Maximal induced colorable subhypergraphs of all uncolorable $BSTS(15)$s

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Abstract

A Bi-Steiner Triple System ($BSTS$) is a Steiner Triple System with vertices colored in such a way that the vertices of each block receive precisely two colors. When we consider all $BSTS(15)$s as mixed hypergraphs, we find that some are colorable while others are uncolorable. The criterion for colorability for a $BSTS(15)$ by Rosa is containing $BSTS(7)$ as a subsysytem. Of the 80 non-isomorphic $BSTS(15)$s, only 23 meet this criterion and are therefore colorable. The other 57 are uncolorable. The question arose of finding maximal induced colorable subhypergraphs of these 57 uncolorable $BSTS(15)$s. This paper gives feasible partitions of maximal induced colorable subhypergraphs of each uncolorable $BSTS(15)$.

1 Introduction

1.1 Mixed Hypergraphs

The concept of mixed hypergraphs was introduced in [4] in 1993 by V. Voloshin. A **mixed hypergraph** is a triple $H=(X,C,D)$, where X is a finite vertex set, C is a family of subsets of X called C-edges, and D is a family of subsets of X called D-edges. If a mixed hypergraph has $C = D$, then it is called a **bi-hypergraph** and its edges are called **bi-edges**. For a proper coloring of a mixed hypergraph, C-edges must contain at least two vertices of the same color and D-edges must contain at least two vertices of different colors. If at least one proper coloring of a mixed hypergraph exists, then it is called **colorable**; otherwise, it is called **uncolorable**. The **strong deletion of a vertex** from a hypergraph is the removal of x from H along with all C-edges and D-edges containing x. An **induced subhypergraph** *H'=(X',C',D')* of a mixed hypergraph *H=(X,C,D)* is obtained through the strong deletion of vertices from *X*. If H is an uncolorable mixed hypergraph and after the deletion of some vertex (or vertices) the induced subhypergraph *H'* becomes colorable, then *H'* is called an **induced colorable subhypergraph** of *H*. It is called a **maximal induced colorable subhypergraph** of H if, with the addition of any of the deleted vertices from *H*, it becomes uncolorable. [5]

1.2 Steiner Triple Systems

A **Steiner System** is a block design of the form $S(t, k, v)$ where v is the total number of vertices, k is the number of vertices that are in each block and t is the number of distinct vertices that appear together in precisely λ blocks. When $\lambda = 1$, $k = 3$, $t = 2$, and $v \equiv 1$ or $3(mod6)$, it is called a **Steiner Triple System** or $STS(v)$ [6]. If a $STS(v)$ is considered as a bi-hypergraph *H=(X,B,B)* where X is a finite vertex set and $|X| = v$, B is the family of 3-element subsets of X known as blocks (which are bi-edges), and each pair of distinct elements of X appear together in precisely one block, then it is called a **Bi-Steiner Triple System** or $BSTS(v)$. Bi-Steiner Triple Systems are also known as bi-colorings of Steiner Triple Systems [1]. Since each block consists of bi-edges and each block contains exactly 3 vertices, the vertices of each block are colored with precisely 2 colors by the definition of a proper coloring of a mixed hypergraph.

2 Method

The number of blocks b in a $BSTS(v)$ is given by the following:

$$b = \frac{v(v-1)}{6}.$$

So in the case of a $BSTS(15)$, $b = 35$. Also, each vertex of a $BSTS(v)$ is contained in r blocks, where

$$r = \frac{v-1}{2}.$$

Therefore, for a $BSTS(15), r = 7$. If just one vertex is strongly deleted from X, then $|X'| = 15 - 1 = 14$ and $b = 35 - 7 = 28$. It follows that there exists a maximal induced colorable subhypergraph *H'=(X',B',B')* of any uncolorable $BSTS(15)$ *H=(X,B,B)* with $|X'| \leq 14$ and $|B'| \leq 28$. In this paper, the case where $|X'| = 14$ and $|B'| = 28$ is proved for every uncolorable $BSTS(15)$. In order to find any *H'* of *H*, we must strongly delete a vertex and test for colorability using any number of colors.

3 Theorem and Proof

Theorem 1. *Every uncolorable* $BSTS(15)$ *H=(X,B,B) has some maximal induced colorable subhypergraph H'=(X',B',B') obtained through the strong deletion of exactly one vertex from X.*

Proof. To show that all 57 uncolorable $BSTS(15)$s have a maximal induced colorable subhypergraph obtained by one vertex deletion, we delete vertex {15}, or as the vertices are labeled in [2] vertex $\{e\}$, and show a proper coloring by listing a feasible partition of *X'*. Note that obviously the partitions listed below are not all of the possible partitions for *H'*, nor is {15} (or $\{e\}$) the only vertex which can be deleted in order to obtain *H'* from *H*. Some systems can have any vertex from *X* deleted to obtain *H'* while others only have certain vertices that can be deleted to obtain *H'*. Also, many of the maximal induced colorable subhypergraphs of these uncolorable $BSTS(15)$s are colorable using 3 and 4 colors; however, here we only show a partition into 3 cells for 3 colors. These calculations were made with the aid of a computer program written by B. Tolbert and myself. The systems below are numbered as in [2] (Note that systems no. 1-22 and no.61 are the 23 colorable $BSTS(15)$s).

