Computer Science Journal of Moldova, vol.18, no.3(54), 2010

Perfect Octagon Quadrangle Systems with an upper C_4-system and a large spectrum *

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To the memory of our dear Lucia

Abstract

An *octagon quadrangle* is the graph consisting of an 8-cycle $(x_1, x_2, ..., x_8)$ with two additional chords: the edges $\{x_1, x_4\}$ and $\{x_5, x_8\}$. An *octagon quadrangle system* of order v and index λ $[OQS]$ is a pair (X, H), where X is a finite set of v vertices and H is a collection of edge disjoint octagon quadrangles (called *blocks*) which partition the edge set of λK_v defined on X. An *octagon quadrangle system* $\Sigma = (X, H)$ of order v and index λ is said to be *upper* $C_4 - perfect$ if the collection of all of the *upper* 4-cycles contained in the octagon quadrangles form a μ-fold 4-cycle system of order v; it is said to be *upper strongly perfect*, if the collection of all of the *upper* 4-cycles contained in the octagon quadrangles form a μ-fold 4-cycle system of order v and also the collection of all of the *outside* 8-cycles contained in the octagon quadrangles form a ϱ-fold 8-cycle system of order v. In this paper, the authors determine the spectrum for these systems, in the case that it is the largest possible.

1 Introduction

A λ-fold *m-cycle system* of order v is a pair $\Sigma = (X, C)$, where X is a finite set of n elements, called *vertices*, and C is a collection of edge disjoint m-cycles which partitions the edge set of λK_v, (the complete graph defined on a set X, where every pair of vertices is joined by λ edges). In this case, $|C| = \lambda v(v-1)/2m$. The integer number λ is also

* Lavoro eseguito nell'ambito di un progetto PRIN 2008.

called the *index* of the system. When $\lambda = 1$, we will simply say that Σ is an *m-cycle system.* Fairly recently the spectrum (the set of all v such that an *m-cycle system* of order v exists) has been determined to be [1][12]:

$$v \geq m, \text{ if } v > 1, \quad v \text{ is odd}, \quad \frac{v(v-1)}{2m} \text{ is an integer.}$$

The spectrum for λ-fold *m-cycle systems* for $\lambda \geq 2$ is still an open problem.

In these last years, G-decompositions of λK_v have been examined mainly in the case in which G is a polygon with some chords forming an inside polygon whose sides joining vertices at distance two. Many results can be found at first in [4,11,13] and after in [5,10,12]. Recently, octagon triple systems and dexagon triple systems have been studied in [3,14]. Generally, in these papers, the authors determine the spectrum of the corresponding systems and study problems of embedding.
In [6,7,8,9], Lucia Gionfriddo introduced another idea: she studied G-decompositions, in which G is a polygon with chords which determine at least a quadrangle. Further, these polygons have the property of *nesting* C_4-systems, *kite-systems*, etc... . In particular, in [8] she studied *perfect dodecagon quadrangle systems.*

In this paper, where the blocks are dodecagons with chords which join vertices at distance three dividing the dodecagon in five quadrangles, the authors study these systems in the case that the spectrum is the *largest* possible.

2 Some definitions

The graph given in the Fig.1 is called an *octagon quadrangle* and will be also denoted by $[(x_1), x_2, x_3, (x_4), (x_5), x_6, x_7, (x_8)]$. The cycle (x_1, x_2, x_3, x_4) will be the *upper* C_4-cycle, the cycle (x_5, x_6, x_7, x_8) will be the *lower* C_4-cycle, while the cycle $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$

will be the *outside* cycle. Obviously, an *upper* C_4-cycle of an octagon quadrangle OQ can be considered as a *lower* C_4-cycle of OQ and viceversa. It depends only on the representation of the OQ in the plane.

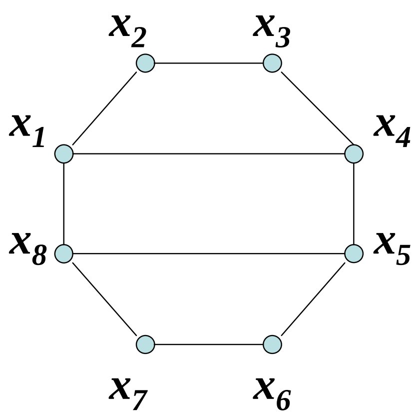

Figure 1. Octagon Quadrangle

An *octagon quadrangle system* of order v and index λ, briefly an OQS, is a pair $\Sigma = (X, B)$, where X is a finite set of v vertices and B is a collection of edge disjoint octagon quadrangles, called *blocks*, which partition the edge set of λK_v, defined on the vertex set X.

