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Complex of abstract cubes and median problem

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Abstract

In this paper a special complex $\mathcal{K}^n$ of abstract cubes [2, 3], which contains only n-dimensional cubes is examined. The border of this complex is an abstract $(n-1)$-dimensional sphere. It is proved that the abstract sphere contains at least one 0-dimensional cube, which belongs to exactly n cubes with dimension 1, if the complex is a homogeneous n-dimensional tree. This result allows to solve, in an efficient way, the problem of median for a skeleton of size 1 of the tree with weighted vertices and edges. The algorithm to calculate the median without using any metric is described. The proposed algorithm can be applied with some modifications, for arbitrary complex of abstract cubes.

Mathematics subject classification: 18F15, 32Q60, 68R10

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1 Introduction

Let (X,d) be a metric space, determined by a finite set X, with ordered elements $X=\{x_1,x_2,...,x_n\}$, and $f(x)=\sum\limits_{i=1}^{n} d(x,x_i)p(x_i)$ be a defined function on X, where $p(x_i)$ is a positive real number, called weight of the element $x_i \in X$.

Definition 1.1. *Point $x^* \in X$, which satisfies the following equality*

$$f(x^*)=\min_{x\in X}\sum_{i=1}^{n} d(x,x_i)p(x_i) \tag{1.1}$$

is called ***median*** *of* X.

Median applications for solving applied problems, especially in service location problems, are well known. But calculating procedure of the median according to the formula (1.1), usually in some circumstances is a difficult issue. In the works [2], [3], [4], efficient algorithms for finding the median without using metric are proposed for some special complexes. This paper largely generalizes results presented in the papers [3] and [4]. However, if in [2] the median is calculated by effective way for weighted tree, then in this paper the problem is studied for an abstract homogeneous tree $\mathcal{K}^n$, $n \geq 1$, with dimension n. For the 1-dimensional skeleton (see below) of this tree an efficient algorithm for finding the median without using any metric is proposed.

In the paper [2] algorithm for calculating median for a finite tree with positive weights of the elements is described. It is an elegant algorithm, which has found many practical applications. The same situation we have in the case of a quadrilateral complex [3] with weighted edges especially for the Euclidean space E^2.

The problem, which is studied in this paper was formulated for the first time, and partially solved in the late '60s, at the Institute of Mathematics with Computer Center of Academy of Sciences of Moldova. At that time, the two-dimensional case of the problem was studied. For the greater dimensions any effective solutions weren't obtained. Last years researches have led to some theoretical results for dimensions greater than 2, using new constructions (such is the complex of abstract cubes). These complexes are studied through their groups of direct homologies [5-7].

2 Homogeneous complexes of abstract cubes

Let $r = \{Q^1_{i_1}, Q^1_{i_2}, ..., Q^1_{i_n}\}$, $n \geqslant 2$, be a landmark of 1-dimensional oriented cubes with a common vertex (a 0-dimensional cube), considered the origin of these cubes. Any one-dimensional and oriented cube can be an ordered pair of 0-dimensional cubes (vertices of 1-dimensional cube). Consider $Q^0_{i_0}$ – the origin of cubes which forms the landmark r

and $Q^1_{i_1} = (Q^0_{i_0}, Q^0_{i_1})$, $Q^1_{i_2} = (Q^0_{i_0}, Q^0_{i_2}), ..., Q^1_{i_n} = (Q^0_{i_0}, Q^0_{i_n})$. Next, we describe the landmark r by ordered tuple of indices $(i_{1,}i_{2,...,}i_n)$, which actually is a permutation of the tuple *(1, 2,..., n)*.

It is clear that any landmark r of n oriented cubes, with dimension 1, determines unequivocally an n-dimensional oriented cube Q^n and vice versa. This cube is also determined by ordered tuple of $(i_{1,}i_{2,...,}i_n)$ indices.

Definition 2.1. *If the number of inversions k of tuple $(i_{1,}i_{2,...,}i_n)$ is even (odd), then the n-dimensional cube Q^n, which is determined by the landmark r, is called **positively (negatively) oriented**.*

Corollary 2.1. *The same tuple of indices $(i_{0,}i_{1,...,}i_n)$ describes unequivocally an n-dimensional simplex. Therefore, we consider that the sign of an n-dimensional cube coincides with the sign of n-dimensional simplex, determined by the same landmark r of 1-dimensional cubes.*

Definition 2.2. *Two cubes with dimension 1 are called **abstract-convex** cubes, if their emptinesses [23] have only one common point $x \in M$ of intersection. A cube Q^m, $2 \leqslant m \leqslant n$, is called abstract-convex if in intersection with another abstract-convex cube Q^1 we obtain at most an abstract-convex cube with dimension 1.*

According to Boltyanski [5] and Hilton [7], any abstract and orientable manifold V^n_p without borders [1], [23], which is determined by a finite number of n-dimensional and abstract-convex cubes, can be oriented if it satisfies the following property: any two arbitrary n-dimensional cubes of V^n_p, which have an intersection of an $(n-1)$-dimensional cube, are both positively or negatively directed. In other words, for these two cubes exists an n-dimensional way [22], that unites them – the so-called **strong orientation**, as a connected and oriented graph [11] is.

Definition 2.3. *Let V^n_p be an abstract manifold, determined by abstract-convex cubes, and x', x'' are two points of infinite set M [22],* *which does not belong to V_p^n, and determines an abstract-convex 1-dimensional cube. Let the edge (x', x'') intersects V_p^n in an odd number of points from M. There are two points x_1 and x_2 in V_p^n which leads to a nonorientable one-dimensional cube (edge) of V_p^n, denoted by (x_1, x_2). We consider that the intersection of edges (x', x'') and (x_1, x_2) is only a single point $y \in M$. Points x' and x'' belong to V_p^n. One of these two vacuums we consider as intern and will denote by $\mathbf{int}V_p^n$, and another – exterior, denoted by $\mathbf{ext}V_p^n$ (a situation which generalizes theorem of Jordan and Holder [14], [24]).*

The abstract and oriented cube is defined by abstract simplexes in inductive way [1], starting with an abstract oriented arc. A complex of abstract-convex cubes

$$\mathcal{K}^n = \left\{Q_\lambda^p : 0 \leq p \leq n,\ \lambda \in \Lambda,\ \dim \Lambda < \infty\right\},$$

where $n = dim\mathcal{K}^n$, and his groups of direct homologies over the group $\mathbb{Z}$ of integers:

$$\Delta^0(\mathcal{K}^n, \mathbb{Z}),\ \Delta^1(\mathcal{K}^n, \mathbb{Z}), ..., \Delta^n(\mathcal{K}^n, \mathbb{Z}), \tag{2.1}$$

is defined as it is shown in [7] (case Euclidean space E^n), and as for the complex of abstract simplexes [16].