$BSTS(15)$ no. 23	$\{1, 2, 9, 13, 14\} \bigcup \{3, 12\} \bigcup \{4, 5, 6, 7, 8, 10, 11\}$
$BSTS(15)$ no. 24	$\{1, 2, 4, 7, 8\} \bigcup \{3, 5, 6\} \bigcup \{9, 10, 11, 12, 13, 14\}$
$BSTS(15)$ no. 25	$\{1, 2, 4\} \bigcup \{3, 5, 6, 7\} \bigcup \{8, 9, 10, 11, 12, 13, 14\}$
$BSTS(15)$ no. 26	$\{1, 2, 4\} \bigcup \{3, 5, 6, 7\} \bigcup \{8, 9, 10, 11, 12, 13, 14\}$
$BSTS(15)$ no. 27	$\{1, 2, 4, 7, 11\} \bigcup \{3, 5, 6\} \bigcup \{8, 9, 10, 12, 13, 14\}$
$BSTS(15)$ no. 28	$\{1, 2, 4, 7, 8\} \bigcup \{3, 5, 6\} \bigcup \{9, 10, 11, 12, 13, 14\}$
$BSTS(15)$ no. 29	$\{1, 2, 4, 7, 12\} \bigcup \{3, 5, 6\} \bigcup \{8, 9, 10, 11, 13, 14\}$
$BSTS(15)$ no. 30	$\{1, 2, 4, 7, 11\} \bigcup \{3, 5, 6\} \bigcup \{8, 9, 10, 12, 13, 14\}$
$BSTS(15)$ no. 31	$\{1, 2, 4, 7, 11\} \bigcup \{3, 5, 6\} \bigcup \{8, 9, 10, 12, 13, 14\}$
$BSTS(15)$ no. 32	$\{1, 2, 4, 7, 11\} \bigcup \{3, 5, 6\} \bigcup \{8, 9, 10, 12, 13, 14\}$
$BSTS(15)$ no. 33	$\{1, 2, 4, 7, 8\} \bigcup \{3, 5, 6\} \bigcup \{9, 10, 11, 12, 13, 14\}$
$BSTS(15)$ no. 34	$\{1, 2, 4, 7, 8\} \bigcup \{3, 5, 6\} \bigcup \{9, 10, 11, 12, 13, 14\}$
$BSTS(15)$ no. 35	$\{1, 2, 7, 10, 13, 14\} \bigcup \{3, 12\} \bigcup \{4, 5, 6, 8, 9, 11\}$
$BSTS(15)$ no. 36	$\{1, 3, 5, 7, 8, 12, 14\} \bigcup \{2, 10, 11\} \bigcup \{4, 6, 9, 13\}$
$BSTS(15)$ no. 37	$\{1, 3, 5, 7, 8, 10\} \bigcup \{2, 4\} \bigcup \{6, 9, 11, 12, 13, 14\}$
$BSTS(15)$ no. 38	$\{1, 2, 7, 9, 10, 13, 14\} \bigcup \{3, 4, 5, 6, 11\} \bigcup \{8, 12\}$
$BSTS(15)$ no. 39	$\{1, 3, 4, 6, 13, 14\} \bigcup \{2, 12\} \bigcup \{5, 7, 8, 9, 10, 11\}$