An *octagon quadrangle system* $\Sigma = (X, B)$ of order v and index λ is said to be:

i) *upper C_4-perfect*, if all of the *upper* C_4-cycles contained in the octagon quadrangles form a μ-fold 4-cycle system of order v;

ii) *C_8-perfect*, if all of the *outside* C_8-cycles contained in the octagon quadrangles form a ϱ-fold 8-cycle system of order v;

iii) *upper strongly perfect*, if the collection of all of the *upper* C_4-cycles contained in the octagon quadrangles form a μ-fold 4-cycle system of order v and the collection of all of the *outside* C_8-cycles contained in the octagon quadrangles form a ϱ-fold 8-cycle system of order v.

In the first two cases, we say that the system has indices (λ, μ) or (λ, ϱ) respectively, in the third case we say that the system has indices (λ, ϱ, μ). It is immediate that any system of order v and index $2k$ can be obtained from a system of the same type of the same order and index k, by a repetition of the blocks.

In the following examples there are OQSs of different types. In them the vertex set is always Z_v and the blocks are given by a given number of base blocks, from which one can obtain their translated blocks and define all the system.

Example 1

The following blocks define a *strongly perfect* $OQS(11)$ of indices (10,8,4): the upper C_4-cycles form an upper C_4-system of index $\mu = 4$ and the outside C_8-cycles form a C_8-system of index $\varrho = 8$.

Base blocks (mod 11):

$$[(0), 6, 4, (10), (8), 3, 7, (1)], [(0), 10, 7, (9), (5), 3, 2, (6)],$$

$$[(0), 10, 7, (3), (9), 5, 2, (6)], [(0), 8, 3, (7), (4), 2, 10, (9)],$$

$$[(1), 0, 4, (6), (7), 5, 8, (9)].$$

Example 2

The following blocks define an upper C_4-*perfect* $OQS(8)$ of indices (10,8,4). It is not C_8-*perfect.* In fact, while the upper C_4-cycles form a C_4-system of index $\mu = 4$, the outside C_8-cycles do not form a C_8-system of index $\varrho = 8$.

Base blocks (mod 7):

$$[(0), 4, 3, (6),\ \ (2), 1, \infty, (5)],$$

$$[(\infty), 3, 5, (2), (4), 1, 6, (0)],$$

$$[(0, 6, 4, (5), (3), 2, \infty, (1)],$$

$$[(\infty), 2, 5, (0), (6), 1, 4, (3)],$$

where ∞ is a fixed vertex and all the others are obtained cyclically in Z_7.

Example 3

The following blocks define a C_8*-perfect* $OQS(8)$ of indices (10,8,4). It is not upper C_4*-perfect.* In fact, while the outside C_8-cycles form a C_8-system of index $\varrho = 8$, it is not possible to find upper or lower C_4-cycles which form a C_4-system of index $\mu = 4$.

Base blocks (mod 7):

$$[(1), 0, 2, (5), \ (4), 6, \infty, (3)],$$

$$[(\infty), 0, 3, (6), (4), 5, 1, (2)],$$

$$[(6), 0, 5, (2), (3), 1, \infty, (4)],$$

$$[(\infty), 0, 4, (1), (3), 2, 6, (5)].$$

where ∞ is a fixed vertex and all the others are obtained cyclically in Z_7.

Remark: It is immediate that any system of order v and index $2k$ can be obtained from a system of the same type of the same order and index k, by a repetition of the blocks. In this paper we will not use this technique and always we will consider OQSs *without* repeated blocks.

3 Necessary existence conditions

In this section we prove some necessary existence conditions.

