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A New Foundation For Knowledge Systems

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Abstract

We propose mathematical foundations for knowledge systems based on the notion of effectively presented domain. The paper aims to give a schematic model for computer's behavior in changing information environment. In this paper, we focus on computational resources rather than on computer's particular actions.

Keywords: multiple valued logic, partial recursive function, semantic domain, continuous operation, epistemic state.

1 Remarks on the use of multiple valued logic

We have to state from the very beginning that, although we start this approach with multiple valued logic, the latter plays merely an auxiliary part in what follows. In other words, we will be using here finite sets of entities, calling them *truth values*, only to mark with them the *degree of truth* or the *degree of trust* that can be assigned to a report. The truth values in use (they may vary) are arranged as a partially ordered set.

The computer (the "agent") which operates a knowledge system is supposed to be placed in information flow. The latter is used in an informal way and can be thought as a stream of messages entering the agent's receiver. The agent receives valuated reports (the notion of a report will be defined later on in Section 3) and revaluates them on the grounds of information that has already been received. How the agent does it does not depend on any many-valued logic understood as a system of valid formulas. Thus multiple valued logic used here is realized as an algebra of a finite type rather than semantics for a non-classical logic system.

The first question, of course, is how the agent process the revaluation of incoming information. This, as the reader will see, depends on how the information the agent possesses is stored in the *space available*.

The last term will be understood here in a qualitative sense rather than as a size of the computer's memory. (This idea will be elaborated in Section 4.) Moreover, since the revaluated information must be stored, the agent has to rearrange the pieces of information in the storage. All this forces us to think of a multiple-task approach. A brief introduction of this approach can be found in [11].

2 Knowledge transformation: Belnap's *Scott Principle*

In [2], which was reprinted in [1], §81, Nuel Belnap proposed what he called *Scott Principle*: All transformations the agent does must be Scott-continuous functions.

Belnap aimed to apply this principle to define operations on the matrix of four values he proposed for the agent to handle situations when it receives two reports about the same statement, valuated as *true* and as *false*. However, the question of rearrangement of the pieces kept in the storage, as well as that of the character of the space available, has not been even raised there.

We remind the reader the Belnap's four-valued logic. It consists of four truth values: $\mathbf{t}$ (true), $\mathbf{f}$ (false), $\top$ (both true and false), and $\bot$ (unknown). In order to define logical matrix in accordance with Scott Principle, we need a formal language in which one can write reports being processed by the agent. Let us assume that all reports the agent works with are built up from an infinite list of atomic sentences, $\mathcal{S}$, by using connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (conditional), and $\neg$ (negation). The actual meaning of the connectives is not important; they are taken as an example. We will keep these connectives in the sequel, but other connectives are also possible.

So far Scott Principle has not been needed. If the reader has the feeling that Scott Principle is not an intrinsic part of logic, this is a right observation. But we always use some justification outside of a logic to define the latter. In Belnap's case, in order to apply Scott Principle, that is to say, to define the connectives as Scott-continuous operations we need that the truth values be arranged as a lattice. Belnap does it as shown on Diagram 1 below.

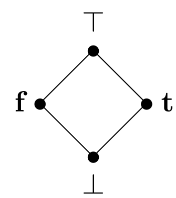

Diagram 1: lattice **A4**

The elements of lattice **A4** are arranged by the partial order $\sqsubseteq$ as seen on the Hasse diagram of **A4**. Before turning to the definition of Scott-continuity (it will be given in the next section), we will say that the only way to make the connectives Scott-continuous operations on **A4** is to define them as follows: $\wedge$ as meet, $\vee$ as join on the following lattice.

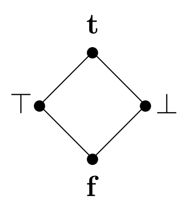

Diagram 2: lattice **L4**

As to $\neg$, in order to be Scott-continuous, it must be defined as follows: $\neg\mathbf{t} = \mathbf{f}$, $\neg\mathbf{f} = \mathbf{t}$, $\neg\bot = \bot$ and $\neg\top = \top$. Then, $\rightarrow$ becomes Scott-continuous if we define $x \rightarrow y = \neg x \vee y$. The last definition is very convenient, since $\rightarrow$ will play a different role in this discussion. Thus, accepting this definition, we limit ourselves to the signature: $\wedge$, $\vee$, and $\neg$.