$BSTS(15)$ no. 40	$\{1,2,7,10,13,14\}\bigcup\{3,12\}\bigcup\{4,5,6,8,9,11\}$
$BSTS(15)$ no. 41	$\{1,2,7,10,13,14\}\bigcup\{3,12\}\bigcup\{4,5,6,8,9,11\}$
$BSTS(15)$ no. 42	$\{1,3,5,7,8,10\}\bigcup\{2,4\}\bigcup\{6,9,11,12,13,14\}$
$BSTS(15)$ no. 43	$\{1,4,6,8,10,12,14\}\bigcup\{2,3,7\}\bigcup\{5,9,11,13\}$
$BSTS(15)$ no. 44	$\{1,2,5,6,13\}\bigcup\{3,8,9,10,11,12,14\}\bigcup\{4,7\}$
$BSTS(15)$ no. 45	$\{1,2\}\bigcup\{3,4,5,6,7\}\bigcup\{8,9,10,11,12,13,14\}$
$BSTS(15)$ no. 46	$\{1,3,5,7,8,10\}\bigcup\{2,4\}\bigcup\{6,9,11,12,13,14\}$
$BSTS(15)$ no. 47	$\{1,3,5,7,8,10\}\bigcup\{2,4\}\bigcup\{6,9,11,12,13,14\}$
$BSTS(15)$ no. 48	$\{1,5\}\bigcup\{2,3,6,7,10,11\}\bigcup\{4,8,9,12,13,14\}$
$BSTS(15)$ no. 49	$\{1,3,5,7,9,11\}\bigcup\{2,10\}\bigcup\{4,6,8,12,13,14\}$
$BSTS(15)$ no. 50	$\{1,2\}\bigcup\{3,4,5,6,7\}\bigcup\{8,9,10,11,12,13,14\}$
$BSTS(15)$ no. 51	$\{1,3,5,7,8,10\}\bigcup\{2,4\}\bigcup\{6,9,11,12,13,14\}$
$BSTS(15)$ no. 52	$\{1,3,4,6,11,13,14\}\bigcup\{2,5,8,9,12\}\bigcup\{7,10\}$
$BSTS(15)$ no. 53	$\{1,4,6,9,10,12\}\bigcup\{2,3,8,11,13,14\}\bigcup\{5,7\}$
$BSTS(15)$ no. 54	$\{1,5\}\bigcup\{2,3,6,7,10,11\}\bigcup\{4,8,9,12,13,14\}$
$BSTS(15)$ no. 55	$\{1,2,4,7,8,11\}\bigcup\{3,5,9\}\bigcup\{6,10,12,13,14\}$
$BSTS(15)$ no. 56	$\{1,3,4,6,11,13,14\}\bigcup\{2,5,8,9,12\}\bigcup\{7,10\}$

$BSTS(15)$ no. 57	$\{1,3,5,7,8,10\}\bigcup\{2,4\}\bigcup\{6,9,11,12,13,14\}$
$BSTS(15)$ no. 58	$\{1,4,6,9,10,12\}\bigcup\{2,3,8,11,13,14\}\bigcup\{5,7\}$
$BSTS(15)$ no. 59	$\{1,11\}\bigcup\{2,3,6,7,8,9\}\bigcup\{4,5,10,12,13,14\}$
$BSTS(15)$ no. 60	$\{1,2,4,9,10,13,14\}\bigcup\{3,8\}\bigcup\{5,6,7,11,12\}$
$BSTS(15)$ no. 62	$\{1,2,4,7,12\}\bigcup\{3,5,6\}\bigcup\{8,9,10,11,13,14\}$
$BSTS(15)$ no. 63	$\{1,2,4,7,11\}\bigcup\{3,5,6\}\bigcup\{8,9,10,12,13,14\}$
$BSTS(15)$ no. 64	$\{1,2,4,7,11\}\bigcup\{3,5,6\}\bigcup\{8,9,10,12,13,14\}$
$BSTS(15)$ no. 65	$\{1,12\}\bigcup\{2,3,8,9,13,14\}\bigcup\{4,5,6,7,10,11\}$
$BSTS(15)$ no. 66	$\{1,2,5,6,8,13,14\}\bigcup\{3,4,7,9,12\}\bigcup\{10,11\}$
$BSTS(15)$ no. 67	$\{1,8,10\}\bigcup\{2,3,4,5,9\}\bigcup\{6,7,11,12,13,14\}$
$BSTS(15)$ no. 68	$\{1,12\}\bigcup\{2,3,8,9,13,14\}\bigcup\{4,5,6,7,10,11\}$
$BSTS(15)$ no. 69	$\{1,8,10\}\bigcup\{2,3,4,5,9\}\bigcup\{6,7,11,12,13,14\}$
$BSTS(15)$ no. 70	$\{1,2,5,9,10,13,14\}\bigcup\{3,8\}\bigcup\{4,6,7,11,12\}$
$BSTS(15)$ no. 71	$\{1,2,5,9,10,13,14\}\bigcup\{3,8\}\bigcup\{4,6,7,11,12\}$
$BSTS(15)$ no. 72	$\{1,5,7,9,11,12\}\bigcup\{2,3,4,10,13,14\}\bigcup\{6,8\}$
$BSTS(15)$ no. 73	$\{1,3,4,7,11,13,14\}\bigcup\{2,5,6,10,12\}\bigcup\{8,9\}$
$BSTS(15)$ no. 74	$\{1,6,9,11,12\}\bigcup\{2,3,4,5,10,13,14\}\bigcup\{7,8\}$