Definition 2.4. *If K^n and K^p are two complexes of abstract-convex cubes, so that $K^p \subset K^n$, then K^p is called subcomplex of K^n.*

Definition 2.5. *A set of all cubes of K^n, with dimensions $p \geq 0$, is called* **p *-dimensional skeleton*** *of K^n and is denoted by $sk(p)K^n$, $0 \leq p \leq n-1$.*

Obviously, $\mathbf{sk}(p)\mathcal{K}^n$ is a subcomplex of $\mathcal{K}^n$ and $\mathbf{sk}(1)\mathcal{K}^n$ is its oriented graph [11]. Next we will make the following denotations:

- $\mathbb{Q}^p$ – the family of all p-dimensional and abstract-convex cubes of $\mathcal{K}^n$, $0 \leq p \leq n$;

- $G = (X; U)$ – a graph which represents the 1-dimensional skeleton of $\mathcal{K}^n$, where $X = \mathbb{Q}^0$, and $U = \mathbb{Q}^1$. Hence $\mathbb{Q}^0$ and $\mathbb{Q}^1$ are the sets of vertices and arcs of G.

Definition 2.6. *The complex K^n of abstract-convex cubes, satisfying the following conditions:*

a) if Q^p, $0 \leq p \leq n$ is an element of K^n, then every facet $Q^k \subset Q^p$, $0 \leq k < p$, is an element of K^n;

b) for every two cubes Q^{p_1} and Q^{p_2} of K^n, the intersection $Q^{p_1} \cap Q^{p_2}$ is empty or is an element of K^n, where $0 \leq p_1, p_2 \leq n$;

c) any cube Q^k of K^n, $0 \leq k \leq n$, belongs at least to one n-dimensional cube Q^n of K^n;

d) for any two subcomplexes K_1^n and K_2^n of K^n, which satisfy conditions a)-c) and $K_1^n \cup K_2^n = K^n$, their intersection is a subcomplex $K^p \subset K^n$, with dimension $p = n - 1$;

e) the homology group of rank zero is isomorphic to the group of integers $\mathbb{Z}$, i.e.

$$\Delta^0(K^n, \mathbb{Z}) \cong \mathbb{Z}; \tag{2.2}$$

f) the homology groups of rank $1, 2, 3, ...n$ are isomorphic to zero, i.e.

$$\Delta^1(K^n, \mathbb{Z}) \cong \Delta^2(K^n, \mathbb{Z}) \cong ... \cong \Delta^n(K^n, \mathbb{Z}) \cong 0, \tag{2.3}$$

is called ***homogeneous*** *n-dimensional abstract complex and is denoted by K_A^n.*

Following the conditions e) and f) of this definition for the complex $\mathcal{K}_A^n$ means that it is **connected** and **acyclic** [5], [7].

Definition 2.7. *The set of all $(n-1)$ -dimensional cubes of the complex K_A^n, which belong exactly to one n-dimensional cube Q^n of K_A^n, will be called the border of this complex and will be denoted by*

bdK_A^n. The set of vacuums of all cubes of K_A^n, which do not belong to the border bdK_A^n will be called interior of this complex and will be denoted by $intK_A^n$.

Definition 2.8. *[22] An orientable variety V_p^n, which is determined by abstract cubes will be called abstract n-dimensional sphere, if V_p^n satisfies the conditions:*

$$\chi(V_p^n) = 2, \quad for\ n = 2m,$$
$$\chi(V_p^n) = 0, \quad for\ n = 2m-1, \tag{2.4}$$

We will denote this sphere by $S^n = V_0^n$, where χ is the Euler characteristic:

$$\chi\left(V_p^n\right) = \sum_{i=0}^{n} (-1)^i \alpha_i, \tag{2.5}$$

and α_i is the number of i-dimensional cubes, $0 \le i \le n$ [5-7].

Theorem 2.1. *If K_A^n is a convex-abstract and homogeneous n-dimensional cubic complex, then its boundary bdK_A^n is an abstract $(n-1)$-dimensional sphere S^{n-1}.*

Proof: First we will show that the border $bd\mathcal{K}_A^n$ is an abstract $(n-1)$-dimensional orientable manifold V_p^{n-1}, with genus $p = 0$.

Suppose the contrary. Let $p > 0$. In these circumstances, taking into account the homogeneity of $\mathcal{K}_A^n$ and the assumption that $p > 0$, we immediately obtain a contradiction: the complex $\mathcal{K}_A^n$ is not acyclic, (contradicts the condition f) of the Definition (2.6)); the variety V_p^{n-1} can be *cut* by a variety V_q^{n-2}, $1 \le q \le p$, and in this case we will have at least $\Delta^{n-1}(V_p^{n-1}, \mathbb{Z}) \ncong 0$ [22]. In this case, satisfying the equalities (2.3), we obtain that V_p^{n-1} is not a variety. Then p loses its meaning. According to the homogeneity of $\mathcal{K}_A^n$ it leads to the fact that this complex does not satisfy the condition d) of the Definition 2.6: there does not exist two subcomplexes $\mathcal{K}_{A(1)}^n$ and $\mathcal{K}_{A(2)}^n$, such that

$\mathcal{K}^n_{A(1)} \cap \mathcal{K}^n_{A(2)} = \mathcal{K}^p_{A(3)}$, where $0 \leq p \leq n-2$. Thus $V_p^{n-1} = V_0^{n-1} = \mathbb{S}^{n-1}$. ■

Theorem 2.2. *If V_p^n is an orientable (abstract) variety, determined by abstract-convex cubes, and any cycle of size k, $1 \leq k \leq n-1$, is homologous to* 0*, then V_p^n, is an abstract sphere.*

Proof: The variety V_p^n is connected [22]. Therefore $\Delta^0(V_p^n, \mathbb{Z}) \cong \mathbb{Z}$. Let C_0^k be an arbitrary cycle of variety V_p^n. If all k-dimensional cycles, $1 \leq k \leq n-1$, are homologous to zero, then in $\mathbf{sk}(k)V_p^n$, each such cycle is the boundary of some subcomplex V_p^K This leads to the fact that the subgroup Z_0^k of the group of cycles Z^k coincides with Z_0^k. Therefore, the factor-group does not contain cycles which are not homologous to 0. This means that $\Delta^k(V_p^n, Z) \cong 0$, $1 \leq k \leq n-1$. Thus we have:

$$\Delta^1(V_p^n, \mathbb{Z}) \cong \Delta^2(V_p^n, \mathbb{Z}) \cong \ldots \cong \Delta^{n-1}(V_p^n, \mathbb{Z}) \cong 0.$$

