose quantum state is stored nonlocally. The state can not be measured with a local measurement on one of the bound states and this is crucial, because the main difficulty with making a quantum computer is preventing the system from accidentally measuring itself.

In two dimensions a number of chiral Majorana edge modes can appear, which resemble chiral modes in the quantum Hall effect, but for the particle-hole redundancy. A spinless superconductor with px + ipy symmetry is the simplest model 2D topological superconductor, in which Majorana bound states appear at the core of vortices

Combining topological insulators with ordinary superconductors leads to an exquisitely correlated interface state that, like a topological superconductor, is predicted to host Majorana fermion excitations and its properties has proposed to be for fault tolerant quantum information processing [5]

Majorana fermion can be created in several ways using topological insulators [5]. The most direct proposal using a 3D topological insulator is to consider the proximity effect from an ordinary *s*-wave superconductor. A magnetic vortex core in such a system will carry a zero-energy Majorana fermion state localized near the vortex in the interface layer (Fig.5).



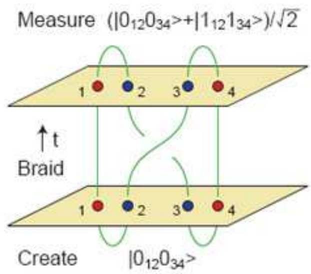


Figure 4. Majorana bound states in topological isolator/superconductor heterojunction and its braiding.

There are analogous ways to create a Majorana fermion using strong spin-orbit quantum wells rather than topological insulators. Recently, a network of 1D semiconductor quantum wires has been proposed [6] as a suitable platform to create, transport, and fuse Majorana fermions at the wire ends. The wire network consists of wire segments in the TS state (shown in red with numbers in Fig. 5) connected by segments in the non-topological superconducting (NTS) state (shown in blue without numbers in Fig. 5).

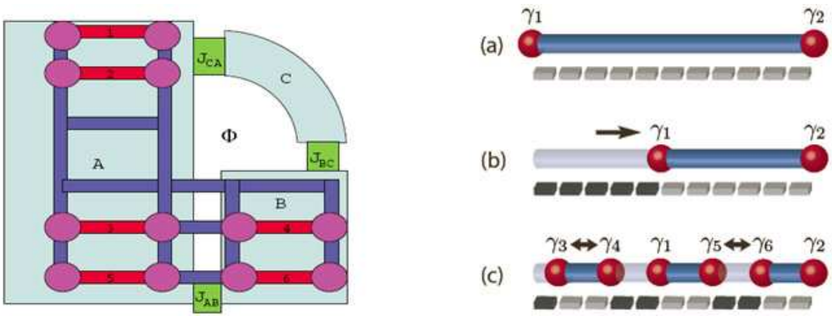


Figure 5. Schematic of entanglement generation and manipulation in quantum wire topological qubits using superconductor Josephson junctions.

Local gates allow Majorana fermions to be transported, created, and fused as outlined in Fig. 5. As one germinates pairs of Majorana fermions, the ground state degeneracy increases as does our capacity to topologically store quantum information in the wire. Specifically, $2N$ Majoranas generate N ordinary zero-energy fermions whose occupation numbers specify topological qubit states. Adiabatically braiding the Majorana fermions would enable manipulation of the qubits, but it is not possible in a single wire [6]. The Majorana fermion states are transported by shifting the end points of the TS segments by applying locally tunable external gate potentials (which control chemical potential).

6 Conclusions

There is a great deal of progress that has been made in the theory of topological quantum computing. Anyons and their braids in the topological matter, such as fractional quantum Hall systems and novel discovered topological insulators, excellently simulate quantum gates to arbitrary accuracy. Combining topological insulators with ordinary superconductors leads to an exquisitely correlated interface state that, like a topological superconductor, is predicted to host Majorana fermion excitations and its properties has proposed to be for fault tolerant quantum information processing.

A network of topological insulator quantum wires in the vicinity of an s-wave superconductor allows universal TQC. Such approach enables the Majorana fusion rules to be probed, along with networks that allow for efficient exchange of arbitrary numbers of Majorana fermions.

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