Theorem 3.1 *: Let $\Omega = (X, B)$ be an upper strongly perfect OQS of order v and let $\Sigma_1 = (X, B_1)$, $\Sigma_2 = (X, B_2)$ be the corresponding outside $C_8 - system$ and upper $C_4 - system$, respectively. If the systems Ω, Σ_1, Σ_2 have indices (λ, ϱ, μ), in the order, then:*

i) $\lambda = 5 \cdot k,\ \varrho = 4 \cdot k,\ \mu = 2 \cdot k,$

for some positive integer k;

ii) the largest possible spectrum for upper strongly perfect OQSs is

$$S = \{v \in N : v \geq 8\},$$

and the corresponding minimum values for the indices are:

$$\lambda = 10, \varrho = 8, \mu = 4.$$

Proof. If $\Omega = (X, B)$ is an upper strongly perfect OQS of order v, $\Sigma_1 = (X, B_1)$ and $\Sigma_2 = (X, B_2)$ the outside $C_8 - system$ and the upper $C_4 - system$ respectively and (λ, ϱ, μ) the indices, since $|B| = |B_1| = |B_2|$, then necessarily:

$$\frac{\lambda}{5} = \frac{\varrho}{4} = \frac{\mu}{2}$$

and the statement *i*) follows.

For $k = 1$ the possible spectrum for strongly perfect OQS is a subset of $S = \{v \in N : v \geq 8\}$. For $k = 2$ the possible spectrum is exactly $S = \{v \in N : v \geq 8\}$. □

Remark: The same conditions are obtained in the case of upper C_4-perfect $OQSs$ but not C_8-perfect, and in the case of C_8-perfect $OQSs$ but not C_4-perfect.

4 Existence of particular octagon systems of indices (10,8,4), without repeated blocks

The systems contained in the following Theorems will be used in what follows.

Theorem 4.1 *: There exist upper strongly perfect OQSs, having order 8,9,10,11,12,13,14,15 and indices (10,8,4).*

Proof. The following OQSs are upper strongly perfect. They have order 8,9,10,11,12,13,14,15 and indices (10,8,4).

i) $\Sigma_9 = (Z_9, B)$, base blocks (mod 9):

$$[(0), 4, 8, (1), (5), 2, 7, (3)], \quad [(0), 1, 7, (2), (5), 4, 6, (8)],$$
$$[(0), 2, 4, (3), (6), 5, 8, (1)], \quad [(0), 1, 7, (4), (6), 2, 3, (5)].$$

ii) $\Sigma_8 = (W_8, B)$, $W_8 = Z_7 \cup \{\infty\}$, $\infty \notin Z_7$, base blocks (mod 7):

$$[(0), 3, 4, (1), (5), 6, \infty, (2)], [(\infty), 4, 2, (5), (3), 6, 1, (0)],$$
$$[(0), 1, 3, (2), (4), 5, \infty, (6)], [(\infty), 5, 0, (4), (1), 6, 3, (2)].$$

iii) $\Sigma_{11} = (Z_{11}, B)$, base blocks (mod 11):

$$[(0), 5, 7, (1), (3), 8, 4, (10)], \quad [(0), 1, 4, (2), (6), 8, 9, (5)],$$
$$[(0), 1, 4, (8), (2), 6, 9, (5)], \quad [(0), 3, 8, (4), (7), 9, 1, (2)],$$
$$[(10), 0, 7, (5), (4), 6, 3, (2)].$$

iv) $\Sigma_{10} = (W_8, B)$, $W_{10} = Z_9 \cup \{\infty\}$, $\infty \notin Z_9$, base blocks (mod 9):

$$[(0), 4, 7, (1), (3), 8, 6, (5)], \quad [(0), 1, 5, (2), (3), 6, \infty, (4)],$$
$$[(\infty), 5, 6, (7), (4), 2, 1, (0)], \quad [(0), 2, 7, (3), (1), 8, \infty, (4)],$$
$$[(\infty), 8, 6, (4), (7), 1, 0, (3)].$$

v) $\Sigma_{13} = (Z_{13}, B)$, base blocks (mod 13):

$[(0), 6, 12, (1), (4), 11, 7, (2)]$, $[(0), 1, 12, (2), (6), 7, 4, (3)]$,
$[(0), 2, 11, (3), (8), 6, 9, (4)]$, $[(0), 3, 11, (4), (10), 8, 6, (5)]$,
$[(0), 4, 8, (5), (12), 11, 7, (6)]$, $[(0), 1, 6, (7), (4), 2, 11, (5)]$.

vi) $\Sigma_{12} = (W_{12}, B)$, $W_{12} = Z_{11} \cup \{\infty\}$, $\infty \notin Z_{11}$, base blocks (mod 11):