Scott Principle has not yet done its job. It is so, because we can use it to define knowledge transformers if we want to have them also Scott-continuous.

Now it is a right time to ask why we care about Scott-continuity. Belnap, I believe, was preoccupied with the idea, although he has never explicitly expressed it, that knowledge, as it is presented to the agent, exists in quanta, regardless of whether it is accessible to the agent or not. In other words, one can talk of states of knowledge which approximate one another. The idea of approximation was explicitly emphasized by Belnap. (Probably, he borrowed this idea from Dana Scott's analysis of computation data types; the earliest source is, perhaps, [12].) Belnap regarded **A4** as an approximation lattice, while **L4** was a logic lattice.

3 Scott-continuous functions

Scott-continuity is an abstract notion. Taking direction toward the abstract, we hope to narrow the concept of knowledge to that which would be applicable to our situation, that is, when a computer-like agent acts in the information flaw of incoming reports. However, first we want to define structures to which we want to apply Scott-continuous functions.

Here are the main definitions.

Definition 3.1 (CPO; cf. [3]). *A partially ordered set* $(P, \sqsubseteq)$ *is a* CPO (*a shorthand for Complete Partially Ordered Set*) *if* P *contains a least element,* $\bot$*, with respect to* $\sqsubseteq$ *and any directed set* $D \subseteq P$ *has a least upper bound* $\bigsqcup D$ (*belonging to* P)*.*

It is customary to use *poset* for "partially ordered set."

Definition 3.2 (finite element; cf. [3]). *Given a* CPO $(P, \sqsubseteq)$*, an element* $a \in P$ *is called finite if for any directed* $D \subseteq P$*,*

$$a \sqsubseteq \bigsqcup D \Rightarrow a \sqsubseteq d \text{ for some } d \in D.$$

Given a CPO P*, the set of all finite elements of* P *is denoted by* $F(P)$*.*

Looking at elements of a CPO as quanta of knowledge, the finite elements have the feature that if any of them, say a, is "absorbed" by a directed "information cloud" D (in the sense of $a \sqsubseteq \bigsqcup D$), then a is a quantum of approximation to one of the elements of the cloud.

Definition 3.3 (Scott domain; cf. [5]). *A* CPO $(P, \sqsubseteq)$ *is a Scott domain if the following conditions are satisfied:*

i) $\emptyset \neq S \subseteq P \Rightarrow$ *a greatest lower bound* $\bigwedge S$ *exists*;
ii) *for every* $a \in P$, *the set* $\{x \in F(P) \mid x \sqsubseteq a\}$ *is directed*;
iii) *for every* $a \in P$, $a = \bigsqcup \{x \in F(P) \mid x \sqsubseteq a\}$.

The last definition is about approximation in letter and in spirit. That is, given Scott domain P, the finite elements are regarded as a source of approximation quanta to every single element of P.

We differ computational issues to the very end of the present paper.

Next we define our main notion due to Dana Scott [13].

Definition 3.4 (Scott-continuous function/operation). *Given* CPOs $(P, \sqsubseteq)$ *and* $(P', \sqsubseteq')$, *a map* $\phi : P \to P'$ *is called Scott-continuous if* ϕ *is monotone, that is, for any* $x, y \in P$,

$$x \sqsubseteq y \Rightarrow \phi(x) \sqsubseteq' \phi(y),$$

and for any directed $D \subseteq P$,

$$\phi\left(\bigsqcup D\right) = \bigsqcup \{\phi(x) \mid x \in D\}.$$

We note that, by virtue of monotonicity of ϕ, *the last set is directed in* P'.

In the sequel we will mainly be dealing with maps $\phi : P \to P$, calling them *operations*.

Let us observe that if the possible states of knowledge are identified with the points of a Scott domain P and a rearrangement machinery is implemented as a Scott-continuous operation ϕ, then the result of such an rearrangement will preserve the approximations which had been stated before ϕ was applied.

4 Knowledge space

Intuitively, if we want to keep "quantum philosophy of knowledge" alive, knowledge space is a set of epistemic states. Some of them can be accessible to the agent, some may not. The latter case is possible simply because we can imagine that there are epistemic states that involve infinite resources. But what is an epistemic state? We will address this question in the present section.

