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All proper colorings of every colorable $BSTS(15)$

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Abstract

A Steiner System, denoted $S(t, k, v)$, is a vertex set X containing v vertices, and a collection of subsets of X of size k, called blocks, such that every t vertices from X are in exactly one of the blocks. A Steiner Triple System, or STS, is a special case of a Steiner System where $t = 2$, $k = 3$ and $v = 1$ or 3 $(mod 6)$ [7]. A Bi-Steiner Triple System, or $BSTS$, is a Steiner Triple System with the vertices colored in such a way that each block of vertices receives precisely two colors. Out of the 80 $BSTS(15)$s, only 23 are colorable [1]. In this paper, using a computer program that we wrote, we give a complete description of all proper colorings, all feasible partitions, chromatic polynomial and chromatic spectrum of every colorable $BSTS(15)$.

1 Introduction

A hypergraph is a generalized graph where an edge, called a hyperedge, can contain more than two vertices. A mixed hypergraph contains two kinds of hyperedges, C-edges and D-edges. A coloring of a mixed hypergraph is proper if every C-edge has at least two vertices mapped to the same color while every D-edge has at least two vertices mapped to different colors [5]. A Steiner Triple System, denoted by $STS(v)$ where v is the number of vertices, is a special case of a Steiner System in which its blocks are made up of exactly three vertices and no two blocks can share a pair of vertices [7]. Here we consider STSs on 15 vertices as bi-hypergraphs, which are mixed hypergraphs such that the **C** family of C-edges and the **D** family of D-edges coincide, or equivalently $\mathbf{C} = \mathbf{D}$. We call these bi-hypergraphs Bi-Steiner Triple Systems of order 15 ($BSTS(15)$) and we consider every block of three vertices to be both a C-edge and a D-edge. Since each block of a $BSTS(15)$ contains exactly three vertices, two of those vertices must be mapped to the same color and the third vertex must be mapped to a different color to satisfy both the C-edge and D-edge requirements. Therefore, each block of a $BSTS(15)$ must be mapped to precisely two colors. The lower chromatic number of a mixed hypergraph is the minimum number of colors for which there exists a proper coloring, and it is denoted by χ [5]. The upper chromatic number of a mixed hypergraph is the maximum number of colors for which there exists a strict proper coloring, and it is denoted by $\bar{\chi}$ [5]. Of the 80 non-isomorphic $BSTS(15)$s, 23 contain $BSTS(7)$ as a subdesign and those 23 $BSTS(15)$s are colorable [1]. They are numbered in [1] as no. $1-22$ and no. 61. For these 23 $BSTS(15)$s, the upper and lower chromatic numbers are equal. They all have $\chi = \bar{\chi} = 4$ [2][3]. This means that all colorable $BSTS(15)$s are only colorable on exactly four colors. The chromatic spectrum of a mixed hypergraph is an integer vector, $R(H)$ whose components are $r_1, r_2, ..., r_k$, where r_i is the number of different feasible partitions into i color classes [5]. It is known that $BSTS(15)$s are only colorable on 4 colors, so $r_i = 0$, when $i \neq 4$. Only r_4 will have a value other than 0, so we can generalize and say the following:

$$R(BSTS(15)) = (0, 0, 0, r_{\bar{\chi}}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \text{ [5]}$$

since $\chi = \bar{\chi} = 4$. Using a computer program that we wrote, we were able to find and display all proper colorings of every colorable $BSTS(15)$. Also we were able to display all feasible partitions and the permutations of colors of every $BSTS(15)$, which when multiplied together gives the number of all proper colorings. In this paper, we show all proper colorings, all feasible partitions, the chromatic polynomial, and the chromatic spectrum of all colorable $BSTS(15)$s.

2 Method

The chromatic polynomial of a mixed hypergraph is simply a polynomial in λ which gives the number of proper λ-colorings of a colorable mixed hypergraph, where λ is the number of available colors [5]. If we let $r_i(H)$ denote the number of feasible partitions of the mixed hypergraph into i color classes or sets, and we let $\lambda^{(i)}$ denote the falling factorial of λ, then we have the following equality:

$$P(H, \lambda) = \sum_{i=\chi(H)}^{\bar{\chi}(H)} r_i(H)\lambda^{(i)} \text{ [5].}$$

