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Correcting Inconsistency in Linear Inequalities by Minimal Change in the Right Hand Side Vector

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Abstract

Correcting an inconsistent set of linear inequalities by minimal changes in problem data is a well studied problem and up to now several algorithms have been developed to do this task. In this paper, we consider doing the minimal correction using the l_2 norm by changing just the right hand vector. A new formulation of the problem is introduced and its relation with the normal solution of the alternative system of the original system is given. Then a generalized Newton algorithm is designed to solve the new formulation. Extensive computational results using this algorithm and conjugate gradient method is reported to demonstrate the advantages and disadvantages of the two algorithms.

Keywords: Linear Inequalities, Convex Optimization, Conjugate Gradient Method, Generalized Newton Method, Barrier Method.

1 Introduction

In this paper we consider the following set of linear inequalities that are inconsistent:

$$Ax \leq b, \tag{1}$$

where $A \in R^{m \times n}$ and $b \in R^m$. In other words, there is no $x \in R^n$ for which (1) is feasible. The inconsistency in system (1) might be due to the various reasons, such as lack of interaction between different groups who are defining the constraints, wrong or inaccurate estimates, error in data, over optimistic goals, and many others. Correcting system (1) to a feasible system by minimal changes in its data have been known for long time and up to now several algorithms have been developed to do it [1, 2]. A very simple approach to form a feasible system from (1) is to consider changes just in the right hand side vector b, which is usually called the resources vector, using l_1 norm i.e.,

$$\begin{aligned} &\min \sum_{i=1}^{m} |r_i| \\ &Ax \leq b + r. \end{aligned} \tag{2}$$

As we know, this problem is easily convertable to an linear programming (LP) problem which is efficiently solvable by either the *Simplex or Interior Point Methods* [3, 5]. It is also worth to note that one may consider the infinity norm in the objective function which results to:

$$\begin{aligned} &\min \ \|r\|_\infty \\ &Ax \leq b + r. \end{aligned} \tag{3}$$

This still is equivalent to an LP problem. In the next section we discuss the minimal correction using the l_2 norm. An equivalent formulation of the problem is given and two efficient algorithms are designed to solve the new formulation.

2 2-Norm Corrections

The minimal correction using the l_2 norm by changing the right hand side vector is:

$$\begin{aligned} &\min_{x,r} \frac{1}{2} \|r\|^2 \\ &Ax \leq b + r. \end{aligned} \tag{4}$$

In the following theorem we show how we compute optimal x and r values.

Theorem 2.1. *Let x^* and r^* be the optimal solution of (4). Then $r^* = (Ax^* - b)_+$, where $a_+ = \max(a, 0)$ and x^* is an optimal solution of*

$$\min_x \frac{1}{2} \left\| (Ax - b)_+ \right\|^2. \tag{5}$$

Proof. Let us write (4) as:

$$\begin{aligned} &\min_x \min_r \frac{1}{2} \|r\|^2 \\ &\qquad Ax \le b + r. \end{aligned} \tag{6}$$

Now for a given $x \in R^n$, let us first consider the inner minimization problem i.e.,

$$\begin{aligned} &\min_r \frac{1}{2} \|r\|^2 \\ &Ax \le b + r. \end{aligned} \tag{7}$$

It is obvious that problem (7) is a convex minimization problem, therefore the KKT conditions are necessary and sufficient for optimality and are given by:

$$\begin{aligned} &r - \lambda = 0, \\ &Ax \le b + r, \\ &\lambda^T (Ax - b - r) = 0, \\ &\lambda \ge 0, \end{aligned}$$

where the vector λ denotes the lagrange multipliers. From the first equation one has $r = \lambda$. Now if $\lambda_i \neq 0$ for some i, then from the third equation $(Ax - b)_i = r_i = \lambda_i$. However, when $\lambda_i = 0$, from the first equation one has $r_i = 0$. All these together imply that $r = (Ax - b)_+$. Therefore, we can write problem (4) as

$$\min_x \frac{1}{2} \left\| (Ax - b)_+ \right\|^2.$$

This completes the proof. □

It is worth mentioning that (5) is the dual of the following optimization problem:

$$\begin{aligned} \max \quad -b^T u - \frac{1}{2}\|u\|^2 & \\ A^T u = 0, & \\ u \geq 0. & \end{aligned} \tag{8}$$

In the following corollary we give an optimal solution of (8) using an optimal solution of (5).