Suppose that the agent receives a message that $A : \tau$ (a *report*), that is, the sentence A is true to the degree τ, where τ is one of the truth values available to the agent and the reporters. How should the agent process this information?

Let us consider the following example. Assume that we have Belnap's four values and they arranged by $\sqsubseteq$ as in **A4**. It looks like we do not need the latter anymore, if we have already defined **L4**, for only **L4** is needed to "decompose" a report. But, as the reader will see very soon, **A4** will play its key part in the definition of an epistemic state below.

Continuing, assume that the report is $p \vee q : \mathbf{t}$. According to **L4**, these cases are possible: $v_1 :\ p = \mathbf{t}$ and $q = \mathbf{t}$; $v_2 :\ p = \mathbf{t}$ and $q = \mathbf{f}$; $v_3 :\ p = \mathbf{f}$ and $q = \mathbf{t}$; $v_4 :\ p = \mathbf{t}$ and $q = \bot$; $v_5 :\ p = \bot$ and $q = \mathbf{t}$; $v_6 :\ p = \mathbf{t}$ and $q = \top$; $v_7 :\ p = \top$ and $q = \mathbf{t}$; $v_8 :\ p = \bot$ and $q = \top$; $v_9 :\ p = \top$ and $q = \bot$.

Each v_i above is called a *valuation*. If the agent does not have any knowledge in its storage (the "memory"), that is, if its initial memory is blank, then it would be wise to store all the valuations $v_1, \ldots, v_9$. We denote $\varepsilon = \{v_1, \ldots, v_9\}$. However, assume that the agent receives also the report $p \vee q : \mathbf{f}$, when it already has $v_1, \ldots, v_9$ in its memory. Decomposing the last report according to **L4**, we get only one valuation: $v_{10} :\ p = \mathbf{f}$ and $q = \mathbf{f}$. Now the question is: How does it impact the epistemic state ε? If we just add v_{10} to ε, what will we achieve by that?

To answer the last question, we have to ask another one: How does the agent's knowledge about sentence $p \vee q$ change when it receives $p \vee q : \mathbf{f}$? Or what is its knowledge about $p \vee q$ at ε?

Belnap suggested to valuate a sentence A at an epistemic state ε

(the definition of which will be given later in this section) as the greatest lower bound with respect to $\sqsubseteq$ in **A4**, denoting this value by $\varepsilon(A)$. It indicates that the value of A at ε cannot be less than $\varepsilon(A)$. Applying this to our example, we get $\varepsilon(p \vee q) = \mathbf{t}$, but the value of $p \vee q$ at $\varepsilon^* = \varepsilon \cup \{v_{10}\}$ is $\perp$. Thus the revaluation of $p \vee q$ occurs when the agent changes its epistemic state from ε to ε^*.

Now let us ask this: Do we really need all $v_1, \ldots, v_9$ in ε to get $\varepsilon(p \vee q) = \mathbf{t}$? Actually, in the sense of $\varepsilon(p \vee q)$, we have the same value if we remove v_1, v_6 and v_7 from ε. The explanation lies on a surface.

Definition 4.1 (relation $v \sqsubseteq w$ on valuations)**.** *Given two valuations v and w, $v \sqsubseteq w$ if for all atomic sentences p, $v(p) \sqsubseteq w(p)$.*

The relation $v \sqsubseteq w$ for valuations is certainly a partial order on the set of all valuations.

Applying the last definition to our example, we can arrange the valuations in ε as follows.

$$
\begin{array}{cccccc}
\multicolumn{2}{c}{v_1} & \multicolumn{2}{c}{v_6} & \multicolumn{2}{c}{v_7} \\
/ & \backslash & / & \backslash & / & \backslash \\
v_4 & v_5 & v_2 & v_8 & v_3 & v_9
\end{array}
$$

Diagram 3: epistemic state ε arranged by $\sqsubseteq$

One can see that the removed valuations are not among the minimal ones in ε. Leaving only the latter, we do minimization of ε, which produces the epistemic state $m(\varepsilon)$ by deleting all non-minimal valuations. It is obvious that with an equal result we can apply minimization to ε^* or apply it first to ε, then add v_{10} to $m(\varepsilon)$ and minimize again; that is,

$$m(m(\varepsilon) \cup \{v_{10}\}) = m(\varepsilon \cup \{v_{10}\}).$$

The last equation becomes trivial if ε is already minimized. It hints at the idea to limit the agent's attention only to minimized epistemic states. Obviously, the agent cannot handle infinite sets. This observation leads to the following definitions.

