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About one algorithm of bidimensional interpolation using splines

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Abstract

In the paper an explicit algorithm for the problem of two-dimensional spline interpolation on a rectangular grid is proposed.

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1 Introduction

Let's assume that the mesh $\Delta = \Delta_x \times \Delta_y$ is given on the domain $\Omega = [a,b] \times [c,d]$, where $\Delta_x = a = x_0 < x_1 < \ldots < x_n = b$ and $\Delta_y = c = y_0 < y_1 < \ldots < y_m = d$. Let us suppose that values $f(x_i, y_j) = f_{ij}$, $i = \overline{0,n}$, $j = \overline{0,m}$, are known at the knots of mesh Δ. The interpolant $S(x,y)$ is to be constructed such that $S(x_i, y_j) = f_{ij}$. In order to solve this problem, bilinear splines, which are local ones, but derivatives are not continuous, are widely used. If cubic splines of two variables [1] are used, we get an interpolating surface $S(x,y) \in C^{2,2}(\Omega)$, where the class of functions $f(x,y)$, which are continuous together with their derivatives $\frac{\partial^{k+l} f(x,y)}{\partial^k x \partial^l y}$, $k = \overline{0,2}$, $l = \overline{0,2}$, on Ω, is denoted by $C^{2,2}(\Omega)$. But in this case you have to solve at least $n + 2m$ or $m + 2n$ (in the case of periodic end conditions) systems of equations for determining unknown coefficients of the spline. This fact may become critical in the case of large set of data from computational point of view, therefore the problem of elaboration of new algorithms which possess the locality property, still remains actual. Below there is an algorithm for explicit two-dimensional spline interpolation for rectangular grid, which is a generalisation of the algorithm for one-dimensional case presented in [2]. Two-dimensional spline is constructed as a tensor product of one-dimensional splines.

2 Definition of splines

Let us introduce splines as follows: on $\Omega_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$

$$S(x, y) = \varphi_i(t) F_{ij} \phi_j(u), \tag{1}$$

where notations $h_i = x_{i+1} - x_i$, $t = (x - x_i)/h_i$, $l_j = y_{j+1} - y_j$ and $u = (y - y_j)/l_j$ are used.

In (1) the matrix F_{ij} is a matrix represented in the following form:

$$F_{ij} = \begin{pmatrix} F_{ij}^{(0,0)}, F_{ij}^{(0,1)}, F_{ij}^{(0,2)} \\ F_{ij}^{(1,0)}, F_{ij}^{(1,1)}, F_{ij}^{(1,2)} \\ F_{ij}^{(2,0)}, F_{ij}^{(2,1)}, F_{ij}^{(2,2)} \end{pmatrix},$$

where the submatrix

$$F_{ij}^{(0,0)} = \begin{pmatrix} f_{ij}, f_{ij+1} \\ f_{i+1j}, f_{i+1j+1} \end{pmatrix},$$

contains given data at the knots of the mesh and submatrices

$$F_{ij}^{(k,l)} = \begin{pmatrix} m_{ij}^{(k,l)}, m_{ij+1}^{(k,l)} \\ m_{i+1j}^{(k,l)}, m_{i+1j+1}^{(k,l)} \end{pmatrix},$$

where $k = \overline{0, 2}$, $l = \overline{0, 2}$ and $k + l \geq 1$ contain unknown coefficients of the spline $m_{ij}^{(k,l)} = \frac{\partial^{k+l} S}{\partial^k x \partial^l y}(x_i, y_j)$. Vector-functions $\varphi_i(t)$ and $\phi_j(u)$ are defined as follows:

$$\begin{aligned} \varphi_i(t) = \; & (1 - \nu(t), \nu(t), h_i(t^4 - 2t^3 + 2t - \nu(t))/2, \\ & h_i(2t^3 - t^4 - \nu(t))/2, h_i^2(3t^4 - 8t^3 + 6t^2 - \nu(t))/12, \\ & h_i^2(3t^4 - 4t^3 + \nu(t))/12) \end{aligned}$$

and

$$\phi_j(u) = \begin{array}{l}(1-\nu(u), \nu(u), l_j(u^4 - 2u^3 + 2u - \nu(u))/2, \\ l_j(2u^3 - u^4 - \nu(u))/2, l_j^2(3u^4 - 8u^3 + 6u^2 - \nu(u))/12, \\ l_j^2(3u^4 - 4u^3 + \nu(u))/12)^T.\end{array}$$

