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# Computer modeling of multidimensional problems of gravitational gas dynamics on multiprocessor computers

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**Abstract**

This article deals with a method, based on total variation diminishing (TVD) scheme, for solving three-dimensional equations of gravitational gas dynamics. For this method a parallel algorithm of the decision is offered. Equations of this kind are a powerful approach to simulating astrophysical problems. Numerical schemes applied for their solving must provide high-resolution capturing of shocks, prevent spurious oscillations and specify the behavior of the matter in the neighborhood of small perturbations beyond shock fronts. Difference schemes have to combine the properties of high resolution in the regions of small perturbations and of monotonicity in the domains of steep gradients in order to satisfy such contradictory conditions.

## 1 Introduction

Modeling supernova explosions is referred to complex dynamic processes, requiring application of difference schemes of a high resolution. These have to describe the behavior of the matter in the neighborhood of the discontinuity at the maximum accuracy and to refer small perturbations far from shock fronts definitely. Conditions of such kind lead to the necessity of loss of dissipative (nonconservative) properties of numerical schemes and therefore to the apparition of large oscillations beyond shock fronts.

TVD, ENO, WENO, PPM schemes refer to the kind of schemes that satisfy all these necessary conditions and possess high resolution in regions of small perturbations combined with monotonicity in domains of steep gradients.

## 1.1 Magnetorotational mechanism of supernova explosion

In [1] a mechanism for the magnetorotational supernova explosion was analyzed. The basic concept of magnetorotational explosion consists in taking account on transition of rotating magnetic field energy into the radial kinetic energy of explosion. Various layers of the star rotate at the different angular velocities during the collapse. Differential rotation of this kind generates and enforces the magnetic field toroidal components. The growth of magnetic field intensity leads to the increase of pressure. Hence a compression shock wave appears in the neighborhood of the extreme magnetic pressure. It starts moving from the center toward the considerably fast falling density of the matter. For a rather short time this leads to the appearance of the fast magnetohydrodynamics (MHD) shock. When the shock wave reaches the surface of the collapsing star it throws out its matter. This emission may be interpreted as an explosion of supernova. Modeling magnetorotational supernova explosion in one-dimensional setting was examined, for example, in [2] and [3]. In one-dimensional case the star may be represented in the form of an infinite cylinder. The equations of ideal MHD with a self-gravitating substance in terms of Lagrangian coordinate system were considered.

The initial magnetic field had only the radial component. Differential rotation led to the appearance and increasing of toroidal component of the magnetic field. Modeling magnetorotational explosion of supernova in one-dimensional case illustrates that differential rotation of toroidal field leads to the apparition of the MHD shocks, moving towards the surface of the star. Modeling supernova explosion in two-dimensional case gives a more realistic flow pattern than in one-dimensional case. The first two-dimensional model of rotating star collapse was analyzed in [4]. The magnetic field magnitude considered in that work was unrealistic large and together with the differential rotating it led to the formation of axial emission.

Simulation of magnetorotational supernova explosion in three-dimensional case is considered in this work. Three-dimensional model of collapse is the most realistic one and does not have any restrictions connected with the assumptions, stated in 1D and 2D models. Three-dimensional models admit simulating of magnerotational supernova explosion in cases, when the axises of rotation do not coincide with the axises of dipole magnetic field (if dipole is taken as the initial value of magnetic field). If numerical schemes, elaborated for simulating two-dimensional cases, are utilized in three-dimensional case, then it will lead to big problems. The substance of the star compresses in the direction of $\varphi$ in two-dimensional case. It is necessary to calculate hundreds and thousands of cycles of rotation for simulating the explosion of protoneutron star. The protoneutron star rotation occurs very differently. If in three-dimensional case Lagrangian mesh contains tetragonal elements, then it should be reorganized on every time step. But grid modification involves reinterpolation of the mesh functions with respect to mesh structure. Utilization of the rectangular Eulerian meshes allows to avoid this problem. A tree-dimensional model of collapsing star in rectangular coordinate system was proposed in [5].