To construct the chromatic polynomial for each colorable $BSTS(15)$s, we need to alter the above equation to specify for $BSTS(15)$s. If we let a colorable $BSTS(15)$ be our H, the chromatic polynomial will give the number of proper λ-colorings of that colorable $BSTS(15)$, where $\lambda \geq 4$ since 4 colors are required for a proper λ-coloring of a colorable $BSTS(15)$. So we adjust the fundamental equality of mixed hypergraph coloring accordingly to accommodate the $BSTS(15)$s. First, [2, 3] showed that $\chi = \bar{\chi} = 4$ in all colorable $BSTS(15)$s; therefore, $\sum_{i=\chi(H)}^{\bar{\chi}(H)}$ is not needed. We simply say that $i = 4$ so now $r_\chi(H) = r_{\bar{\chi}}(H) = r_4(H)$. Therefore, these adjustments yield the following:

$$P(BSTS(15), \lambda) = r_4(BSTS(15))\lambda^{(i)}.$$

We also know that $BSTS(15)$s are colorable if and only if they contain $BSTS(7)$ as a subsystem or subdesign [4]. There are 21 feasible partitions in $BSTS(7)$ [6, 8] and there are four cases of $(7, 3, 1)$-subdesigns in various colorable $BSTS(15)$s. A $(7, 3, 1)$-subdesign is a subsystem on 7 vertices where each block contains 3 vertices and each vertex appears precisely once with every other vertex in a block. $BSTS(7)$ is the same as the finite projective plane of order 2, called the Fano Plane [7]. In all 23 cases of colorable $BSTS(15)$s, there exist(s):

1. 1 case in which a colorable $BSTS(15)$ has 15 (7,3,1)-subdesigns,
2. 1 case in which a colorable $BSTS(15)$ has 7 (7,3,1)-subdesigns,
3. 5 cases in which a colorable $BSTS(15)$ has 3 (7,3,1)-subdesigns, and
4. 16 cases in which a colorable $BSTS(15)$ has 1 (7,3,1)-subdesign.

[1]

This covers all colorable $BSTS(15)$s. Let the number of $(7, 3, 1)$-subdesigns in a $BSTS(15)$ be denoted by s. The number of feasible partitions in a particular colorable $BSTS(15)$ is equal to the number of feasible partitions in $BSTS(7)$ which is 21 using three colors, times the number of $(7, 3, 1)$-subdesigns in the $BSTS(15)$, or equivalently, $r_4 = 21s$. We will show this in the next section. Now we can write the fundamental equality of colorable $BSTS(15)$s as the following:

Proposition 1. *The number of proper λ-colorings of a colorable $BSTS(15)$ is a polynomial with the following form:*

$$P(BSTS(15), \lambda) = 21s(\lambda^{(4)}).$$

Now we will look at all colorable $BSTS(15)$s and arrive at each of their minimum number of proper colorings, their number of feasible partitions, and their chromatic spectrums.

3 Theorem and Proof

Theorem 1. *When $\lambda = 4$, the minimum number of proper colorings, the number of feasible partitions, and the minimum number of permutations of each partition of each colorable $BSTS(15)$ can be obtained by the following equality:*

$$P(BSTS(15), 4) = 21s(4!)$$

Proof. Let s denote the number of $(7,3,1)$-subdesigns in a particular colorable $BSTS(15)$. It is known that the number of feasible partitions of $BSTS(7)$ is 21 [6]. By an exhaustive search of all possible colorings using a program that we wrote and by applying the splitting-contraction algorithm [8] as defined in [5], the feasible partitions of $BSTS(7)$ are the following:

The blocks for $BSTS(7)$ are

`{1,2,4} {2,3,5} {3,4,6} {4,5,7} {5,6,1} {6,7,2} {7,1,3}`

The set of available colors is `{0,1,2}`

Vertices	1 2 3 4 5 6 7	Vertices	1 2 3 4 5 6 7
Partition 1	0 1 2 1 2 2 2	Partition 11	0 1 1 0 2 0 0
Partition 2	0 1 2 0 2 2 2	Partition 12	0 0 0 1 2 0 2
Partition 3	0 0 2 1 2 2 2	Partition 13	0 0 2 1 0 2 0
Partition 4	0 1 2 1 1 1 2	Partition 14	0 0 2 1 0 1 0
Partition 5	0 1 1 1 2 2 1	Partition 15	0 0 1 1 0 2 0
Partition 6	0 1 2 1 1 1 0	Partition 16	0 0 0 1 1 0 2
Partition 7	0 1 0 1 1 1 2	Partition 17	0 0 0 1 2 0 1
Partition 8	0 1 2 0 2 0 0	Partition 18	0 1 0 0 0 1 2
Partition 9	0 1 0 0 0 2 2	Partition 19	0 1 0 0 0 2 1
Partition 10	0 1 2 0 1 0 0	Partition 20	0 1 1 1 2 0 1
		Partition 21	0 1 1 1 0 2 1