A function ν, called generating function for the spline (1), has to satisfy the following conditions:

$$\nu(1) = 1,\ \nu(0) = \nu'(0) = \nu'(1) = \nu''(0) = \nu''(1) = 0,$$
$$\nu^{(3)}(0) = \nu^{(3)}(1) = 24 \text{ and } \nu \in C^3[0,1]. \qquad (2)$$

Some examples of generating functions ν (see [2]) are presented below. Conditions (2) are held by functions

$$\nu(t) = t^3(4 + 15t - 48t^2 + 42t^3 - 12t^4),$$

$$\nu(t) = -48+120t-84t^2+106t^3-75t^4+30t^5-48t/(2-t)+48(1-t)/(1+t)$$

or

$$\nu(t) = \begin{cases} 4t^3 + 6t^4 - 12t^5,\ t \in [0,1/2] \\ 1 - 4(1-t)^3 - 6(1-t)^4 + 12(1-t)^5,\ t \in [1/2,1]. \end{cases}$$

Taking into account (2) we have that $\varphi(0) = \phi^T(0) = (1,0,0,0,0,0)$ and $\varphi(1) = \phi^T(1) = (0,1,0,0,0,0)$, therefore from (1) it immediately follows that

$$S(x_i, y_j) = \varphi_i(0)F_{ij}\phi_j(0) = f_{ij},$$
$$S(x_{i+1}, y_j) = \varphi_i(1)F_{ij}\phi_j(0) = f_{i+1j},$$
$$S(x_i, y_{j+1}) = \varphi_i(0)F_{ij}\phi_j(1) = f_{ij+1},$$
$$S(x_{i+1}, y_{j+1}) = \varphi_i(1)F_{ij}\phi_j(1) = f_{i+1j+1},$$

i.e. interpolation conditions are fulfilled.

Due to the fact that $\varphi'(0) = (\phi'(0))^T = (0,0,1,0,0,0)$, $\varphi'(1) = (\phi'(1))^T = (0,0,0,1,0,0)$, $\varphi''(0) = (\phi''(0))^T = (0,0,0,0,1,0)$ and $\varphi''(1) = (\phi''(1))^T = (0,0,0,0,0,1)$ it follows that the following equalities

$$\varphi_{i-1}^{(k)}(1)F_{i-1j} = \varphi_i^{(k)}(0)F_{ij}, \quad k = \overline{0,2},$$

$$F_{ij-1}\phi_{j-1}^{(l)}(1) = F_{ij}\phi_j^{(l)}(0), \quad l = \overline{0,2}$$

are valid. As a result it can be concluded that the function $S(x,y)$ and its derivatives $\frac{\partial^{k+l}S}{\partial^k x \partial^l y}(x,y)$, where $k = \overline{0,2}$, $l = \overline{0,2}$, $k + l \geq 1$, are the continuous ones.

3 Computing formulae for unknown coefficients

In this section we'll present computing formulae for unknown coefficients of the spline (1). Components of submatrix $F_{ij}^{(1,0)}$ are computed using formula

$$m_{ij}^{(1,0)} = \alpha_{1i}\delta_{i-2j}^{(1,0)} + \alpha_{2i}\delta_{i-1j}^{(1,0)} + \alpha_{3i}\delta_{ij}^{(1,0)} + \alpha_{4i}\delta_{i+1j}^{(1,0)}, \; i = \overline{2, n-2}, \; j = \overline{0, m},$$

where $\delta_{ij}^{(1,0)} = (f_{i+1j} - f_{ij})/h_i$, $i = \overline{0, n-1}$, $j = \overline{0, m}$ and

$$\alpha_{1i} = -\frac{h_{i-1}h_i(h_i + h_{i+1})}{(h_{i-2} + h_{i-1})(h_{i-2} + h_{i-1} + h_i)(h_{i-2} + h_{i-1} + h_i + h_{i+1})}$$