Corollary 2.2. *Let x^* be an optimal solution of problem (5). Then $u^* = (Ax^* - b)_+$ is an optimal solution of (8).*

Proof. Let x^* be an optimal solution of (5). It is obvious that for $u^* = (Ax^* - b)_+$, $A^T u^* = 0$, which is the optimality condition for (5) and $u \geq 0$. Now we further show that the objective values of (5) and (8) are equal i.e.,

$$-b^T(Ax^* - b)_+ = \|(Ax^* - b)_+\|^2 .$$

Since $A^T(Ax^* - b)_+ = 0$, therefore, $-b^T(Ax^* - b)_+ = (Ax^* - b)^T(Ax^* - b)_+ = (Ax^* - b)_+^T(Ax^* - b)_+ = \|(Ax^* - b)_+\|^2$. □

In the sequel we further show that using an optimal solution of (5) we can construct an optimal solution for

$$\begin{aligned} \min \quad \frac{1}{2}\|u\|^2 & \\ A^T u = 0, & \\ b^T u = -\rho, & \\ u \geq 0, & \end{aligned} \tag{9}$$

where ρ is an arbitrary strictly positive parameter.

Remark 2.3. *It is worth to note that the constraints of (9) are the alternative system of (1).*

Corollary 2.4. *Let x^* be an optimal solution of problem (5), then $u^* = -\frac{\rho(Ax^*-b)_+}{\|(Ax^*-b)_+\|^2}$ is the normal solution of*

$$A^T u = 0, \quad b^T u = -\rho, \ u \geq 0$$

namely a solution of (9).

Proof. The Lagrangian dual of (9) is

$$\max_{\lambda,\mu} \ -\frac{1}{2}\left\|(A\lambda - b\mu)_+\right\|^2 + \mu\rho, \tag{10}$$

where $u = (A\lambda - b\mu)_+$. At optimality the objective values of (9) and (10) should be equal. This implies that

$$\|(A\lambda^* - b\mu^*)_+\|^2 = \rho\mu^*.$$

From this we further can deduce that $\mu > 0$, then

$$\mu^* \left\| \left(A\left(\frac{\lambda^*}{\mu^*}\right) - b \right)_+ \right\| = \rho.$$

By further defining $x = \frac{\lambda^*}{\mu^*}$ we have $\mu^* = \frac{\rho}{\|(Ax-b)_+\|}$. Now let x^* be the optimal solution of (5) and also let $\mu^* = \frac{\rho}{\|(Ax^*-b)_+\|}$. This implies that $\lambda^* = \mu^* x^*$. Now for this choice of variables the two objective values are equal. Thus we have the optimal solutions of both problems. □

To solve (5) we use conjugate gradient algorithm and the so called generalized Newton algorithm that is discussed in the sequel. As it is obvious, the objective function of (5) is a convex function, but it just has the first derivative not the second one [6]. However, the generalized hessian is defined for this function that follows:

$$\nabla f(x) = A^T (Ax - b)_+$$

and

$$\nabla^2 f(x) = A^T D A,$$

where D is an $n \times n$ diagonal matrix for which $D(i,i) = 1$ when $(Ax - b)_i > 0$, $D(i,i) = 0$, when $(Ax-b)_i < 0$, and in $[0,1]$ when $(Ax-b)_i = 0$.

Obviously the generalized Hessian is a set and for simplicity in this article we consider a specific element of this set, namely $D(i,i) = 0$ when $(Ax - b)_i = 0$. Now the generalized Newton algorithm can be outlined as follows:

Generalized Newton Algorithm

- **Inputs:** An accuracy parameter $\epsilon > 0$, a regularization parameter[1], $\delta = 10^{-4}$ and a starting point $x_0 \in R^n$.
- i=0;
- **While** $\|\nabla f(x_i)\|_\infty \geq \epsilon$.
- $x_{i+1} = x_i - (\nabla^2 f(x_i) + \delta I)^{-1} \nabla f(x_i)$.
- i=i+1.
- **End.**

Remark 2.5. *It is worth to note that one may use line search techniques such as Armijo or Wolf in the structure of the algorithm. Moreover the finite global convergence of generalized Newton algorithm with Armijo line search is proved in [6].*