Definition 4.2 (finite valuation)**.** *A valuation v is called finite if it maps only a finite subset* (*maybe empty*) *of $\mathcal{S}$ truth values unequal to $\perp$.*

Definition 4.3 (finite epistemic state)**.** *A finite epistemic state is a non-empty finite set of finite valuations which form an antichain with respect to $\sqsubseteq$ in Definition 4.1.*

Thus we arrive at the model according to which the agent's knowledge resides within the set of finite epistemic states, occupying one of them until a change occurs. However, there is the question: What can make a change? Namely, how can a Scott-continuous operation be defined on the set of finite epistemic states?

Let us be more precise. So far we have used the word *space* in a common language. Scott-continuous functions as defined in Definition 3.4 are not seen as continuous in the sense of topology, though they are. Indeed, they are continuous in Scott topology which can be defined in Scott domain. (Cf. [4].) From our perspective, it forces one to think of a bigger space which would include all finite epistemic states. Lack of such a space does not allow us to define knowledge transformers as continuous operations.

5 Finishing with the Belnap's logic

In this section we show how a bigger space incorporating the finite epistemic states can be defined. Then or even now, the reader may have the question: What if one wants to arrange the truth values of the Belnap's logic in a different way or use more or less values? We address this issue in Section 6. The other question that can be raised is about operations. When the bigger space is defined and the finite states play the part they should, any Scott-continuous operation is possible to be used. But for practical purposes, not all of them are actually needed. In this section we will consider three operations, the justification of which will be clear from the context. In a more general setting, these operations will be reviewed in Section 6.

We will be using [6] as a main reference in this section.

First we notice that two given finite (epistemic) states, say ε and ε', are formula indistinguishable, that is, $\varepsilon(A) = \varepsilon'(A)$ for any A, if and only if they are equal, i.e. $\varepsilon = \varepsilon'$. (Cf [6, Theorem 3.4].) This observation gives rise to the following definitions (four in one).

Definition 5.1 (epistemic state, generalized (epistemic) state)**.** *An epistemic state is any collection of valuations.*

A generalized epistemic state is an equivalence class of formula indistinguishable epistemic states. By $\overline{\varepsilon}$ we denote the generalized state generated by ε.

Definition 5.2 (**AFE**, **AGE**)**.** **AFE** *and* **AGE** *are the collections of all finite states and generalized ones, respectively. Given $\varepsilon, \varepsilon' \in$* **AFE** *(***AGE***), we define a partial ordering as follows:*

$$\varepsilon \ll \varepsilon' \Rightarrow \text{ for any } A,\ \varepsilon(A) \sqsubseteq \varepsilon'(A).$$

We collected the information about **AFE** and **AGE** in the following proposition.

Proposition 5.1. *The following holds.*

- **AGE** *is a CPO and complete co-atom Heyting algebra; hence* **AGE** *is a distributive lattice.* ([8], Theorem 7)
- *The map $\varepsilon \mapsto \overline{\varepsilon}$ restricted to* **AFE** *is a lattice embedding.* ([6], Theorem 4.3)
- **AGE** *is a Scott domain with* **AFE** *as all finite elements of the former.* ([9], Theorem 1, Corollary 8.2.)

Definition 5.3. *Let $(P, \sqsubseteq)$ be a Scott domain. A subset $B \subseteq P$ is called a basis of P if*

$$\begin{array}{rl} i) & B \cap \{y \mid y \sqsubseteq x\} \text{ is directed, for all } x \in H; \\ ii) & x = \bigsqcup \{y \mid y \in B, y \sqsubseteq x\}. \quad \text{([4], Definition III-4.1)} \end{array}$$

A basis is called effective if the relation $\sqsubseteq$ on the elements of the basis is decidable. ([5])

In the light of Definition 5.3, we have the following.