$$\alpha_{4i} = \frac{(h_{i-2} + h_{i-1})^2(h_{i-2} + h_{i-1} + h_i)\alpha_{1i}}{(h_{i-1} + h_i + h_{i+1})(h_i + h_{i+1})^2}$$

$$\alpha_{3i} = \frac{h_{i-1} + (h_{i-2} + h_{i-1})\alpha_{1i} - (h_{i-1} + 2h_i + h_{i+1})\alpha_{4i}}{h_{i-1} + h_i}$$

$$\alpha_{2i} = 1 - \alpha_{1i} - \alpha_{3i} - \alpha_{4i}.$$

For components of submatrix $F_{ij}^{(0,1)}$ we have formula of the similar form:

$$m_{ij}^{(0,1)} = \beta_{1j}\delta_{ij-2}^{(0,1)} + \beta_{2j}\delta_{ij-1}^{(0,1)} + \beta_{3j}\delta_{ij}^{(0,1)} + \beta_{4j}\delta_{ij+1}^{(0,1)}, \; i = \overline{0, n}, \; j = \overline{2, m-2}$$

where, respectively, $\delta_{ij}^{(0,1)} = (f_{ij+1} - f_{ij})/l_j$, $i = \overline{0, n}$, $j = \overline{0, m-1}$ and

$$\beta_{1j} = -\frac{l_{j-1}l_j(l_j + l_{j+1})}{(l_{j-2} + l_{j-1})(l_{j-2} + l_{j-1} + l_j)(l_{j-2} + l_{j-1} + l_j + l_{j+1})},$$

$$\beta_{4j} = \frac{(l_{j-2} + l_{j-1})^2(l_{j-2} + l_{j-1} + l_j)\beta_{1j}}{(l_{j-1} + l_j + l_{j+1})(l_j + l_{j+1})^2},$$

$$\beta_{3j} = \frac{l_{j-1} + (l_{j-2} + l_{j-1})\beta_{1j} - (l_{j-1} + 2l_j + l_{j+1})\beta_{4j}}{l_{j-1} + l_j},$$

$$\beta_{2j} = 1 - \beta_{1j} - \beta_{3j} - \beta_{4j}.$$

In order to compute $m_{ij}^{(1,1)}$ we can use values of $m_{ij}^{(1,0)}$ or values of $m_{ij}^{(0,1)}$. In the first case we get

$$m_{ij}^{(1,1)} = \beta_{1j}\delta_y m_{ij-2}^{(1,0)} + \beta_{2j}\delta_y m_{ij-1}^{(1,0)} + \beta_{3j}\delta_y m_{ij}^{(1,0)} + \beta_{4j}\delta_y m_{ij+1}^{(1,0)},$$

$$i = \overline{2, n-2}, \quad j = \overline{2, m-2},$$

where $\delta_y m_{ij}^{(1,0)} = (m_{ij+1}^{(1,0)} - m_{ij}^{(1,0)})/l_j$.

In the second case the computational formula has the form

$$m_{ij}^{(1,1)} = \alpha_{1i}\delta_x m_{i-2j}^{(0,1)} + \alpha_{2i}\delta_x m_{i-1j}^{(0,1)} + \alpha_{3i}\delta_x m_{ij}^{(0,1)} + \alpha_{4i}\delta_x m_{i+1j}^{(0,1)},$$

$$i = \overline{2, n-2}, \quad j = \overline{2, m-2},$$

where $\delta_x m_{ij}^{(0,1)} = (m_{i+1j}^{(0,1)} - m_{ij}^{(0,1)})/h_i$.

Let us require the continuity of $S^{(3,0)}(x, y)$ along $x = x_i$. This requirement has the form

$$\varphi_{i-1}^{(3)}(1) F_{i-1j} \phi_j(u) = \varphi_i^{(3)}(0) F_{ij} \phi_j(u)$$

or

$$\varphi_{i-1}^{(3)}(1) F_{i-1j} = \varphi_i^{(3)}(0) F_{ij}. \tag{3}$$

Taking into account that

$$\varphi_{i-1}^{(3)}(1) = (-24/h_{i-1}^3, 24/h_{i-1}^3, -6/h_{i-1}^2, -18/h_{i-1}^2, 0, 6/h_{i-1})),$$