$BSTS(15)$ no. 75	$\{1,2,5,6,10,12\} \bigcup \{3,7,11,13,14\} \bigcup \{4,8,9\}$
$BSTS(15)$ no. 76	$\{1,9\} \bigcup \{2,3,6,7,12,13,14\} \bigcup \{4,5,8,10,11\}$
$BSTS(15)$ no. 77	$\{1,4,6,8,10\} \bigcup \{2,3,9\} \bigcup \{5,7,11,12,13,14\}$
$BSTS(15)$ no. 78	$\{1,4,6,8,10,13\} \bigcup \{2,3,5\} \bigcup \{7,9,11,12,14\}$
$BSTS(15)$ no. 79	$\{1,3\} \bigcup \{2,6,7,12,13,14\} \bigcup \{4,5,8,9,10,11\}$
$BSTS(15)$ no. 80	$\{1,3\} \bigcup \{2,8,9,12,13,14\} \bigcup \{4,5,6,7,10,11\}$

Therefore, each uncolorable $BSTS(15)$ has a maximal induced colorable subhypergraph using 3 colors with the deletion of exactly one vertex.

$\square$

4 How the Program Works

This program was written in the C++ language and contains several sub-programs and functions. We created incidence matrices for each colorable $BSTS(15)$ as text files in the source code. We added display functions for all relevant data to check our results and to double check the computer results. We started by creating files that would find partitions and collect them with different permutations of colors being collected as a single partition. The main program calls the incidence matrix that is specified in a subprogram and displays it along with each block of vertices. The program then prompts the user to enter the number of colors that are to be used and then the number of colorings the user wishes to find (first 10 or first 200 for example, or the user can enter -1 for all colorings). If the user wants to find all proper colorings, the program runs an exhaustive search of all possible colorings from a string of all 0s to a string of all 3s. If a coloring is proper, then the coloring is displayed and counted; and if it is not a proper coloring, then that coloring is skipped. Also, if the coloring is proper, then that feasible partition is stored. After all proper colorings have been found and displayed and counted, the monitor prompts the user to press any key to see the feasible partitions displayed and counted and the number of colorings of each partition. All of the different permutations of colors of the partitions that were stored from the proper colorings are grouped together by the computer and only the first permutation of colors is displayed. For example, 011222233333333 would be displayed and 122333300000000 would not be displayed because it is a permutation of the same partition where vertex 1 is mapped to one color, vertices $2, 3$ are mapped to one color, vertices $4 - 7$ are mapped to one color, and vertices $8 - 15$ are mapped to one color. By altering the incidence matrices to account for vertex deletions, we were able to use this program to test the colorability of these induced subhypergraphs of every uncolorable $BSTS(15)$. This enabled us to check the accuracy of our hypothesis and our results; and by displaying all of the relevant data on the monitor, we were able to check the accuracy of the computer results [3].

5 Concluding Remarks

This paper shows that the minimum number of vertex deletions required for a maximal induced colorable subhypergraph of each uncolorable $BSTS(15)$ is precisely one. Obviously with the deletion of vertex {15}, there are more partitions than those listed in the proof; however, they are not needed to show the existence of one *H'* from *H*. However, it remains to find a maximal partial colorable subhypergraph for each uncolorable $BSTS(15)$. The **weak removal of edges** is the deletion of edges without changing the vertex set *X*. A **partial subhypergraph** *H'=(X,B',B')* of an uncolorable $BSTS(15)$ *H=(X,B,B)* is obtained through the weak removal of C-edges and/or D-edges resulting in *B'*. It is called a **partial colorable subhypergraph** if with the weak removal of edges, *H'* becomes colorable. Given H, which is an uncolorable $BSTS(15)$, then *H'*, which is a partial colorable subhypergraph, is called a **maximal partial colorable subhypergraph** of H if adding any C-edge or D-edge of H to *H'* makes it uncolorable [5].

References

[1] C. J. Colbourn, J. Dinitz, A. Rosa. *Bi-coloring Steiner Triple Systems.* The Electronic Journal of Combinatorics 6, (1999), R25.

[2] C. J. Colbourn, A. Rosa. *Triple Systems.* (Oxford University Press, Inc.), New York, (1999).

[3] J. Mathews, B. Tolbert. *All proper colorings of every colorable $BSTS(15)$.* Computer Science Journal of Moldova, 1 (2010), 41–53.

[4] V. I. Voloshin. *The mixed hypergraphs.* Computer Science Journal of Moldova, 1 (1993), 49–52.

[5] V. I. Voloshin. *Introduction to Graph and Hypergraph Theory.* (Nova Science Publishers, Inc.), New York, (2009), 202–204.

[6] E. W. Weisstein, *Steiner Triple System*, Mathworld – A Wolfram Web Source, http://mathworld.wolfram.com/SteinerTripleSystem.html, (2009).

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