Considering the Euler-Poancare equality [22]:

$$\chi(V_p^n) = \sum_{i=0}^{n} (-i)^i \alpha_i = \sum_{i=0}^{n} (-1)^i r_i,$$

where α_i is the number of i-dimensional cubes of V_p^n, and r_i is the rank of the group $\Delta^i(V_p^n, \mathbb{Z})$, $0 \leq i \leq n$, we obtain:

$$\chi(V_p^n) = 1 + 0 + \ldots + (-1)^n 1 = \left\{ \begin{array}{ll} 2, & \text{for } n = 2m; \\ 0, & \text{for } n = 2m-1 \end{array} \right\}.$$

However, according to the Definition 2.8, the variety V_p^n is an abstract n-dimensional sphere $\mathbb{S}^n$.■

In the paper [9] it is defined the notion of emptiness *(vacuum)* of a p-dimensional cube, $1 \leq p \leq n$. For the 0-dimensional cube Q^0, we consider that it coincides with its emptiness, and it will be called 0-dimensional emptiness.

Corollary 2.2. *Let $Q^k \in \mathbb{Q}^k$ be a convex-abstract, k-dimensional cube, $0 \leq k \leq n$, of some homogeneous complex. Then the variety bdQ^k is an abstract $(k-1)$-dimensional sphere.*

The proof of this assertion follows immediately from the Theorem 2.1.

3 n-dimensional homogeneous tree

Let us first explain some auxiliary issues.

Theorem 3.1. *If $\mathbb{S}^{n-1}$ is an abstract sphere determined by the border $bd\mathcal{K}_A^n$ of a homogeneous complex $\mathcal{K}_A^n$ of abstract-convex cubes, then it has at least one cube (vertex) $Q^0 \in \mathbb{Q}^0$, that exactly belongs to n 1-dimensional cubes (edges).*

To prove this theorem some additional examinations are necessary.

Lemma 3.1. *There exists at least one homogeneous n -dimensional complex $\mathcal{K}_A^n$ with the property that every cube $Q^p \in \mathbb{Q}^p$ from $\mathbf{int}K_A^n$, which intersects the border $\mathbf{bd}\mathcal{K}_A^n$ at most through a cube of the dimension $(p-1)$, $1 \leq p \leq n$, is incident to a number not less than 2^{n-p} of n-dimensional cubes.*

Proof is done in a constructive way. Let Q_1^n be a cube. Let us stick the cube Q_2^n to the cube Q_1^n so that $Q_1^n \cap Q_2^n = Q^{n-1}$ and continue this process in a way not contrary to the Definition 2.2. It is enough to stop this process when we obtain **the complex** $\mathcal{K}_A^n$.

According to the Definition 2.5. we may consider that any n-dimensional abstract sphere $\mathbb{S}^n$ is determined by a complex of n-dimensional abstract cubes or at least of abstract simplexes. Obviously, for any abstract sphere $\mathbb{S}^n$ there always exists a complex of n-dimensional abstract cubes $\mathcal{K}^{n+1}$ (not necessary homogeneous), so that its border is $\mathbb{S}^n$. The interior of the complex $\mathcal{K}^{n+1}$ will be considered to be the interior of the n-dimensional sphere $\mathbb{S}^n$ and will be denoted by $int\mathbb{S}^n$. The union $\mathbb{S}^n \cup int\mathbb{S}^n$ is called an *n-dimensional disk***.**

To examine the possibility of using the homogeneous complex of abstract cubes, in solving some practical problems, some additional issues are necessary to be examined. First let define the notion of parallel edges class of a homogeneous complex $\mathcal{K}_A^n$. Let Q^n be an n-dimensional abstract cube and $\mathcal{F}_1^{n-1}$ and $\mathcal{F}_2^{n-1}$ two n-dimensional opposite **facets** of this cube. The cube Q^n contains 2^{n-1} edges between the facet vertices $\mathcal{F}_1^{n-1}$ and $\mathcal{F}_2^{n-1}$.

Definition 3.1. *Edges of the cube Q^n, that merge vertices of the facets F_1^{n-1} and F_2^{n-1} are called parallel edges of this cube. The set of all parallel edges between two $(n-1)$-dimensional facets we will denote by $C(Q^n)$.*

Obviously the set content is unequivocally determined by the pair of opposite facets $\mathcal{F}_1^{n-1}$ and $\mathcal{F}_2^{n-1}$. In the cube Q^n there exist n different sets of parallel edges.

Let us iteratively choose a special family of n-dimensional cubes. We denote by i the numbers of cubes of this family. Initially we will consider this family empty, i.e. $i = 0$. To make this kind of family we should follow the following 4 steps:

p.1. *Let us choose an n-dimensional cube Q^n of the complex K_A^n. So, we may consider that $i = 1$. Let us denote by $Q_T^n(1)$ the family of cubes and by $C(Q_T^n(1))$ – one of the parallel edges set of the chosen cube Q^n.*

p.2. *Suppose that some $i \geqslant 1$ n-dimensional cubes from K_A^n were selected. Thus we obtained family of cubes $Q_T^n(i)$, for which the parallel edges set $C(Q_T^n(i))$ is known.*

p.3. *Let us choose a new cube (if there exists one) $Q_*^n \in \mathbb{Q}^n \backslash \mathbb{Q}_T^n(i)$, which contains at least an edge from $\mathbb{C}(\mathbb{Q}_T^n(i))$. We denote by $\mathbb{C}(Q_*^n)$ the parallel edges set of this cube, that satisfies the following relation: $\mathbb{C}(\mathbb{Q}_T^n(i)) \cap \mathbb{C}(Q_*^n) \neq 0$, and forms new sets*

$$\mathbb{Q}_T^n(i+1) = \mathbb{Q}_T^n(i) \cup \{Q_*^n\}$$

$$\mathbb{C}(\mathbb{Q}_T^n(i+1)) = \mathbb{C}(\mathbb{Q}_T^n(i)) \cup \mathbb{C}(Q_*^n).$$

p.4. *Repeat step 3 until it is possible. Since only finite n-dimensional homogeneous complex is studied, at a certain point we will reach the situation when we cannot select an n-dimensional cube from K_A^n that satisfies the step 3. In this case we will consider the searched family of n-dimensional cube formed.*

Definition 3.2. *The family of cubes, constructed according to the steps p.1-p.4, will be called* ***n-dimensional transversal*** *of the complex $\mathcal{K}_A^n$. We will denote this family by T^n, and by $\mathbb{C}(T^n)$ – the respective class (set) of parallel edges.*

By the Definition 2.6. any n-dimensional homogeneous complex $\mathcal{K}_A^n$ contains $m \geq n$ sets of parallel edges, that we will denote by $\mathbb{C}_1, \mathbb{C}_2, ..., \mathbb{C}_m$. The equality $m = n$ is true only if $\mathcal{K}_A^n$ is formed from a single n-dimensional cube Q^n.