Proposition 5.2 ([6], Theorem 6.4)**.** *Lattice* **AFE** *is an effective basis of* **AGE**.

What have we achieved by this? We have two things in our hands. First, given finite state ε and generalized state ε', if $\varepsilon \ll \varepsilon'$ then, according to the Scott topology, [4] ε is contained in a neighborhood of ε'. That is, within Scott topology on **AGE**, one can approach any element of **AGE** with elements of **AFE** as close as possible.

Secondly, we prepared a space, that is **AGE**, for defining continuous operations on it. We differ our discussions about operations to Section 7.

6 Leaving the Belnap's logic

So far we have been tied to **A4**, our starting point. It looks like that if we decide to consider another structure, we have have to go all way down all over again by using structures like **L4**, **AFE** and **AGE**. Therefore, we have to think of a unifying framework. We are going to discuss this in the present section.

Definition 6.1 (epistemic structure)**.** *An epistemic structure is an algebraic system* $\mathfrak{F} = (\mathcal{F}, \wedge, \vee, \neg, \mathbf{f}, \mathbf{t}, \bot, \sqsubseteq)$ *of the type* $(2, 2, 1, 0, 0, 0, 2)$*, satisfying the following conditions:*

i) $(\mathcal{F}, \sqsubseteq)$ *is a finite CPO with the least element* $\bot$;
ii) *operations* $\wedge$, $\vee$ *and* $\neg$ *are monotone with respect to* $\sqsubseteq$;
iii) $\neg\mathbf{f} = \mathbf{t}$ *and* $\neg\mathbf{t} = \mathbf{f}$.

Lattice **A4** is an example of an epistemic structure. In fact, it is the simplest four-element epistemic structure, but not the simplest one. The structure depicted below in Diagram 4, known as Kleene's strong logic, [7] is the simplest epistemic structure.

Definition 6.2 (complete semilattice; cf [4])**.** *A poset P is a complete semilattice if any non-empty subset Q of P has a greatest lower bound,* $\sqcap Q$*, and any non-empty directed subset R has a least upper bound,* $\sqcap R$.

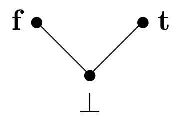

Diagram 4: lattice **A3**

Definition 6.3 (poset $\mathcal{V}$). *The set of the valuations arranged by* $\sqsubseteq$ *in the sense of Definition 4.1 is denoted by* $\mathcal{V}$*, where the valuations under consideration are into a fixed epistemic structure* $\mathfrak{F}$*.*

Proposition 6.1. *Poset* $\mathcal{V}$ *is a complete semilattice. Moreover, for any* $p \in \mathcal{S}$*,*

$$\sqcap\{v_i \mid i \in I\}(p) = \sqcap\{v_i(p) \mid i \in I\};$$
$$\sqcup\{v_i \mid i \in I_0\}(p) = \sqcup\{v_i(p) \mid i \in I_0\},$$

where $\{v_i \mid i \in I_0\}$ *is a non-empty directed set of valuations of* $\mathcal{V}$*.*

In addition, we obtain the following.

Proposition 6.2. *A valuation* $v \in \mathcal{V}$ *(wth a fixed* $\mathfrak{F}$*) is finite in the sense of Definition 4.2 if and only if* v *is a finite element in* $(\mathcal{V}, \sqsubseteq)$*. Therefore,* $(\mathcal{V}, \sqsubseteq)$ *is a Scott domain, where the set of the finite elements is an effective basis.*

Definition 6.4 (poset **ME**). *The poset of minimized finite epistemic states (with respect to an* $\mathfrak{F}$*) is denoted by* **ME***.*

The next proposition gives us a desirable knowledge space. We obtain it through the notion of a powerdomain. Some details of its definition will be given in Section 7. A full definition can be found, e.g., in [5].