3 Linear Inequalities with Nonnegativity Constraints

In this section we consider the set of linear inequalities (1) by adding extra nonnegativity constraint to them i.e.,

$$\begin{aligned} Ax &\leq b \\ x &\geq 0. \end{aligned} \tag{11}$$

It is obvious that one can consider (11) as a special case of (1), but due to its special structure it is reasonable to do the correction of this sort

[1] It guarantees the nonsingularity of hessian matrix.

of inconsistent set of linear inequalities specifically. Here we consider the case where the correction is done by just correcting the right hand side of the first set of inequalities not the $x \geq 0$ i.e.,

$$\begin{aligned} \min \quad & \frac{1}{2}\|r\|^2 \\ & Ax \leq b + r \\ & x \geq 0. \end{aligned} \tag{12}$$

In the following theorem we show how one can compute optimal x and r values.

Theorem 3.1. *Let x^* and r^* be optimal solutions of (12). Then $r^* = (Ax^* - b)_+$, where x^* is an optimal solution of*

$$\min_{x \geq 0} \quad \frac{1}{2}\|(Ax - b)_+\|^2 . \tag{13}$$

Proof. Similar to the proof of Theorem 2.1. □

As it is obvious the only difference between (13) and (5) is the nonnegativity constraint and it makes the problem a constraint optimization problem. To solve (13) we use the logarithmic barrier [7] approach by bringing the $x \geq 0$ to the objective functions as:

$$\min_{x} \quad \frac{1}{2}\|(Ax - b)_+\|^2 - \mu \sum_{i=1}^{n} \log(x_i), \tag{14}$$

where μ is the barrier parameter. Then we apply the generalized Newton method by starting from a strictly positive vector x and $\mu_0 = 1$. The logarithmic term does not allow the components of variable x to get negative and the value of μ approaches to zero during the iterations of the algorithm, for example $\mu_{k+1} = 0.8\mu_k$. Another approach which one might consider to solve (13) is the penalty function method as:

$$\min_{x} \quad \frac{1}{2}\|(Ax - b)_+\|^2 + \frac{1}{2}M\|(-x)_+\|^2 , \tag{15}$$

where M is a very big number, for example 10^{10}. This does not allow to have big $\|(-x)_+\|^2$. It is worth mentioning that vector x might have very small negative values in the optimal solution which can be rounded to zero.

4 Computational Results

In this section we present numerical results for the generalized Newton and conjugate gradient algorithms on various randomly generated problems. Test problems are generated using the following MATLAB code:

MATLAB random insolvable linear inequalities generator

```
% Generates random inconsistent system Ax <= b;
% Input:m,n,d(density); Output:A ∈ R^{m×n}, b ∈ R^m;
pl=inline('(abs(x)+x)/2');%pl(us) function;
m=input('enter m= '); n=input('enter n= '); d=input('enter
d= ');
m1=max(m-round(0.5*m),m-n);
A1=sprand(m1,n,d);A1=1*(A1-0.5*spones(A1));
x=spdiags(rand(n,1),0,n,n)*1*(rand(n,1)-rand(n,1));
x=spdiags(ones(n,1)-sign(x),0,n,n)*10*(rand(n,1)-rand(n,
1));
m2=m-m1;u=randperm(m2);A2=A1(u,:);
b1=A1*x+spdiags((rand(m1,1)),0,m1,m1)*1*ones(m1,1);
b2=b1(u)+spdiags((rand(m2,1)),0,m2,m2)*10*ones(m2,1);
A=100*[A1;-A2]; b=[b1;-b2];
```

In Tables 1 and 2 we present comparison between the gradient based algorithm (**GR**) and our new generalized Newton algorithm (**GNewton**) with Armijo linesearch for various randomly generated problems with different densities. Our numerical experiments show that the generalized Newton method finds an optimal solution much faster than the gradient based algorithm for majority of problems and for all problems the optimal objective values are much smaller than the gradient algorithm. It is worth mentioning that we run both algorithms for at most 500 seconds with the tolerance equal to 10^{-5} for gradient algorithm and 10^{-8} for the generalized Newton method.