From those mentioned above follows that any class of parallel edges $\mathbb{C}_i, 1 \leq i \leq m$, determines an n-dimensional transversal and vice versa, any n-dimensional transversal determines a set of parallel edges. Also we will consider that any class of parallel edges $\mathbb{C}_i$, $1 \leq i \leq m$, generates unequivocally an n-dimensional homogeneous subcomplex of abstract cubes (see definition 2.6). This subcomplex is determined by the facets family, both own and unfit, of all n-dimensional cubes from transversal T_i^n. We will denote this subcomplex by $\mathcal{K}_i^n$.

Corollary 3.1. *The n-dimensional subcomplex $\mathcal{K}_i^n$, generated by class of parallel edges $\mathbb{C}_i$, $1 \leq i \leq m$, of the complex $\mathcal{K}_A^n$, is an n-dimensional subcomplex from $\mathcal{K}_A^n$.*

The border of the complex $\mathcal{K}_i^n$ contains exactly two maximal $(n-1)$-dimensional and acyclic subcomplexes, that don't contain an edge from the respective class of parallel edges $\mathbb{C}_i$. We denote these subcomplexes by $\mathcal{K}_{i(1)}^{n-1}$ and $\mathcal{K}_{i(2)}^{n-1}$. Prior let call $\mathcal{K}_{i(1)}^{n-1}$ "left" facet, and $\mathcal{K}_{i(2)}^{n-1}$ - "right" facet of the transversal T_i^n.

Definition 3.3. *The off-empty union of all abstract-convex cubes with dimension k, $1 \leq k \leq n$, that belong to the complex K_i^n, but do not belong to the left and right facets of transversal T_i^n, will be called vacuum of transversal T_i^n and will be denoted by $V(T_i^n)$.*

Corollary 3.2. *Any transversal T_i^n of an abstract and homogeneous complex $\mathcal{K}_A^n$ divides this complex, through its vacuum $V(T_i^n)$, in two connected complexes of abstract cubes. Each of these complexes is not necessary homogeneous, and has at least the dimension equal to $n-1$.*

Definition 3.4. *n-dimensional transversals $T_{i_1}^n, T_{i_2}^n, ..., T_{i_q}^n$ of an abstract and homogeneous complex K_A^n, determined by the classes of parallel edges $\mathbb{C}_{i_1}^n, \mathbb{C}_{i_2}^n, ..., \mathbb{C}_{i_q}^n$, $2 \leq q \leq m$, are called pairwise neighbors transversals if any two transversals $T_{i_r}^n, T_{i_s}^n$, $0 \leq i_r, i_s \leq q$, satisfy the following conditions:*

1) $V(T_{i_r}^n) \cap V(T_{i_s}^n) = \varnothing$;

2) *the classes of parallel edges $\mathbb{C}_{i_r}$ and $\mathbb{C}_{i_s}$ contain each at least one edge $Q_1' \in \mathbb{C}_i$ and $Q_1'' \in \mathbb{C}_j$, that at their turn have a common vertex.*

From the Definition 3.4. it follows: For a transversal $T_{i_k}^n$ we will denote the left and the right facets by $T_{i_k(1)}^n$ and $T_{i_k(2)}^n$. If transversals $T_{i_1}^n, T_{i_2}^n, ..., T_{i_q}^n$ are *pairwise neighbors, then there exists a forked transversal,* $T_{i_{1,q}}^n = \bigcup\limits_{j=1}^{q} T_{i_j}^n \cap T_{i_k}^n$, that divides $\mathcal{K}_A^n$ in many connected subcomplexes. Let us denote by $st\mathcal{K}_{i_k(1)}^n$ and $dr\mathcal{K}_{i_k(2)}^n$ the components respectively determined by the "left" and "right" facets of transversal $T_{i_k}^n$. For these components the relation $st\mathcal{K}_{i_k(1)}^n = \mathcal{K}_A^n \backslash dr\mathcal{K}$ is true.

Let T_i^n, $1 \leq i \leq m$, be any transversal of the complex $\mathcal{K}_A^n$ and $\mathcal{K}_{i(1)}^{n-1}$ be its "left" facet

Definition 3.5. *$(n-1)$-dimensional maximal connected subcomplex of the complex $\mathcal{K}_A^n$, with the following properties:*

1) *any subcomplex contains the facet $\mathcal{K}_{i(1)}^{n-1}$ of the transversal T_i^n;*

2) *any two $(n-1)$-dimensional cubes of this subcomplex do not belong to an n-dimensional cube from $\mathcal{K}_A^n$,*

is called an $(n-1)$-dimensional transversal, determined by n-dimensional transversal T_i^n, $1 \le i \le m$.

According to the Definition 3.5., any $n-$dimensional transversal of K_A^n determines exactly two $(n-1)$-dimensional transversals. Also an $(n-1)$-dimensional transversal could be determined by more than one n-dimensional transversal.

By analogy with the notation of n-dimensional transversal, we will denote the $(n-1)$-dimensional transversal by T^{n-1}.

$(n-1)$-Dimensional transversal, determined by the transversal T_i^n and containing the facet $\mathcal{K}_{i(1)}^{n-1}$ will be denoted by $T_{i(1)}^{n-1}$; and one that contains the facet $\mathcal{K}_{i(2)}^{n-1}$, will be denoted by $T_{i(2)}^{n-1}$.

Lemma 3.2. *If T_i^n is an $(n-1)$-dimensional transversal from K_A^n, and $T_{i(1)}^{n-1}$, $T_{i(2)}^{n-1}$ are the $(n-1)$-dimensional transversals of the K_A^n, determined by the transversal T_i^n, then $T_{i(1)}^{n-1} \cap T_{i(2)}^{n-1} = \varnothing$, $1 \le i \le m$.*

Proof. Let $T_{i(1)}^{n-1}$ and $T_{i(2)}^{n-1}$ be $(n-1)$-dimensional transversals, determined by an n-dimensional T_i^n. Their intersection is not empty. This intersection may coincide with a cube $Q^0 \in \mathbb{Q}^0$ or at least with a cube $Q^{n-2} \in \mathbb{Q}^{n-2}$. If this intersection is a cube $Q^0 \in \mathbb{Q}^0$, then two n-dimensional cubes from T_i^n and incident to Q^0 are degraded, what is excluded. The same situation is for the cube $Q^{n-2} \in \mathbb{Q}^{n-2}$: Q^{n-2} is the opposite facet with the dimension $n-1$ of the n-dimensional cube. Thus we have at least an n-dimensional degraded cube.