$[(0), 5, 10, (1), (4), 8, 3, (2)]$, $[(0), 2, 9, (3), (8), 5, \infty, (4)]$,
$[(\infty), 4, 8, (7), (5), 6, 2, (0)]$, $[(0), 1, 10, (2), (6), 7, 4, (3)]$,
$[(0), 3, 10, (4), (7), 6, \infty, (2)]$, $[(\infty), 3, 6, (5), (1), 7, 2, (0)]$.

vii) $\Sigma_{15} = (Z_{15}, B)$, base blocks (mod 15):

$[(0), 7, 13, (1), (4), 6, 5, (2)]$, $[(0), 1, 13, (2), (6), 12, 10, (3)]$,
$[(0), 2, 13, (3), (8), 14, 11, (4)]$, $[(0), 3, 13, (4), (10), 12, 6, (5)]$,
$[(0), 4, 13, (5), (12), 7, 11, (6)]$, $[(0), 5, 4, (6), (8), 12, 11, (1)]$,
$[(14), 1, 8, (7), (4), 11, 10, (3)]$.

viii) $\Sigma_{14} = (W_{14}, B)$, $W_{14} = Z_{13} \cup \{\infty\}$, $\infty \notin Z_{13}$, base blocks (mod 13):

$[(0), 6, 10, (1), (4), 5, 8, (2)]$, $[(0), 1, 12, (2), (6), 4, 9, (3)]$,
$[(0), 2, 11, (3), (8), 5, 10, (4)]$, $[(0), 3, 10, (4), (6), 2, \infty, (1)]$,
$[(0), 1, 11, (5), (10), 6, \infty, (4)]$, $[(\infty), 3, 4, (9), (8), 7, 5, (2)]$,
$[(\infty), 6, 8, (3), (1), 2, 7, (0)]$. □

Theorem 4.2 : *There exist upper C_4- perfect OQSs, having order 8,9,10,11,12,13,14,15 and indices (10,4), which are not C_8- perfect.*

Proof.

The following OQSs are upper C_4- perfect, have order 8,9,10,11, 12, 13,14,15 and indices (10,4), but they are not C_8- perfect.

i) $\Omega_9 = (Z_9, B)$, base blocks (mod 9):

$[(0), 4, 8, (1), (5), 2, 7, (3)]$, $[(0), 1, 7, (2), (5), 4, 6, (8)]$,
$[(0), 2, 4, (3), (6), 5, 8, (1)]$, $[(0), 1, 7, (4), (3), 8, 6, (5)]$.

ii) $\Omega_8 = (W_8, B)$, $W_8 = Z_7 \cup \{\infty\}$, $\infty \notin Z_7$, base blocks (mod 7):

$[(0), 3, 4, (1), (5), 6, \infty, (2)]$,$[(\infty), 4, 2, (5), (3), 6, 1, (0)]$,
$[(0), 1, 3, (2), (4), 5, \infty, (6)]$,$[(\infty), 5, 2, (0), (1), 6, 3, (4)]$.

iii) $\Omega_{11} = (Z_{11}, B)$, base blocks (mod 11):

$[(0), 5, 7, (1), (3), 8, 4, (10)]$, $[(0), 1, 4, (2), (6), 8, 9, (5)]$,
$[(0), 1, 4, (8), (2), 6, 9, (5)]$, $[(0), 3, 8, (4), (7), 9, 6, (5)]$,
$[(10), 0, 7, (5), (4), 6, 3, (2)]$.

iv) $\Omega_{10} = (W_8, B)$, $W_{10} = Z_9 \cup \{\infty\}$, $\infty \notin Z_9$, base blocks (mod 9):

$[(0), 4, 7, (1), (3), 8, 6, (5)]$, $[(0), 1, 5, (2), (3), 6, \infty, (4)]$,
$[(\infty), 5, 6, (7), (4), 2, 1, (0)]$, $[(0), 2, 7, (3), (1), 8, \infty, (4)]$,
$[(\infty), 8, 6, (4), (7), 3, 0, (1)]$.

v) $\Omega_{13} = (Z_{13}, B)$, base blocks (mod 13):

$[(0), 6, 12, (1), (4), 11, 7, (2)]$, $[(0), 1, 12, (2), (6), 7, 4, (3)]$,
$[(0), 2, 11, (3), (8), 6, 9, (4)]$, $[(0), 3, 11, (4), (10), 8, 6, (5)]$,
$[(0), 4, 8, (5), (12), 1, 2, (6)]$, $[(0), 1, 6, (7), (8), 12, 11, (5)]$.