Therefore, for any number of $(7,3,1)$-subdesigns in a colorable $BSTS(15)$, the number of the subdesigns times the number of feasible partitions of the subdesign will equal the number of feasible partitions of the colorable $BSTS(15)$. In the 4 cases of $(7,3,1)$-subdesigns mentioned earlier, the number of proper colorings, the number of feasible partitions, and the permutations of colors of each partition have been calculated by a computer program we wrote. The number of proper colorings, the number of feasible partitions, and the number of permutations of colors of each partition from the program for each of $(1-4)$ above are as follows:

1. 7560 proper colorings, 315 feasible partitions, and 24 permutations of colors of each partition;

2. 3528 proper colorings, 147 feasible partitions, and 24 permutations of colors of each partition;
3. 1512 proper colorings, 63 feasible partitions, and 24 permutations of colors of each partition; and
4. 504 proper colorings, 21 feasible partitions, and 24 permutations of colors of each partition.

This will also give us the chromatic spectrum. This is a simple proof by cases:

Case 1. $15(7,3,1)$-*subdesigns*

$$P(BSTS(15),4) = 21s(4!) = 21(15)(24) = 315(24) = 7560$$

This shows that the minimum number of proper colorings for $BSTS(15)$ no. 1 is 7560. It also gives the number of feasible partitions as $21(15) = 315$. The vertices of $BSTS(7)$ are mapped to this $BSTS(15)$, while keeping the same colorings and having the other eight vertices of this $BSTS(15)$ mapped to the fourth color. The vertices of $BSTS(7)$ are mapped to the vertices of $BSTS(15)$ no. 1 in the following way:

$BSTS(7)$:	1	2	3	4	5	6	7
$BSTS(15)$: first 21 partitions	1	2	4	3	6	7	5
$BSTS(15)$: second 21 partitions	1	2	8	3	10	11	9
$BSTS(15)$: third 21 partitions	1	2	12	3	14	15	13
$BSTS(15)$: fourth 21 partitions	1	4	8	5	12	13	9
$BSTS(15)$: fifth 21 partitions	1	4	10	5	14	15	11
$BSTS(15)$: sixth 21 partitions	1	6	10	7	12	13	11
$BSTS(15)$: seventh 21 partitions	1	6	8	7	14	15	9
$BSTS(15)$: eighth 21 partitions	2	4	8	6	12	14	10
$BSTS(15)$: ninth 21 partitions	2	4	9	6	13	15	11
$BSTS(15)$: tenth 21 partitions	2	5	9	7	12	14	11
$BSTS(15)$: eleventh 21 partitions	2	5	8	7	13	15	10

$BSTS(15)$: `twelfth 21 partitions`	3	4	9	7	13	14	10
$BSTS(15)$: `thirteenth 21 partitions`	3	4	8	7	12	15	11
$BSTS(15)$: `fourteenth 21 partitions`	3	5	8	6	13	14	11
$BSTS(15)$: `fifteenth 21 partitions`	3	5	9	6	12	15	10

Therefore, the above vertices in this $BSTS(15)$ are mapped to the same colors as in $BSTS(7)$ and the other eight vertices are mapped to the fourth color. This is minimum because $\lambda = 4$ is the minimum number of available colors needed to properly color any colorable $BSTS(15)$. This also shows that the number of feasible partitions is equal to the number of partitions in $BSTS(7)$ multiplied by the number of times the $(7,3,1)$-subdesign appears in the $BSTS(15)$, which is $21(15) = 315$ feasible partitions; therefore, the chromatic spectrum of this $BSTS(15)$ is:

$$R(BSTS(15), 4) = (0, 0, 0, 315, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

Case 2. $7(7,3,1)$-*subdesigns*

$$P(BSTS(15), 4) = 21s(4!) = 21(7)(24) = 147(24) = 3528$$

This shows that the minimum number of proper colorings for $BSTS(15)$ no. 2 is 3528. It also gives the number of feasible partitions as $21(7) = 147$. The vertices of $BSTS(7)$ are mapped to this $BSTS(15)$, while keeping the same colorings and having the other eight vertices of this $BSTS(15)$ mapped to the fourth color. The vertices of $BSTS(7)$ are mapped to the vertices of $BSTS(15)$ no. 2 in the following way:

$BSTS(7)$:	1	2	3	4	5	6	7
$BSTS(15)$: `first 21 partitions`	1	2	4	3	6	7	5
$BSTS(15)$: `second 21 partitions`	1	2	8	3	10	11	9
$BSTS(15)$: `third 21 partitions`	1	2	12	3	14	15	13
$BSTS(15)$: `fourth 21 partitions`	1	4	8	5	12	13	9
$BSTS(15)$: `fifth 21 partitions`	1	4	10	5	14	15	11
$BSTS(15)$: `sixth 21 partitions`	1	6	10	7	12	13	11
$BSTS(15)$: `seventh 21 partitions`	1	6	8	7	14	15	9

Therefore, the above vertices in this $BSTS(15)$ are mapped to the same colors as in $BSTS(7)$ and the other eight vertices are mapped to the fourth color. This is minimum because $\lambda = 4$ is the minimum number of available colors needed to properly color any colorable $BSTS(15)$. This also shows that the number of feasible partitions is equal to the number of partitions in $BSTS(7)$ multiplied by the number of times the $(7, 3, 1)$-subdesign appears in the $BSTS(15)$, which is $21(7) = 147$ feasible partitions; therefore, the chromatic spectrum of this $BSTS(15)$ is:

$$R(BSTS(15), 4) = (0, 0, 0, 147, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

Case 3. $3(7, 3, 1)$-*subdesigns*

$$P(BSTS(15), 4) = 21s(4!) = 21(3)(24) = 63(24) = 1512$$

This shows that the minimum number of proper colorings for $BSTS(15)$s no. $3 - 7$ is 1512. It also gives the number of feasible partitions as $21(3) = 63$. The vertices of $BSTS(7)$ are mapped to this $BSTS(15)$, while keeping the same colorings and having the other eight vertices of this $BSTS(15)$ mapped to the fourth color. The vertices of $BSTS(7)$ are mapped to the vertices of $BSTS(15)$ no. $3 - 7$ in the following way:

$BSTS(7)$:	1	2	3	4	5	6	7
$BSTS(15)s$: first 21 partitions	1	2	4	3	6	7	5
$BSTS(15)s$: second 21 partitions	1	2	8	3	10	11	9
$BSTS(15)s$: third 21 partitions	1	2	12	3	14	15	13

Therefore, the above vertices in these $BSTS(15)$s are mapped to the same colors as in $BSTS(7)$ and the other eight vertices are mapped to the fourth color. This is minimum because $\lambda = 4$ is the minimum number of available colors needed to properly color any colorable $BSTS(15)$. This also shows that the number of feasible partitions is equal to the number of partitions in $BSTS(7)$ multiplied by the number of times the $(7,3,1)$-subdesign appears in the $BSTS(15)$, which is $21(3) = 63$ feasible partitions; therefore, the chromatic spectrum of these $BSTS(15)$s is:

$$R(BSTS(15),4) = (0,0,0,63,0,0,0,0,0,0,0,0,0,0,0).$$

Case 4. $1(7,3,1)$-*subdesign*

$$P(BSTS(15),4) = 21s(4!) = 21(1)(24) = 21(24) = 504$$

This shows that the minimum number of proper colorings for $BSTS(15)$s no. $8-22$ and no. 61 is 504. It also gives the number of feasible partitions as $21(1) = 21$. The vertices of $BSTS(7)$ are mapped to these $BSTS(15)$, while keeping the same colorings and having the other eight vertices of this $BSTS(15)$ mapped to the fourth color. The vertices of $BSTS(7)$ are mapped to the vertices of $BSTS(15)$ no. $8-22$ and no. 61 in the following way:

$BSTS(7)$:		1	2	3	4	5	6	7
$BSTS(15)s$:	21 partitions	1	2	4	3	6	7	5

Therefore, the above vertices in these $BSTS(15)$s are mapped to the same colors as in $BSTS(7)$ and the other eight vertices are mapped to the fourth color. This is minimum because $\lambda = 4$ is the minimum number of available colors needed to properly color any colorable $BSTS(15)$. This also shows that the number of feasible partitions is equal to the number of partitions in $BSTS(7)$ multiplied by the number of times the $(7,3,1)$-subdesign appears in the $BSTS(15)$, which is $21(1) = 21$ feasible partitions; therefore, the chromatic spectrum of these $BSTS(15)$s is:

$$R(BSTS(15),4) = (0,0,0,21,0,0,0,0,0,0,0,0,0,0,0).$$

Thus, all cases have been satisfied and all 23 colorable $BSTS(15)$s have been covered. Also, the $(2v+1)$ construction as described in [6] follows.