In Tables 3 and 4 we report numerical experiments on various randomly generated problems with different densities for inconsistent linear inequalities that involve nonnegativity of variables. To solve these

Table 1. **Comparison of Generalized Newton method and Gradient algorithm with Armijo line-search**

m, n, d		$\|(Ax^* - b)_+\|$	$\|x^*\|$	$\|\nabla f(x^*)\|_\infty$	time(sec)
$2 \times 10^6, 10^3, 0.001$	GNewton	1.9416×10^2	1.7966	1.6584×10^{-10}	33
	GR	1.9450×10^2	1.7975	6.7686×10^2	500
$5 \times 10^6, 2000, 0.001$	GNewton	2.7879×10^2	2.5694	1.7558×10^{-11}	134
	GR	1.2976×10^4	9.5597	5.1477×10^5	500
$2000, 1500, 0.01$	GNewton	1.2657×10^2	2.2207×10^2	6.442×10^{-7}	10
	GR	1.4245×10^2	22.236	3.1181×10^2	4
$3000, 2000, 0.01$	GNewton	1.613×10^2	2.2550×10^2	2.1859×10^{-7}	23
	GR	1.8045×10^2	22.855	7.9328×10^2	6
$5000, 2500, 0.01$	GNewton	2.0322×10^2	5.0537×10	2.1461×10^{-9}	75
	GR	2.6368×10^2	13.356	6.7538×10^2	8
$7000, 3000, 0.01$	GNewton	2.4224×10^2	4.3524	1.4739×10^{-8}	42
	GR	2.7263×10^2	6.8396	5.5115×10^2	10
$10000, 5000, 0.01$	GNewton	2.8706×10^2	1.1382×10^2	2.4625×10^{-9}	571
	GR	3.4522×10^2	14.383	4.7877×10^2	19
$10^5, 500, 0.01$	GNewton	1.3636×10^2	1.2889	1.4504×10^{-10}	3
	GR	1.3636×10^2	1.2905	2.2150×10^2	71
$10^6, 1000, 0.01$	GNewton	1.9512×10^2	1.7957	1.3308×10^{-11}	64
	GR	4.9737×10^2	1.8019	4.7845×10^4	501

Table 2. **Comparison of Generalized Newton method and Gradient algorithm with Armijo line-search**

m, n, d		$\|(Ax^* - b)_+\|$	$\|x^*\|$	$\|\nabla f(x^*)\|_\infty$	time(sec)
$1000, 500, 0.1$	GNewton	9.1251×10	1.3206×10	3.0594×10^{-9}	1
	GR	1.1227×10^2	5.6497	3.1439×10^2	5
$2000, 1500, 0.1$	GNewton	1.3063×10^2	2.2709×10^2	1.8781×10^{-5}	7
	GR	1.3138×10^2	22.788	2.7351×10^2	13
$3000, 2000, 0.1$	GNewton	1.5668×10^2	2.1775×10^2	2.044×10^{-5}	14
	GR	1.5736×10^2	21.884	2.4234×10^2	17
$4000, 3000, 0.1$	GNewton	1.8075×10^2	3.2590×10^2	7.8024×10^{-8}	163
	GR	1.8134×10^2	32.697	2.1083×10^2	29
$6000, 4000, 0.1$	GNewton	2.2436×10^2	3.1374×10^2	1.5982×10^{-5}	91
	GR	2.4760×10^2	31.531	9.1938×10^3	55
$5000, 1000, 1$	GNewton	1.6045×10^2	1.8174	8.3030×10^{-9}	25
	GR	1.8707×10^2	1.8241	1.7451×10^4	72
$5000, 3000, 1$	GNewton	1.7489×10^2	1.8673	1.0023×10^{-9}	49
	GR	1.9926×10^2	1.8644	1.8771×10^4	149
$10000, 1000, 1$	GNewton	2.0643×10^2	2.2497×10^2	9.8919×10^{-6}	132
	GR	1.4414×10^3	23.696	3.0855×10^4	216