Proof of the Theorem 3.1.

Let $T_{i_k}^{n-1}$, $1 \leq k \leq m$ be an $(n-1)$-dimensional transversal. If $k \neq m$, then we will consider the transversal $T_{i_{k+1}}^{n-1}$. According to lemma 3.2. we obtain $T_{i_k}^{n-1} \cap T_{i_{k+1}}^{n-1} = \varnothing$. Let us consider the following $(n-1)$-dimensional transversals till $T_{i_m}^{n-1}$. We obtain again $T_{i_{m-1}}^{n-1} \cap T_{i_m}^{n-1} = \varnothing$. Let us consider such $n-1$ sections of n-dimensional and abstract-connected cubes from $\mathcal{K}_A^n$, that two neighbors are intersected through an $(n-1)$-dimensional facet, opposite to $(n-1)$-dimensional facet of the first and the second cube, which is at the border of T_m^n and intersects through the n-dimensional cube $Q^n \in \mathbb{Q}^n$. This cube is abstract-convex. Cube Q^n contains a vertex from circular border of $(n-1)$-dimensional cubes, that has a vertex $Q^0 \in \mathbb{Q}^0$ with $n+1$ edges incident to $st\mathbb{S}^{n-1}$.■

Let $\mathcal{K}_A^n$ be a homogeneous abstract complex (see Definition 2.6.), but non-oriented.

As for the Theorem 2.2, the border $\mathbf{bd}\mathcal{K}_A^n$ is an $(n-1)$-dimensional cubical sphere $\mathbb{S}^{n-1}$. Let $\mathbf{st}(Q^0)\mathbb{S}^{n-1}$ be the star of the vertex Q^0, calculated on sphere $\mathbb{S}^{n-1}$.

Definition 3.6. *An n-dimensional homogeneous undirected abstract complex $\mathcal{K}_A^n$, that satisfies the following conditions:*

1) *any cube $Q^k \in \mathbf{int}\mathcal{K}_A^n$ belongs at least to 2^{n-k} n-dimensional cube from $\mathcal{K}_A^n$, $0 \leq k \leq n$;*

2) *if Q^0 is a vertex from $\mathbf{bd}\mathcal{K}_A^n$, that has exactly n incident arcs of $\mathbf{bd}\mathcal{K}_A^n$, and the star $\mathbf{st}(Q^0)\mathbf{bd}\mathcal{K}_A^n$ contains exactly n cubes of $(n-1)$ dimension of $\mathbf{bd}\mathcal{K}_A^n$, then the n-dimensional cube Q^n determined by this star belongs to the complex K_A^n;*

is called ***n-dimensional homogeneous tree****. We will denote this tree by A^n.*

By the Definition 3.6. we exclude the situation from Figure 1 (Figure 1 is shown for a better understanding of the examined situation.)

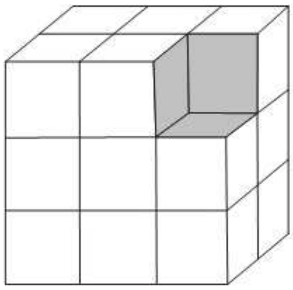

Figure 1. A complex, which does not satisfy the condition 2) of the Definition 3.6.

Note 3.1. *The first condition of the Definition 3.6. is assured by the Lemma 3.1.*

Let us next consider the following function:

$$P : \mathbb{Q}^0 \to R^+ \tag{3.1}$$

and the length of the edges from the classes of parallel edges $\mathbb{C}_1, \mathbb{C}_2, ..., \mathbb{C}_m$ equal to the numbers

$$d_1, d_2, ..., d_p, \tag{3.2}$$

where $d_i > 0$, for $1 \leq i \leq m$. Thus all edges that belongs to a class have the same length.

For any $Q^0 \in \mathbb{Q}^0$ number $p(Q^0)$ is the share of Q^0.

4 The representation of an n-dimensional tree in a normed space

Let $\mathbb{Q}^0 = \left\{Q^0_\lambda : \lambda \in \Lambda\right\}$ be the vertices set and $\mathbb{Q}^1 = \left\{Q^1_\mu : \mu \in \mathcal{M}\right\}$ be the edges set of the tree A^n. Let us fix an arbitrary chain $L^1 = (Q^1_{\mu_1}, Q^1_{\mu_2}, ..., Q^1_{\mu_k})$ with the origin $Q^0_{\lambda_i}$ and the extremity in $Q^0_{\lambda_j}$;

$\mu_1, \mu_2, ..., \mu_k \in \mathcal{M}$; $\lambda_i, \lambda_j \in \Lambda$. Next we will form an integer non-negative number (see conditions (3.1.) and (3.2.)).

$$d(Q^0_{\lambda_i}, Q^0_{\lambda_j}) = t_1 d_1 + t_2 d_2 + .. + t_k d_k, \tag{4.1}$$

where $t_i = 0$ (or $t_i = 1$), if L^1 passes an even (odd) number of times through the edges from $\mathbb{Q}$, that belongs to the class $\mathbb{C}_i$, $1 \leq i \leq p$.

Theorem 4.1. *The relation (4.1) represents the Hamming metric [25].*

Proof. From the relation (4.1) and the condition (3.2) we have

1) $d(Q^0_{\lambda_i}, Q^0_{\lambda_j}) \geq 0$, and from the equality $d(Q^0_{\lambda_i}, Q^0_{\lambda_j}) = 0$ it follows that $t_1 = t_2 = ... = t_k = 0$, and $Q^0_{\lambda_i} = Q^0_{\lambda_j}$;

2) the equality $d(Q^0_{\lambda_i}, Q^0_{\lambda_j}) = d(Q^0_{\lambda_j}, Q^0_{\lambda_i})$ is obvious;

3) according to (4.1) for three different vertices $Q^0_{\lambda_i}$, $Q^0_{\lambda_j}$, $Q^0_{\lambda_k}$, it is easy to prove the equality

$$d(Q^0_{\lambda_i}, Q^0_{\lambda_j}) = d(Q^0_{\lambda_j}, Q^0_{\lambda_k}) + d(Q^0_{\lambda_k}, Q^0_{\lambda_j}).$$