vi) $\Omega_{12} = (W_{12}, B)$, $W_{12} = Z_{11} \cup \{\infty\}$, $\infty \notin Z_{11}$, base blocks (mod 11):

$[(0), 5, 10, (1), (4), 8, 3, (2)]$, $[(0), 2, 9, (3), (8), 5, \infty, (4)]$,
$[(\infty), 4, 8, (7), (5), 6, 2, (0)]$, $[(0), 1, 10, (2), (6), 7, 4, (3)]$,
$[(0), 3, 8, (4), (5), 10, \infty, (2)]$, $[(\infty), 3, 6, (5), (1), 7, 2, (0)]$.

vii) $\Omega_{15} = (Z_{15}, B)$, base blocks (mod 15):

$[(0), 7, 13, (1), (4), 6, 5, (2)]$, $[(0), 1, 13, (2), (6), 12, 11, (3)]$,

$[(0), 2, 13, (3), (8), 14, 11, (4)]$, $[(0), 3, 13, (4), (10), 12, 6, (5)]$,
$[(0), 4, 13, (5), (12), 7, 11, (6)]$, $[(0), 5, 4, (6), (8), 12, 11, (1)]$,
$[(14), 1, 8, (7), (6), 13, 11, (3)]$.

viii) $\Omega_{14} = (W_{14}, B)$, $W_{14} = Z_{13} \cup \{\infty\}$, $\infty \notin Z_{13}$, base blocks (mod 13):

$[(0), 6, 10, (1), (4), 5, 8, (2)]$, $[(0), 1, 12, (2), (6), 4, 9, (3)]$,
$[(0), 2, 11, (3), (8), 5, 10, (4)]$, $[(0), 3, 10, (4), (6), 2, \infty, (1)]$,
$[(0), 1, 11, (5), (10), 6, \infty, (4)]$, $[(\infty), 3, 4, (9), (8), 7, 5, (2)]$,
$[(\infty), 10, 8, (3), (2), 1, 6, (0)]$. □

Theorem 4.3 *: There exist C_8- perfect OQSs, having order 8,9,10,11, 12, 13,14,15 and indices (10,8), which are not upper C_4- perfect.*

Proof.

The following OQSs are C_8- perfect, have order 8,9,10,11,12,13,14,15 and indices (10,8), but they are not upper C_4- perfect.

i) $\Delta_9 = (Z_9, B)$, base blocks (mod 9):

$[(0), 4, 8, (1), (5), 2, 7, (3)]$, $[(0), 1, 7, (2), (5), 4, 6, (8)]$,
$[(0), 2, 4, (3), (6), 5, 8, (1)]$, $[(0), 4, 7, (5), (2), 3, 8, (1)]$.

ii) $\Delta_8 = (W_8, B)$, $W_8 = Z_7 \cup \{\infty\}$, $\infty \notin Z_7$, base blocks (mod 7):

$[(0), 6, 5, (1), (3), 2, \infty, (4)]$,$[(\infty), 6, 3, (5), (4), 1, 2, (0)]$,
$[(0), 3, 5, (2), (6), 4, \infty, (1)]$,$[(\infty), 2, 1, (4), (6), 0, 5, (3)]$.

iii) $\Delta_{11} = (Z_{11}, B)$, base blocks (mod 11):

$[(0), 5, 3, (1), (10), 4, 11, (6)]$, $[(0), 1, 4, (2), (6), 8, 9, (5)]$,
$[(0), 1, 4, (8), (2), 6, 3, (10)]$, $[(2), 5, 3, (9), (6), 10, 7, (1)]$,
$[(10), 0, 7, (5), (4), 6, 3, (2)]$.

iv) $\Delta_{10} = (W_8, B)$, $W_{10} = Z_9 \cup \{\infty\}$, $\infty \notin Z_9$, base blocks (mod 9):

$[(0), 4, 7, (1), (3), 8, 6, (5)]$, $[(0), 1, 5, (2), (3), 6, \infty, (4)]$,
$[(\infty), 5, 6, (7), (4), 2, 1, (0)]$, $[(0), 2, 7, (3), (1), 8, \infty, (4)]$,
$[(\infty), 0, 2, (8), (6), 3, 4, (1)]$.