Table 3. **Comparison between Generalized Newton method and Barrier approach**

m, n, d		$\|(Ax^* - b)_+\|$	$\|x^*\|$	$\|\nabla f(x^*)\|_\infty$	time(sec)
$2 \times 10^6, 10^3, 0.001$	GNewton	1.0186×10^3	1.2894	1.5717×10^5	20
	Barrier	3.1282×10^3	4.1182	1.5370×10^5	503
$5 \times 10^6, 2000, 0.001$	GNewton	2.3461×10^3	1.7933	3.7410×10^5	97
	Barrier	1.0125×10^4	7.7755	3.6690×10^5	511
$2000, 1500, 0.01$	GNewton	1.6669×10^2	4.6664	1.4246×10^3	10
	Barrier	1.6669×10^2	4.6664	1.12670×10^{-2}	503
$3000, 2000, 0.01$	GNewton	2.5783×10^2	3.7585	3.3761×10^3	19
	Barrier	2.8079×10^2	4.1085	3.9359×10^3	513
$5000, 2500, 0.01$	GNewton	3.5457×10^2	3.1902	4.4665×10^3	28
	Barrier	1.6086×10^3	11.942	2.3028×10^4	520
$7000, 3000, 0.01$	GNewton	4.5226×10^2	3.1648	8.3185×10^3	43
	Barrier	2.4054×10^3	14.101	43.4104×10^4	514
$10000, 5000, 0.01$	GNewton	6.1538×10^2	4.3014	8.8359×10^3	211
	Barrier	1.9856×10^3	10.725	2.9873×10^4	519
$10^5, 500, 0.01$	GNewton	5.8234×10^2	0.83951	7.1131×10^4	3
	Barrier	1.07228×10^3	1.9616	6.8137×10^4	105
$10^6, 1000, 0.01$	GNewton	2.4497×10^3	1.2288	1.3796×10^5	51
	Barrier	9.7547×10^3	5.1913	7.3701×10^5	509

Table 4. **Comparison between Generalized Newton method and Barrier approach**

m, n, d		$\|(Ax^* - b)_+\|$	$\|x^*\|$	$\|\nabla f(x^*)\|_\infty$	time(sec)
$1000, 500, 0.1$	GNewton	1.7786×10^2	1.4306	7.4710×10^3	1.2
	Barrier	1.9887×10^2	1.5843	7.5234×10^3	66
$2000, 1500, 0.1$	GNewton	3.2366×10^2	3.2806	1.0015×10^4	21
	Barrier	3.4735×10^2	3.3738	1.1146×10^4	500
$3000, 2000, 0.1$	GNewton	4.2772×10^2	3.0036	1.9189×10^4	45
	Barrier	7.0578×10^2	3.7875	2.5369×10^4	500
$4000, 3000, 0.1$	GNewton	5.2605×10^2	4.9115	1.8869×10^4	150
	Barrier	1.3361×10^3	7.4745	4.8884×10^4	512
$6000, 4000, 0.1$	GNewton	8.5661×10^2	431.15	3.4293×10^4	282
	Barrier	2.9507×10^4	87.763	1.0627×10^6	540
$5000, 1000, 1$	GNewton	1.3920×10^3	1.4183	2.4942×10^5	26
	Barrier	4.6499×10^3	4.3363	5.1778×10^5	505
$5000, 3000, 1$	GNewton	1.6982×10^3	3.4684	1.7544×10^5	330
	Barrier	5.7628×10^3	6.8423	631×10^5	561
$10000, 1000, 1$	GNewton	1.9394×10^3	1.4330	4.7339×10^5	50
	Barrier	7.0689×10^3	4.8063	8.5838×10^5	508

problems we have employed the generalized Newton (**GNewton**) and barrier methods (**Barrier**). In the optimal solution obtained by the generalized Newton method we might have very small components of x that are negative. In this case we rounded them to zero and this is the reason for having norm infinity of the gradient vector far away than zero, however at optimality it is usually around $O(10^{-8})$. On the other hand this shows the sensitivity of these problems to very small changes in the optimal solution. As we observe from these tables, the generalized Newton method beats the barrier approach both in time and quality of solution for all of the problems.

5 Conclusion

In this paper we have furthermore investigated how to correct an inconsistent set of linear inequalities by minimal changes in its data. A new formulation of the original model is given and its relation to normal solution of alternative system for original system is discussed. Then we have presented a generalized Newton based algorithm to solve the new formulation. We also discussed inconsistent set of inequalities that involve nonnegativity of variables. To solve this specific case we have utilized the generalized Newton method and barrier approach. At last, our computational experiments on several randomly generated problems show the superior performance of the generalized Newton algorithm to the classical gradient based algorithm and barrier approach.

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