The last equality proves that (4.1) is the Hamming metric. ■

Suppose that the tree A^n has classes of parallel edges $\mathbb{C}_1, \mathbb{C}_2, ..., \mathbb{C}_m$ with the fixed length (3.2). Let us also consider space R^m_1 over real numbers set with norm $\|x\| = \sum_{1=i}^{m} |x_i|$. We will fix on the coordinate axes $OY_1, OY_2, ..., OY_p$, from the origin $O \in R^m_1$, segments with lengths $d_1, d_2, ..., d_m$. Let us make unequivocally a parallelepiped $\mathbb{P}^m$ on these segments. The set of all k-dimensional facets of this parallelepiped forms a complex of k-dimensional parallelepipeds, $0 \leq k \leq m$. We will denote this complex by $\mathcal{P}^k = \{\mathbb{P}^k \subset \mathbb{P}^m, 0 \leq k \leq m\}$. For the case $k = 1$ we obtain the complex $\mathcal{P}^1$, that represents union of all 0- and 1-dimensional facets from $\mathbb{P}^m$. This complex $\mathcal{P}^1$ is a subcomplex of the $\mathcal{P}^m$. The complex $\mathcal{P}^1$ represents a connected, metric and undirected graph. This graph will be denoted by $H = (Y; V)$, where Y is the vertices set from $\mathcal{P}^1$, and V – the edges set from $\mathcal{P}^1$.

Theorem 4.2. *For the tree A^n there exists an unequivocal application $\alpha : A^n \to P^m$, that intersects A^n on a subcomplex from P^m, so that $\alpha : G \to P^1$ represents an isometry.*

Proof. The truth of the theorem follows from the construction method of the complex $\mathbb{P}^m$ and relation (4.1).

We will denote by $Y = \{y_1, y_2, ..., y_n\} \subset \alpha(A^n)$ the vertices set $\alpha(X)$ in the space R_1^m. Let us consider the following function:

$$f(y) = \sum_{i=1}^{n} p(y_i) \parallel y - y_i \parallel, \tag{4.1*}$$

where y_i is the image of $\alpha(x_i)$ and has the weight $p(y_i) = p(\alpha(x_i))$, $1 \leq i \leq n$.

We will prove that the point $y_* \in R_1^m$, that minimizes the function (4.1*), is **median** of the graph $H = (Y; V)$.

Note 4.1. The condition 2) from the Definition 3.6. is necessary in calculation of the median of skeleton $\mathbf{sk}(1)K_A^n$, that may not be on $\mathbf{sk}(0)K_A^n$. For example, let us consider the complex K_A^2 with $\mathbb{Q}^2 = \{Q_1^2, Q_2^2, Q_3^2\}$, $\mathbb{Q}^1 = \{(x_1, x_2), (x_1, x_4), (x_1, x_6), (x_2, x_3), (x_3, x_4), (x_4, x_5), (x_5, x_6), (x_6, x_7), (x_7, x_2)\}$ and $\mathbb{Q}^0 = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ (The geometric representation of this complex is given in Figure 2.)

For this complex we consider the edges length 1, and the vertices weight $p(x_1) = p(x_2) = p(x_4) = p(x_6) = 1$, $p(x_3) = p(x_5) = p(x_7) = 100$. According to the formula (1.1) the vertex x_8^* that does not belong to $\mathbb{Q}^0 = \mathbf{sk}(0)K_A^2$ is a median.

Thus we denote by y the vectors from R_1^m. So for the vector $y = (y^1, y^2, ..., y^m) \in R_1^m$ let us form the differences $y^i - y_1^i$, $y^i - y_2^i$, ..., $y^i - y_n^i$, $1 \leq i \leq m$, and the set of indices

$$\begin{aligned} I_i^+ &= j : y^i - y_j^i > 0, \\ I_i^0 &= j : y^i - y_j^i = 0, \\ I_i^- &= j : y^i - y_j^i < 0. \end{aligned} \tag{4.2}$$

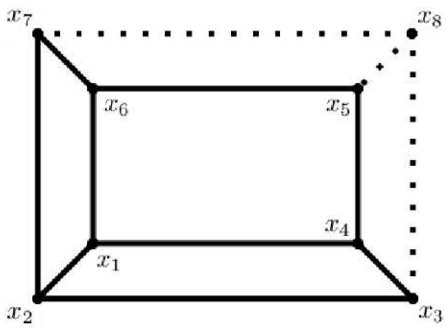

Figure 2. A complex, which does not contain the median.

The relations (4.2) are formed as in [4].

Theorem 4.3. *The vector $y_j \in R_1^m$ minimizes the function (4.1*) if and only if the following relations are satisfied:*

$$\sum_{j\in I_i^+\cup I_i^0} p(y_j) \geq \sum_{j\in I_i^-} p(y_j),$$

$$\sum_{j\in I_i^+} p(y_j) \geq \sum_{j\in I_i^0\cup I_i^-} p(y_j). \tag{4.3}$$

The Proof of the theorem is analogous to that of the Theorem 3.1. from [3].

The Theorem 4.3. permits us to state an important fact: *median of metric graphic H does not depend on the edges lengths $d_1, d_2, ..., d_m$ from V*. Thus, the parallelepiped $\mathbb{P}^m$ could be replaced by a unitary cube of the R_1^m, using the operation of the expansion or constraint for each edge with lengths d_i, $1 \leq i \leq m$.

So we obtain mapping

$$\beta : \mathbb{P}^m \to I^m, \tag{4.4}$$

where I^m is a *unitary* cube the vertices of which have the coordinates formed from 0 and 1 of the space R_1^m. As a result of this mapping the graph H passes in a metric graph $\beta(H)$ that we will denote by $\mathcal{G} = (Z; W) \subset I^m$. We have Hamming metric both on the cube I^m and on the graph G. The vertices set Z of this graph is $\beta\alpha(\mathbb{Q}^0)$, and the edges set W represents the edges set $\beta\alpha(\mathbb{Q}^1)$. The classes of parallel edges $\mathbb{C}_1, \mathbb{C}_2, ..., \ \mathbb{C}_m$ are transformed into the following classes of parallel edges from the cube I^m:

$$\mathbb{C}_1^1 = \beta\alpha(\mathbb{C}_1), \mathbb{C}_2^1 = \beta\alpha(\mathbb{C}_2), ..., \beta\alpha(\mathbb{C}_m^1) = \beta\alpha(\mathbb{C}_m).$$

The union of these classes does not necessary cover all the edges from unitary cube I^m. It could be a cover only if $A^n = \mathcal{P}^m$.

For the set of vertices $Z = \{z_1, z_2, ..., z_n\}$ the same weights are kept:

$$p(z_1), \quad p(z_2), ..., p(z_n), \tag{4.5}$$

where $p(z_i) = p(\beta\alpha(x_i), 1 \leq i \leq m$.