v) $\Delta_{13} = (Z_{13}, B)$, base blocks (mod 13):

$[(0), 6, 12, (1), (4), 11, 7, (2)]$, $[(0), 1, 12, (2), (6), 7, 4, (3)]$,
$[(0), 2, 11, (3), (8), 6, 9, (4)]$, $[(0), 5, 10, (4), (11), 12, 1, (3)]$,
$[(0), 4, 8, (5), (12), 11, 7, (6)]$, $[(0), 1, 6, (7), (4), 2, 11, (5)]$.

vi) $\Delta_{12} = (W_{12}, B)$, $W_{12} = Z_{11} \cup \{\infty\}$, $\infty \notin Z_{11}$, base blocks (mod 11):

$[(0), 5, 10, (1), (4), 8, 3, (2)]$, $[(0), 2, 9, (3), (8), 5, \infty, (4)]$,
$[(\infty), 9, 10, (8), (5), 6, 2, (0)]$, $[(0), 1, 10, (2), (6), 7, 4, (3)]$,
$[(0), 3, 10, (4), (7), 6, \infty, (2)]$, $[(\infty), 8, 1, (0), (4), 10, 5, (3)]$.

vii) $\Delta_{15} = (Z_{15}, B)$, base blocks (mod 15):

$[(0), 7, 13, (1), (4), 6, 5, (2)]$, $[(2), 5, 11, (0), (4), 7, 9, (1)]$,
$[(0), 2, 13, (3), (8), 14, 11, (4)]$, $[(0), 3, 13, (4), (10), 12, 6, (5)]$,
$[(0), 4, 13, (5), (12), 7, 11, (6)]$, $[(0), 5, 4, (6), (8), 12, 11, (1)]$,
$[(14), 1, 8, (7), (4), 11, 10, (3)]$.

viii) $\Delta_{14} = (W_{14}, B)$, $W_{14} = Z_{13} \cup \{\infty\}$, $\infty \notin Z_{13}$, base blocks (mod 13):

$[(0), 6, 10, (1), (4), 5, 8, (2)]$, $[(0), 1, 12, (2), (6), 4, 9, (3)]$,
$[(0), 2, 11, (3), (8), 5, 10, (4)]$, $[(0), 3, 10, (4), (6), 2, \infty, (1)]$,
$[(0), 1, 11, (5), (10), 6, \infty, (4)]$, $[(\infty), 11, 10, (9), (1), 4, 6, (7)]$,
$[(\infty), 6, 8, (3), (1), 2, 7, (0)]$. □

5 Construction $v \to v + 8$

In this section we give a construction for *OQSs* having indices (10,8,4),(10,4),(10,8), for all possible orders.

Theorem 5.1 *: An upper strongly perfect OQS of order v+8 and indices (10,8,4) can be constructed starting from an upper strongly perfect OQS of order v and indices (10,8,4).*

Proof. Let $A = \{1', 2', 3', 4', 5', 6', 7', 8'\}$, $Z_v = \{0, 1, 2..., v-1\}$, where $A \cap Z_v = \emptyset$. Let $\Sigma = (Z_v, B)$, $\Sigma' = (A, B')$ be two upper strongly perfect OQSs both of indices $(10, 8, 4)$. Define on $Z_v \cup A$ the family H of octagon quadrangles as follows.
Define a partition of A in two sets $L = \{\alpha, \beta, \gamma, \delta\}$, $M = \{a, b, c, d\}$ such that $L \cap M = \emptyset$. Then, H is the family having the blocks:

$$[(\alpha), i, \beta, (i{+}1), (\gamma), i{+}2, \delta, (i{+}3)],\ [(\beta), i{+}1, \alpha, (i{+}2), (\delta), i{+}3, \gamma, (i{+}4)],$$

$$[(\gamma), i, \delta, (i{+}1), (\alpha), i{+}2, \beta, (i{+}3)],\ [(\delta), i{+}1, \gamma, (i{+}2), (\beta), i{+}3, \alpha, (i{+}4)],$$

$$[(a), i, b, (i{+}1), (c), i{+}2, d, (i{+}3)],\ [(b), i{+}1, a, (i{+}2), (d), i{+}3, c, (i{+}4)],$$

$$[(c), i, d, (i{+}1), (a), i{+}2, b, (i{+}3)],\ [(d), i{+}1, c, (i{+}2), (b), i{+}3, a, (i{+}4)],$$

where i belongs to Z_v.