Proposition 6.3. *Let* $\mathcal{V}^*$ *be the upper powerdomain of the domain* $\mathcal{V}$*. Then* $\mathcal{V}^*$ *is a Scott domain that has an effective basis isomorphic to* **ME**. *Moreover, given* $\varepsilon, \varepsilon' \in \mathbf{ME}$*,* $\varepsilon \sqcup \varepsilon' \in \mathbf{ME}$*, providing* $\varepsilon \sqcup \varepsilon'$ *exists in* $\mathcal{V}^*$*, which is the case when the set* $\{v \sqcup v' \mid v \in \varepsilon, v' \in \varepsilon'\}$ *is non-empty, and then*

$$\varepsilon \sqcup \varepsilon' = m(\{v \sqcup v' \mid v \in \varepsilon, v' \in \varepsilon'\})$$

and

$$\varepsilon \sqcap \varepsilon' = m(\varepsilon \cup \varepsilon').$$

As an illustration of the construction described in the last proposition, we have the following.

Proposition 6.4. *If $\mathcal{V}$ in Proposition 6.3 is related to* **A4** *as $\mathfrak{F}$, then $\mathcal{V}^*$ and* **AGE***, on the one hand, and* **ME** *and* **AFE***, on the other, are isomorphic.*

7 Knowledge transformers

If one believes that he is sunk in a heresy, he has to go to an end. That is, if we find (as we do) that Scott Principle is good to arrange quanta of knowledge, then we can use the obtained space to define knowledge transformers on it. We already gave our argument in favor of Scott-continuous operations and now we sum up with the following definition.

Definition 7.1 (knowledge transformer)**.** *Given epistemic structure $\mathfrak{F}$, ϕ is a knowledge transformer on $\mathcal{V}^*$ (associated with $\mathfrak{F}$) if it satisfies the following conditions:*

i) *ϕ is a Scott-continuous operation on $\mathcal{V}^*$;*
ii) *ϕ is closed on* **ME**.

Below we consider three operations for the knowledge space associated with a fixed epistemic structure $\mathfrak{F}$. The first two operations are associated with reports $A : \mathbf{t}$ and $A : \mathbf{f}$, that is, with "A is true" and "A is false," respectively. Then, we will discuss one more operation, the idea of which was proposed by Belnap. We could characterize the first two transformers as belonging to the kind of those which are used when the agent communicates with the outer world. The third operation will illustrate the kind of those transformers which the agent uses when, according to some requirements, a current state of knowledge is to be corrected.

7.1 Transformers for communication

Given a statement A, we define:

$$\mathbf{t}(A) = \{v \in \mathcal{V} \mid \mathbf{t} \sqsubseteq v(A), E(v) \subseteq E(A)\},$$
$$\mathbf{f}(A) = \{v \in \mathcal{V} \mid \mathbf{f} \sqsubseteq v(A), E(v) \subseteq E(A)\},$$

where $E(v) = \{p \in \mathcal{S} \mid v(p) \neq \bot\}$ and $E(A)$ is the set of atomic statements occurring in A.

To proceed we need some details of the powerdomain in Proposition 6.3.

Definition 7.2 ($\mathbf{FE}$ and $(\mathbf{FE}, \leq)$)**.** *Let us denote by* $\mathbf{FE}$*, the set of all finite epistemic states* (*associated with an epistemic structure* $\mathfrak{F}$)*. Then, for any* $\varepsilon, \varepsilon' \in \mathbf{FE}$*, we define:*

$$\varepsilon \leq \varepsilon' \Leftrightarrow \textit{for every } v \in \varepsilon, \textit{ there is a } v' \in \varepsilon' \textit{ such that } v \sqsubseteq v'.$$

The relation $\leq$ *is a pre-order on* $\mathbf{FE}$*.*

Next for any $\varepsilon, \varepsilon' \in \mathbf{FE}$, we define:

$$\varepsilon \sim \varepsilon' \Leftrightarrow \varepsilon \leq \varepsilon' \text{ and } \varepsilon' \leq \varepsilon,$$

and also for any $\varepsilon \in \mathbf{FE}$,

$$\widetilde{\varepsilon} = \{\varepsilon' \mid \varepsilon' \sim \varepsilon\}.$$

The two last definitions in this series are the following.

For any $\varepsilon, \varepsilon' \in \mathbf{FE}$,

$$\widetilde{\varepsilon} \preceq \widetilde{\varepsilon'} \Leftrightarrow \varepsilon \leq \varepsilon'.$$

(The correctness of the last definition can be easily obtained.)