Thus we have an isometric mapping:

$$\beta\alpha : G \to \mathcal{G}.$$

Let the vertex $z_i \in Z$ of the cube I^m has the coordinates:

$$z_i = (z_i^1, z_i^2, ..., z_i^m),$$

where $z_i^j = 1$ or $z_i^j = 0$.

Because the median from R_1^m does not depend on metric, let consider that all lengths $d_1, d_2, ..., d_m$ are equal to 1. We will denote A^n by $A^n(1)$. Thereby the mapping $\beta\alpha(A^n(1)) \to I^m$ is also an isometry, where $\beta\alpha(A(1))$ is a subcomplex of the complex, formed from the facets of I^m.

Let us form the matrix as we did in [3]:

$$N = \begin{array}{cc}
\begin{array}{cccccc} C_1^1 & C_2^1 & ... & C_j^1 & ... & C_m^1 \\ \downarrow & \downarrow & & \downarrow & & \downarrow \end{array} & \\
\begin{pmatrix}
z_1^1 & z_1^2 & ... & z_1^j & ... & \varepsilon_1^m \\
z_2^1 & z_2^2 & ... & z_2^j & ... & z_2^m \\
... & ... & ... & ... & ... & ... \\
z_i^1 & z_i^2 & ... & z_i^j & ... & z_i^m \\
... & ... & ... & ... & ... & ... \\
z_n^1 & z_n^2 & ... & z_n^j & ... & z_n^m
\end{pmatrix} &
\begin{array}{c} \leftarrow z_1 \\ \leftarrow z_2 \\ ... \\ \leftarrow z_i \\ ... \\ \leftarrow z_n \end{array}
\end{array}$$

For every column from the matrix N we calculate a new tuple $r = (r_1, r_2, ..., r_j, ..., r_m)$, considering $r_j = 1$ or $r_j = 0$:

a) if the scalar product $(z_1^j, z_2^j, ..., z_n^j)(p(z_1), p(z_2), ..., p(z_n)) = \sum_{i=1}^n z_i^j p(z_i)$ is a number bigger than $\frac{1}{2}\sum_{i=1}^n p(z_i)$, then $r_j = 1$, and if this product is less than the sum, then $r_j = 0$;

b) if the following equality:

$$\sum_{i=1}^{n} Z_i^j p(z_i) = \frac{1}{2}\sum_{i=1}^{n} p(z_i), \tag{4.6}$$

is true, then the value of r_j is chosen arbitrarily: 0 or 1.

Theorem 4.4. *Any vector $r = (r_1, r_2, ..., r_m)$ is a line of the matrix N.*

To prove this theorem some additional issues are necessary.

Let T_i^{n-1} be an $(n-1)$-dimensional transversal of the complex $A^n(1)$. This transversal divides $A^n(1)$ in two different parts $T_{i(1)}^n$ and $T_{i(2)}^n$, with a non-empty intersection T_i^{n-1}. These represent some distinct subcoplexes $A_1^n(j)$, $j = 1, 2$ of the A_1^n. We will call $T_{i(1)}^n$ and $T_{i(2)}^n$ n-dimensional subcomplexes, determined by T_i^{n-1} (let call them *left* and *right* subcomplexes).

Next let us denote by $\mathbb{Q}_{i(j)}^0$, $1 \le i \le B$, $j \in \{1, 2\}$, $\mathbb{Q}_{i(1)}^0$ and $\mathbb{Q}_{i(2)}^0$ the respective sets of vertices of $A^n(1)$, $T_{i(1)}^n$ and $T_{i(2)}^n$. Obviously the following relations are true:

$$\mathbb{Q}^0_{i(1)} \cap \mathbb{Q}^0_{i(2)} = \mathbb{Q}^0_{i(j)},$$
$$\mathbb{Q}^0_{i(1)} \cup \mathbb{Q}^0_{i(2)} = \mathbb{Q}^0, \tag{4.7}$$

If $\mathbb{Q}^0_{i(1)}$ and $\mathbb{Q}^0_{i(2)}$ are the sets of vertices of n-dimensional complexes $T^n_{i(1)}$ and $T^n_{i(2)}$, determined by the transversals $T^n_{i(j)}$, $j \in \{1,2\}$, $1 \leq i \leq B$, where B is the number of all n-dimensional transversals, then we will denote by $p(\mathbb{Q}^0_{i(1)})$, $p(\mathbb{Q}^0_{i(2)})$ the weight sum of the respective left and right set of vertices, i.e.:

$$p(\mathbb{Q}^0_{i(1)}) = \sum_{x_i \in \mathbb{Q}^0_{i(1)}} p(x_i),$$
$$p(\mathbb{Q}^0_{i(2)}) = \sum_{x_i \in \mathbb{Q}^0_{i(2)}} p(x_i). \tag{4.8}$$

The numbers $p(\mathbb{Q}^0_{i(1)})$ and $p(\mathbb{Q}^0_{i(2)})$ will be called weights of the respective n-dimensional complexes.

Theorem 4.5. *A vertex x_* of the graph $G = (X; U) = sk(1)A^n(1)$ is a median of G if and only if this vertex represents in $A^n(1)$ the intersection of n transversals $T^n_{i(j)}$, $j \in \{1,2\}$, $1 \leq i \leq B$, of distinct directions pairwise, for which the weight sum of the pair of n-dimensional complexes $A^n_{i_1(1)}$, $A^n_{i_1(2)}$,..., $A^n_{i_n(1)}$, $A^n_{i_n(2)}$, determined by the mentioned and accommodated transversals at the graph G, satisfies the following relations:*

$$\mathrm{p}(\mathrm{Q}^0_{i_1(1)}) = \mathrm{p}(\mathrm{Q}^0_{i_1(2)}),$$
$$p(\mathbb{Q}^0_{i_2(1)}) = p(\mathbb{Q}^0_{i_2(2)}), \tag{4.9}$$

....