$$\varphi_i^{(3)}(0) = (-24/h_i^3, 24/h_i^3, -18/h_i^2, -6/h_i^2, 6/h_i, 0)$$

from (3) we get

$$\begin{aligned} m_{ij}^{(2,0)} &= 4\left(\frac{\mu_i \delta_{ij}^{(1,0)}}{h_i} - \frac{\lambda_i \delta_{i-1j}^{(1,0)}}{h_{i-1}}\right) + \frac{\lambda_i (m_{i-1j}^{(1,0)} + 3m_{ij}^{(1,0)})}{h_{i-1}} - \\ &\quad - \frac{\mu_i (3m_{ij}^{(1,0)} + m_{i+1j}^{(1,0)})}{h_i}, \ i = \overline{3, n-3}, \ j = \overline{0, m}, \end{aligned}$$

$$m_{ij}^{(2,1)} = 4(\frac{\mu_i \delta_x m_{ij}^{(0,1)}}{h_i} - \frac{\lambda_i \delta_x m_{i-1j}^{(0,1)}}{h_{i-1}}) + \frac{\lambda_i (m_{i-1j}^{(1,1)} + 3m_{ij}^{(1,1)})}{h_{i-1}} - \\ - \frac{\mu_i (3m_{ij}^{(1,1)} + m_{i+1j}^{(1,1)})}{h_i}, \; i = \overline{3, n-3}, \; j = \overline{2, m-2},$$

$$m_{ij}^{(2,2)} = 4(\frac{\mu_i \delta_x m_{ij}^{(0,2)}}{h_i} - \frac{\lambda_i \delta_x m_{i-1j}^{(0,2)}}{h_{i-1}}) + \frac{\lambda_i (m_{i-1j}^{(1,2)} + 3m_{ij}^{(1,2)})}{h_{i-1}} - \\ - \frac{\mu_i (3m_{ij}^{(1,2)} + m_{i+1j}^{(1,2)})}{h_i}, \; i = \overline{3, n-3}, \; j = \overline{3, m-3}. \quad (4)$$

The notations $\lambda_i = h_i/(h_{i-1} + h_i)$, $\mu_i = 1 - \lambda_i$ are used above.

Similarly, from the requirement of continuity of $S^{(0,3)}(x,y)$ along $y = y_j$, which has the form

$$F_{ij-1} \phi_{j-1}^{(3)}(1) = F_{ij} \phi_j^{(3)}(0),$$

where

$$\phi_{j-1}^{(3)}(1) = (-24/l_{j-1}^3, 24/l_{j-1}^3, -6/l_{j-1}^2, -18/l_{j-1}^2, 0, 6/l_{j-1}),$$

$$\phi_j^{(3)}(0) = (-24/l_j^3, 24/l_j^3, -18/l_j^2, -6/l_j^2, 6/l_j, 0)$$

the following formulae are derived:

$$m_{ij}^{(0,2)} = 4(\frac{\mu'_j \delta_{ij}^{(0,1)}}{l_j} - \frac{\lambda'_j \delta_{ij-1}^{(0,1)}}{l_{j-1}}) + \frac{\lambda'_j (m_{ij-1}^{(0,1)} + 3m_{ij}^{(0,1)})}{l_{j-1}} - \\ - \frac{\mu'_j (3m_{ij}^{(0,1)} + m_{ij+1}^{(0,1)})}{l_j}, \; i = \overline{0, n}, \; j = \overline{3, m-3},$$

$$m_{ij}^{(1,2)} = 4(\frac{\mu'_j \delta_y m_{ij}^{(1,0)}}{l_j} - \frac{\lambda'_j \delta_y m_{ij-1}^{(1,0)}}{l_{j-1}}) + \frac{\lambda'_j (m_{ij-1}^{(1,1)} + 3m_{ij}^{(1,1)})}{l_{j-1}} - \\ - \frac{\mu'_j (3m_{ij}^{(1,1)} + m_{ij+1}^{(1,1)})}{l_j}, \; i = \overline{2, n-2}, \; j = \overline{3, m-3},$$

$$m_{ij}^{(2,2)} = 4\left(\frac{\mu'_j \delta_y m_{ij}^{(2,0)}}{l_j} - \frac{\lambda'_j \delta_y m_{ij-1}^{(2,0)}}{l_{j-1}}\right) + \frac{\lambda'_j (m_{ij-1}^{(2,1)} + 3m_{ij}^{(2,1)})}{l_{j-1}} - $$
$$- \frac{\mu'_j (3m_{ij}^{(2,1)} + m_{ij+1}^{(2,1)})}{l_j}, \; i = \overline{3, n-3}, \; j = \overline{3, m-3}, \quad (5)$$

where $\lambda'_j = l_j/(l_{j-1} + l_j)$, $\mu'_j = 1 - \lambda'_j$.

