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Convexity preserving interpolation by splines of arbitrary degree

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Abstract

In the present paper an algorithm of C^2 interpolation of discrete set of data is given using splines of arbitrary degree, which preserves the convexity of given set of data.

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1 Introduction

It is well known that problems concerning nonnegativity, monotonicity, or convexity preserving interpolation have received considerable attention, because of their interest in computer aided design and in other practical applications [1]. Problem of construction of interpolating curve which preserves the convexity of the initial discrete set of data still remains in the focus of investigators [2]. In what follows an algorithm preserving the convexity of given set of data is presented.

2 Interpolating splines of arbitrary degree

Let us assume that the mesh $\Delta : a = x_0 < x_1 < ... < x_n = b$ is given on the interval $[a, b]$ and $f_i = f(x_i)$, $i = 0(1)n$, are the corresponding data points. Problem of construction of an interpolation function S, such that interpolation conditions $S(x_i) = f_i$, $i = 0(1)n$, are held and $S \in C^2[a, b]$, is considered.

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Let us introduce splines as follows: on $[x_i, x_{i+1}]$

$$S(x) = f_i + (f_{i+1} - f_i)t + + \frac{h_i^2 M_i (1-t)((1-t)^{\alpha_i} - 1)}{\alpha_i(\alpha_i + 1)} + \frac{h_i^2 M_{i+1} t(t^{\alpha_i} - 1)}{\alpha_i(\alpha_i+1)}, \quad (1)$$

where the following notations are used:

$$t = (x - x_i)/h_i, h_i = x_{i+1} - x_i, S''(x_i) = M_i.$$

The α_i is a free parameter of the splines (1) and has to satisfy the condition $\alpha_i > 1$.

From (1) for the first derivative of spline we get:

$$S'(x) = \delta_i^{(1)} - - \frac{h_i(M_i((\alpha_i + 1)(1-t)^{\alpha_i} - 1) - M_{i+1}((\alpha_i + 1)t^{\alpha_i} - 1))}{\alpha_i(\alpha_i+1)}, (2)$$

where

$$\delta_i^{(1)} = (f_{i+1} - f_i)/h_i,$$

and for the second derivative, respectively:

$$S''(x) = M_i(1-t)^{\alpha_i - 1} + M_{i+1} t^{\alpha_i - 1}. \quad (3)$$

Obviously, at the knots of the mesh the second derivative is the continuous one.

For the first derivative at the knots of the mesh we have

$$S'(x_i-) = \delta_{i-1}^{(1)} + \frac{h_{i-1}M_{i-1}}{\alpha_{i-1}(\alpha_{i-1}+1)} + \frac{h_{i-1}M_i}{\alpha_{i-1}+1}$$

and

$$S'(x_i+) = \delta_i^{(1)} - \frac{h_i M_i}{\alpha_i + 1} - \frac{h_i M_{i+1}}{\alpha_i(\alpha_i+1)}.$$

Requiring the continuity of the first derivative of the spline at the knots of the mesh we obtain the following system of linear algebraic equations:

$$c_i M_{i-1} + a_i M_i + b_i M_{i+1} = \delta_i^{(2)}, i = 1(1)n-1, \tag{4}$$

where

$$c_i = \frac{h_{i-1}}{\alpha_{i-1}(\alpha_{i-1}+1)},$$

$$a_i = \frac{h_{i-1}}{\alpha_{i-1}+1} + \frac{h_i}{\alpha_i+1},$$

$$b_i = \frac{h_i}{\alpha_i(\alpha_i+1)},$$

and

$$\delta_i^{(2)} = \delta_i^{(1)} - \delta_{i-1}^{(1)}.$$

The system (4), presented above, is the undetermined one. Since the system (4) provides only $n-1$ linear equations in $n+1$ parameters M_i, it follows that two additional linearly independent conditions are needed in order to have a determined system of equations.

In what follows we'll consider that the end conditions of the type $M_0 = f_0''$ and $M_n = f_n''$ are used as additional conditions.

In fact, it is easy to prove that the system of equations (4) has the diagonally dominant matrix of coefficients, therefore the solution of this system exists and it is the unique one for fixed parameters of the spline.

3 Convexity preserving algorithm

In this section it is considered that the initial set of data is the convex one, namely,

$$\delta_i^{(2)} \geq 0, i = 1(1)n-1.$$

From the formulae (3) it immediately follows, that in order to preserve the convexity of initial data the solution of the system (4) has to be the nonnegative one.

So, let's choose the value of free parameter as it follows:

$$\alpha_i \geq \max\left(\frac{2\delta_i^{(2)}}{\delta_{i+1}^{(2)}}, \frac{2\delta_{i+1}^{(2)}}{\delta_i^{(2)}}\right). \tag{5}$$

There is no problem to prove that in this case the solution of the system (4) is the nonnegative one.

This conclusion is based on the fact that for the coefficients of the matrix of linear algebraic equations (4) the following relations are valid:

$$\frac{c_i}{a_{i-1}} < \frac{1}{2}$$

and

$$\frac{b_i}{a_{i+1}} < \frac{1}{2}.$$

In this case we get that

$$\delta_i^{(2)} - \frac{c_i\delta_{i-1}^{(2)}}{a_{i-1}} - \frac{b_i\delta_{i+1}^{(2)}}{a_{i+1}} \geq 0.$$

As a result, taking into account [3] it can be concluded that the solution of the system (4) is the nonnegative one. As a result the spline, constructed using condition (5), preserves the convexity of the initial set of data.

4 Conclusions

In fact not only the problem of convexity preserving interpolation represents the interest, but also the problem of construction of interpolants which have the same number of inflection points as the initial set of data and preserve the convexity or concavity of data.

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