If $X{=}Z_v{\cup}A$ and $C = B{\cup}B'{\cup}H$, then $\Omega = (X, C)$ is an upper strongly perfect *OQS* of order $v + 8$ and indices (10,8,4).
If $x, y \in Z_v$ [resp. A], then the edge $\{x, y\}$ is in a block of B [resp. B']: exactly in ten octagon quadrangles, in eight outside C_8-cycles and in four upper C_4-cycles.
If $x \in Z_v$ and $y \in A$, then the edge $\{x, y\}$ is contained in the octagon quadrangles of H. Each vertex $y \in A$ has degree 3 in $2v$ blocks and degree 2 in the other $2v$ blocks, also the edge $\{x, y\}$ is contained exactly in ten octagon quadrangles of H, in eight outside C_8-cycles and in four upper C_4-cycles.

We also observe that the number of blocks of C is:

$|C| = |B| + |B'| + |H| =$

$\frac{v(v-1)}{2} + \frac{8 \cdot 7}{2} + 8 \cdot v = \frac{1}{2} \cdot (v^2 + 15v + 56),$

which is exactly the number of blocks of an $OQS(v + 8)$ of indices (10,8,4):

$\frac{(v+8)(v+7)}{2} = \frac{1}{2} \cdot (v^2 + 15v + 56).$

So, the proof is complete. □

Theorem 5.2 *: An upper C_4-perfect OQS of order $v + 8$ and indices (10,4), which is not C_8-perfect, can be constructed starting from an upper C_4-perfect OQS of order v and indices (10,4).*

Proof. Let $\Sigma' = (A, B')$ be the $OQS(8)$ of indices (10,4), isomorphic to the $OQS(8)$ defined on $Z_7 \cup \{\infty\}$ and defined by the translated one of the following

base blocks (mod 7):
$[(0), 3, 4, (1), (5), 6, \infty, (2)]$,$[(\infty), 4, 2, (5), (3), 6, 1, (0)]$,
$[(0), 1, 3, (2), (4), 5, \infty, (6)]$,$[(\infty), 5, 2, (0), (1), 6, 3, (4)]$.

Following the proof of Theorem 5.1, since Σ' is an upper C_4-perfect $OQS(8)$, but not C_8-perfect (see Theorem 4.2), the statement is proved. □

Theorem 5.3 *: A C_8-perfect OQS of order $v + 8$ and indices (10,8), which is not C_4-perfect, can be constructed starting from a C_8-perfect OQS of order v and indices (10,8).*

Proof. Let $\Sigma' = (A, B')$ be the $OQS(8)$ of indices (10,8), isomorphic to the $OQS(8)$ defined on $Z_7 \cup \{\infty\}$ and defined by the translated one of the following

base blocks (mod 7):
$[(0), 6, 5, (1), (3), 2, \infty, (4)]$,$[(\infty), 6, 3, (5), (4), 1, 2, (0)]$,
$[(0), 3, 5, (2), (6), 4, \infty, (1)]$,$[(\infty), 2, 1, (4), (6), 0, 5, (3)]$.

Following the proof of Theorem 5.1, since Σ' is a C_8-perfect $OQS(8)$, but it is not upper C_4-perfect (see Theorem 4.3), the statement is proved. □

6 Conclusive Existence Theorems

Collecting together the results of the previous sections, we have the following conclusive theorems:

Theorem 6.1 *: There exist upper strongly perfect OQS(v)s of indices (10,8,4) for every positive integer v, $v \geq 8$.*

Proof. The statement follows from Theorems 4.1 and 5.1. □

Theorem 6.2 *: There exist OQS(v)s of indices (10,4), which are upper C_4-perfect but not C_8-perfect, for every positive integer v, $v \geq 8$.*

Proof. The statement follows from Theorems 4.2 and 5.2. □

Theorem 6.3 *: There exist OQS(v)s of indices (10,8), which are C_8-perfect but not upper C_4-perfect, for every positive integer v, $v \geq 8$.*

Proof. The statement follows from Theorems 4.3 and 5.3. □

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L. Berardi, M. Gionfriddo, R. Rota, Received January 12, 2011

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