□

4 How the Program Works

This program was designed in C++ and contains several sub-programs and functions. We created files that were added to the source code of the program. We also created incidence matrices for each colorable $BSTS(15)$ as text files in the source code. We added display functions for all relevant data to check our results and to double check the computer results.

We started by creating headers that would find partitions and collect them not counting different permutations of colors. The main program calls the incidence matrix that is specified in a subprogram and displays it along with each block of vertices. The program then prompts the user to enter the number of colors that are to be used and then the number of colorings the user wishes to find (first 10 or first 200 for example, or the user can enter -1 for all colorings). If the user wants to find all proper colorings, the program runs an exhaustive search of all possible colorings from a string of all 0s to a string of all 3s. If a coloring is proper, then the coloring is displayed and counted; and if it is not a proper coloring, then that coloring is skipped. Also, if the coloring is proper, then that feasible partition is stored. After all proper colorings have been found and displayed and counted, the monitor prompts the user to press any key to see the feasible partitions displayed and counted and the number of colorings of each partition. All of the different permutations of colors of the partitions that were stored from the proper colorings are grouped together by the computer and only the first permutation of colors is displayed. For example, 011222233333333 would be displayed and 122333300000000 would not be displayed because it is a permutation of the same partition where vertex 1 is mapped to one color, vertices $2, 3$ are mapped to one color, vertices $4-7$ are mapped to one color, and vertices $8-15$ are mapped to one color. When complete, the user can see the $BSTS(7)$ subsystem(s) and its coloring, and the expansion of colors to the remaining eight vertices. We were able to use this program to check the accuracy of our hypothesis and our results; and by displaying all of the relevant data on the monitor, we were able to check the accuracy of the computer results.

5 Concluding Remarks

This paper shows two things: 1. the minimal number of colorings over all feasible sets of colors, the number of feasible partitions, and the chromatic polynomial and chromatic spectrum for every colorable $BSTS(15)$; and 2. that computer science can be an invaluable part of research in mathematics. Of course, the findings on the number of proper colorings, the number of feasible partitions, and the number of permutations of colors of each partition can be generalized for any number of colors in a set of available colors by the equality in Proposition 1. It is true that all colorable $BSTS(15)$s are colorable only with 4 colors, but if you have 5 colors in the set of available colors, then you can choose to use some subset of colors from $1, 2, 3, 4, 5$ such as colors $1, 2, 3, 4$, or colors $1, 2, 3, 5$, or colors $2, 3, 4, 5$, etc. The generalization would simply be to change λ in the fundamental equality of colorable $BSTS(15)$s to whatever number of colors are available in the set of available colors. If we take the above example of 5 available colors on the $BSTS(15)$ with 15 $(7, 3, 1)$-subdesigns, we have:

Example 1. $P(BSTS(15), \lambda) = 21s(\lambda^{(4)}) = 21(15)(5^{(4)}) = 315(120) = 37,800$

This shows that the number of proper colorings with 5 colors available to use is $37,800$. This, of course, is not minimum. We still have $21(15) = 315$ feasible partitions and $(5^{(4)}) = 120$ permutations of colors of each partition. The chromatic spectrum would be:

$$R(BSTS(15)) = (0, 0, 0, 315, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

since the number of feasible partitions would not change. There would simply be more permutations of colors of each partition.

We strongly believe that working with someone in a different field can help find answers and solutions to problems that have yet to be solved. We both benefited from working with one another on this project. When one may have strength in one area but very little development in another area, the other can bring balance in the areas needed. Also, the amount that you learn in the other's field is incredible. Working together also gives additional viewpoints and ideas that one may not think of on his or her own.

This paper leads obviously into the discussion of the 57 uncolorable $BSTS(15)$s and their induced and/or partial colorable $BSTS(15)$s. How many vertex deletions or edge deletions are needed to obtain an induced or partial colorable $BSTS(15)$? Is this result universal for all uncolorable $BSTS(15)$s? Much research is still needed in the area of uncolorable $BSTS(15)$s.

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