In order to compute values of $m_{ij}^{(2,2)}$ formula (4) or formula (5) can be used.

Taking into account values which indices take in the formulae presented above, we can conclude that we have an explicit algorithm for the subdomain $\Omega' = [x_3, x_{n-3}] \times [y_3, y_{m-3}]$.

4 Case of uniform mesh

In the case when the mesh is the uniform one, i.e. $h_i = h, \forall i$ and $l_j = l, \forall j$, the computational formulae are more simple. Taking into account that $\lambda_i = \mu_i = \lambda'_j = \mu'_j = 1/2$ and $\alpha_{1i} = \alpha_{4i} = \beta_{1j} = \beta_{4j} = 1/12$, $\alpha_{2i} = \alpha_{3i} = \beta_{2j} = \beta_{3j} = 5/12$ from the previous section we get

$$m_{ij}^{(1,0)} = (-f_{i-2j} - 4f_{i-1j} + 4f_{i+1j} + f_{i+2j})/12h,$$

$$m_{ij}^{(0,1)} = (-f_{ij-2} - 4f_{ij-1} + 4f_{ij+1} + f_{ij+2})/12l,$$

$$m_{ij}^{(1,1)} = (-m_{i-2j}^{(0,1)} - 4m_{i-1j}^{(0,1)} + 4m_{i+1j}^{(0,1)} + m_{i+2j}^{(0,1)})/12h \text{ or}$$

$$m_{ij}^{(1,1)} = (-m_{ij-2}^{(1,0)} - 4m_{ij-1}^{(1,0)} + 4m_{ij+1}^{(1,0)} + m_{ij+2}^{(1,0)})/12l,$$

$$m_{ij}^{(2,0)} = 2(f_{i-1j} - 2f_{ij} + f_{i+1j})/h^2 + (m_{i-1j}^{(1,0)} - m_{i+1j}^{(1,0)})/2h,$$

$$m_{ij}^{(0,2)} = 2(f_{ij-1} - 2f_{ij} + f_{ij+1})/l^2 + (m_{ij-1}^{(0,1)} - m_{ij+1}^{(0,1)})/2l,$$

$$m_{ij}^{(2,1)} = 2(m_{i-1j}^{(0,1)} - 2m_{ij}^{(0,1)} + m_{i+1j}^{(0,1)})/h^2 + (m_{i-1j}^{(1,1)} - m_{i+1j}^{(1,1)})/2h,$$

$$m_{ij}^{(1,2)} = 2(m_{ij-1}^{(1,0)} - 2m_{ij}^{(1,0)} + m_{ij+1}^{(1,0)})/l^2 + (m_{ij-1}^{(1,1)} - m_{ij+1}^{(1,1)})/2l,$$

$$m_{ij}^{(2,2)} = 2(m_{i-1j}^{(0,2)} - 2m_{ij}^{(0,2)} + m_{i+1j}^{(0,2)})/h^2 + (m_{i-1j}^{(1,2)} - m_{i+1j}^{(1,2)})/2h \text{ or}$$

$$m_{ij}^{(2,2)} = 2(m_{ij-1}^{(2,0)} - 2m_{ij}^{(2,0)} + m_{ij+1}^{(2,0)})/l^2 + (m_{ij-1}^{(2,1)} - m_{ij+1}^{(2,1)})/2l.$$

Obviously, for each of the above formulae the indices take the same values as for the corresponding formulae from the previous section.

5 Conclusions

In this paper we restricted ourselves to the interpolation problem on subdomain Ω'. If you want to construct an interpolation surface on the domain Ω you have to use boundary conditions in order to determine coefficients of the spline which remain unknown.

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