Finally,

$$\widetilde{\mathbf{FE}} = \{\widetilde{\varepsilon} \mid \varepsilon \in \mathbf{FE}\}.$$

Now

$$\mathcal{V}^* = (\widetilde{\mathbf{FE}}, \preceq).$$

We return to knowledge transformers corresponding to the two types of reports: "A is true" and "A is false."

First we note that both $\mathbf{t}(A)$ and $\mathbf{f}(A)$ are finite epistemic states, that is, $\mathbf{t}(A), \mathbf{f}(A) \in \mathbf{FE}$.

Then, for any $\varepsilon \in \mathcal{V}^*$, we define

$$[A^+](\varepsilon) = \varepsilon \sqcup \widetilde{\mathbf{t}(A)},$$
$$[A^-](\varepsilon) = \varepsilon \sqcup \widetilde{\mathbf{f}(A)}.$$

It is interesting to notice the following properties:

$$[A^-](\varepsilon) = \neg[A^+](\varepsilon),$$
$$[A^+](\varepsilon) = \neg[A^-](\varepsilon),$$
$$[A^+]([A^+](\varepsilon)) = [A^+](\varepsilon),$$
$$[A^-]([A^-](\varepsilon)) = [A^-](\varepsilon),$$
$$[A^+]([B^+](\varepsilon)) = [B^+]([A^+](\varepsilon)).$$

Proposition 7.1. *Operations* $[A^+]$ *and* $[A^-]$ *are closed on* $\mathbf{ME}$*. Moreover, for any* $\varepsilon \in \mathbf{ME}$*,*

$$[A^+](\varepsilon) = \varepsilon \sqcup m(\mathbf{t}(\varepsilon)),$$
$$[A^-](\varepsilon) = \varepsilon \sqcup m(\mathbf{f}(\varepsilon))$$

Proposition 7.2. *Operations* $[A^+]$ *and* $[A^-]$ *are Scott-continuous on* $\mathcal{V}^*$*.*

The two last propositions were proved for **A4** in [6] as Theorems 7.1 and 7.2, respectively.

7.2 Transformers for adjustment

Suppose we find it useful that when the agent arrives at a state at which a statement A is true, it makes "minimal" changes in the valuations of the state to make a statement B true too. It can be done as follows.

Let us consider an epistemic state ε. Consider a valuation $v \in \varepsilon$. If $\mathbf{t} \sqsubseteq v(A)$ then we adjust v by replacing it with the set of $v \sqcup w$, for each $w \in \mathbf{t}(A)$. Otherwise, we keep the same v.

This definition is not correct, since it in general does not satisfy the requirements of Definition 7.1. To take the latter in the account we have to do the following.

First we define for any valuation v:

$$[A \to B]^*(v) = \begin{cases} \{v \sqcup w \mid w \in \mathbf{t}(A)\} \text{ if } \mathbf{t} \sqsubseteq v(A) \\ \{v\} \text{ otherwise.} \end{cases}$$

Then, we continue to define for any epistemic state ε:

$$[A \to B]^*(\varepsilon) = \cup\{[A \to B]^*(v) \mid v \in \varepsilon\}$$

and then:

$$[A \to B](\widetilde{\varepsilon}) = \widetilde{[A \to B]^*(\varepsilon)}.$$

Proposition 7.3. *Operation* $[A \to B]$ *is closed on* **ME** *and Scott-continuous on* $\mathcal{V}^*$.

8 Computational issues

We have to note that the abstract character of the definitions of lattices **ME**, $\mathcal{V}^*$ and operations $[A^-]$, $[A^+]$ and $[A \to B]$ raises questions of computational aspect. For instance, is there a good algorithm for determining whether $\varepsilon \leq \varepsilon'$, for any two $\varepsilon, \varepsilon' \in$ **ME**? Or are there good algorithms for obtaining $[A^-](\varepsilon)$, $[A^+](\varepsilon)$, $[A \to B](\varepsilon)$, for any $\varepsilon \in$ **ME**. By the present point, it must be clear that in order to have efficient algorithms, we have to look for good representations of the lattices in question. Also, one should take into consideration the complexity of an initial epistemic structure $\mathfrak{F}$. The reader can find solutions to these questions for some cases of $\mathfrak{F}$ in [9, 10].

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