$$p(\mathbb{Q}^0_{i_n(1)}) = p(\mathbb{Q}^0_{i_n(2)}).$$

Proof. Necessity. Let x_* be the median vertex of the graph $G = (X; U) \subset A^n(1)$. This vertex is the median of the tree $A^n(1)$ also. We observe first that according to the condition c) of the cubes complex, any its vertex is situated at the intersection of at least n $(n-1)$-dimensional transversals pairwise. As for the Theorem 4.2, it follows unequivocally that through $T^{n-1}_{i_1}, T^{n-1}_{i_2}, ..., T^{n-1}_{i_b}$ there exist some n transversals at the intersection of which there is the median vertex x_*, for which it is necessary to take place relations analogous to (4.3). Let us be more explicit. Let $A^n_{i_1(1)}$, $A^n_{i_1(2)}$, $A^n_{i_2(1)}$, $A^n_{i_2(2)}$,..., $A^n_{i_n(1)}$, $A^n_{i_n(2)}$ be the pairs of subcomplexes of the $A^n(1)$, determined by the mentioned transversals; $Q^0_{i_1(1)}$, $Q^0_{i_1(2)}$; $Q^0_{i_2(1)}$, $Q^0_{i_2(2)}$; $Q^0_{i_n(1)}$, $Q^0_{i_n(2)}$ – the sets of vertices of the respective subcomplexes; and $p(Q^0_{i_1(1)})$, $p(Q^0_{i_1(2)})$; $p(Q^0_{i_2(1)})$, $p(Q^0_{i_2(2)})$;..., $p(Q^0_{i_n(1)})$, $p(Q^0_{i_n(2)})$ – the pairs of sums from the theorem. Some of the mentioned subcomplexes may be even some transversals $T^{n-1}_{i_1}, T^{n-1}_{i_2}, ..., T^{n-1}_{i_n}$. Obviously the inequalities (4.6) determine the vertex x_*, i.e.

$$T^{n-1}_{i_1} \cap T^{n-1}_{i_2} \cap ... \cap T^{n-1}_{i_n} = x_*, \tag{4.10}$$

because the relations (4.9) are the adapted ones to the A^n from (4.3).

Let the relations (4.9) be again verified.

$$T^{n-1}_{i_1} \cap T^{n-1}_{i_2} \cap ... \cap T^{n-1}_{i_n} = \oslash. \tag{4.11}$$

Through a simple syllogism we get to a contradiction that the condition 2 from the Definition 3.1. is not satisfied.

According to the results from [4] it follows that the set of all medians of the graph $\mathcal{G}$= $(Z; W)$ represents a facet-cube of the I^m. So, if we would be interested in those medians of the graph $\mathcal{G}$, that are also the vertices of this graph, then these are the vertices of the respective facets. More than that, *their existence does not depend on the distances (3.2).*

Now let us return to the Theorem 4.4.

Proof. Suppose the opposite. That means that there exists a vertex $r = (r_1, r_2, ..., r_m)$ of the cube I^m, that does not belong to the graph $\beta\alpha(G) = \mathcal{G}$, where β is the mapping

$$\beta : \mathbb{P}^m \to I^m.$$

Let $r = z_* \in I^n \backslash \mathcal{G}$ minimize the function $f(z)$, that is analogous to (4.1). The point z_* has m facets of dimension $m-1$. These facets for the cubic complex of all facets from I^m, represent some $(n-1)$-dimensional transversals, that have the point (vector) z_* as intersection. Each of these facets contains an $(n-1)$-dimensional transversal of the complex $\beta\alpha(A_n)$, that is isometric with $A_n(1)$. Now let us mention that z_* satisfies the pair of relations (4.9), adapted to the complex $\beta\alpha(A_n(1))$. Also, in the complex $\beta\alpha(A_n(1))$ any m transversals $\beta\alpha(T_{i_1}^{n-1})$, $\beta\alpha(T_{i_2}^{n-1})$, ..., $\beta\alpha(T_{i_m}^{n-1})$ of $(n-1)$-dimension has an empty intersection, because the vector z_* does not belong to them. This is in contradiction with the Theorem 4.2.

5 The algorithm of median calculation for an n-dimensional tree

Let $A^n(1)$ be an n-dimensional tree. From those studied above follows that the median of the tree $A^n(1)$ could be determined by the median z_* calculated in the m-dimensional cube I^m for a special graph $\mathcal{G}$. This graph could be obtained as a result of two consecutive mappings α and β. Thus, if z_* is the median of the graph $\mathcal{G}$ in the cube I^m then median x^* of the tree $A^n(1)$ is determined by the relation

$$x^* = \alpha^{-1}\beta^{-1}(z^*).$$

The obtained results let us describe as an efficient algorithm of median calculation in $A^n(1)$ which does not depend on metric.

So the searched algorithm is as follows:

1) Let $X = \{x_1, x_2, ..., x_m\} = Q^0$

2) To find the classes of parallel edges in the tree $A^n(1)$. Suppose we have m classes of parallel edges

$$\mathbb{C}_1, \mathbb{C}_2, ..., \mathbb{C}_m;$$

3) To establish an arbitrary vertex from X, for example x_1, and put in correspondence the tuple $x_1 = (0, 0, ...0)$, formed from m zeros;

4) For any other vertex $x_i \in X$ we choose an arbitrary chain $L^1(x_1, x_i)$, that merge together two vertices x_1 and x_i;

5) The vertex x_i will have in correspondence the tuple

$$x_i = (\varepsilon_i^1, \varepsilon_i^2, ..., \varepsilon_i^m),$$

for $\varepsilon_i^j = 1$, if the chain $L^1(x_1, x_i)$ passes an odd number of times through the edges of the class $\mathbb{C}_j$, and for $\varepsilon_i^j = 0$, if this number is even, $i \in \overline{1, n}$.

From tuples we form a matrix M, the lines of which represent the tuples, proper to the vertices x_i, $1 \leq i \leq n$:

$$M = \begin{array}{c} \begin{array}{cccc} \mathbb{C}_1 & \mathbb{C}_2 & ... & \mathbb{C}_m \\ \downarrow & \downarrow & & \downarrow \end{array} \\ \begin{pmatrix} \varepsilon_1^1 & \varepsilon_1^2 & ... & \varepsilon_1^m \\ \varepsilon_2^1 & \varepsilon_2^2 & ... & \varepsilon_2^m \\ ... & ... & ... & ... \\ \varepsilon_n^1 & \varepsilon_n^2 & ... & \varepsilon_n^m \end{pmatrix} \end{array} .$$

6) To calculate a new tuple $r^* = (r_1^*, r_2^*, ..., r_m^*)$ using the matrix M elements and the vertices weight $p(x_i)$ from X, $1 \leq i \leq n$, according to the rules:

$$r_j = \begin{cases} 1, \text{ if } \sum\limits_{i=1}^{n} \varepsilon_i^j p(x_i) > \frac{1}{2} \sum\limits_{i=1}^{n} p(x_i), \\ 0, \text{ if } \sum\limits_{i=1}^{n} \varepsilon_i^j p(x_i) < \frac{1}{2} \sum\limits_{i=1}^{n} p(x_i), \\ 0 \text{ or } 1 \text{ (unconcerned), if } \sum\limits_{i=1}^{n} \varepsilon_i^j p(x_i) = \frac{1}{2} \sum\limits_{i=1}^{n} p(x_i). \end{cases}$$

7) As for the the Theorem 4.4. the tuple r^* belongs to the matrix M, and the the vertex x^* which corresponds to this tuple is the median A^n.

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