### 1.2 TVD schemes

TVD-type schemes of the first and second order of accuracy are considered in this article. First order accurate difference schemes retain the property of monotonicity, but lead to the smearing of the shock fronts. Second order accurate nonlinear schemes with the diminishing of total variation allow to carry out calculations of high resolution and to prevent nonphysical oscillations beyond shock wave fronts. Schemes of this type are of different order of accuracy in the domains with steep and low gradients. Application of these schemes in tree-dimensional case produces especially good results while simulating collapsing stars.

Equations that govern hydrodynamic motion are conservation laws for mass, momentum, [6] and energy. The conservation form of hydrodynamic equations in terms of Eulerian coordinate system is the following:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_i}(\rho v_i) = 0, \tag{1}$$

$$\frac{\partial \rho v_i}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i v_j + P\delta_{ij}) = 0, \tag{2}$$

$$\frac{\partial e}{\partial t} + \frac{\partial}{\partial x_i}[(e + P)v_i] = 0. \tag{3}$$

The influence of gravitational field is omitted in equations (1) - (3) as well as the action of other sources of energy, for example, neutrino radiation. The equation of state may be written as follows

$$P = (\gamma - 1)\varepsilon, \tag{4}$$

here $\rho$ is the density, $v$ is the vector of speed and $P$ is the pressure, besides that the total energy is $e = \frac{1}{2}\rho v^2 + \varepsilon$.

A TVD scheme was applied to the equations (1) - (3) in [7, 8]. A common restriction of oscillations is a nonlinear condition of stability. The discrete solution for TVD scheme may be defined in the following way

$$TV(u^t) = \sum_{i=1}^{N} |u_{i+1}^t - u_i^t| \tag{5}$$

as a measure of total amount of oscillations.

Thus using second order accurate fluxes $F_{i+1/2}^{(2)t}$ across cells boundaries a nonlinear TVD scheme may be presented in another way. Second order fluxes are derived from first order accurate fluxes $F_{i+1/2}^{(1)t}$ for the upwind scheme applying second order accurate correction. First order accurate flux is obtained, in turn, from the flux mean values. Second order accurate correction is introduced in order to bound spurious oscillations. Hence the number of oscillations on the current time step must not exceed the number of oscillations on the previous one. $TV(u_{i+1}) \leq TV(u_i)$.

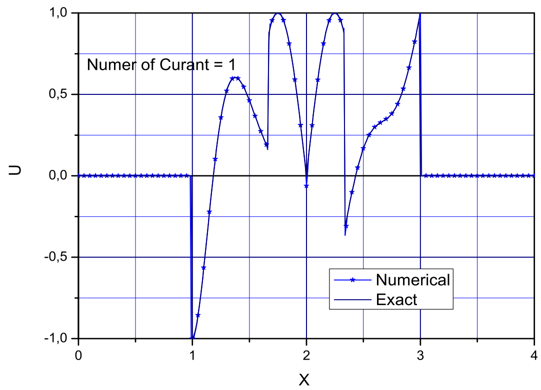


Figure 1. TVD scheme using **vanLeer** flux limiter (stared line) in comparison with analytical solution (solid line)

Different flux limiters are used in order to limit oscillations , specifically, **minmod**, **superbee**, **vanLeer**. The former limiter chooses the smallest absolute value from between the left and right corrections:

$$minmod(a,b) = \frac{1}{2}[sign(a) + sign(b)]\ min(|a|,|b|). \tag{6}$$

The **superbee** limiter choses between the larger correction and 2 times the smallest correction, whichever is smaller in magnitude

$$superbee(a,b) = \begin{cases} minmod(a,2b), & if|a| \geq |b|, \\ minmod(2a,b), & if|a| < |b|. \end{cases} \tag{7}$$

The **vanLeer** limiter is the most moderate of all limiters and finds a harmonic mean between left and right corrections

$$vanleer(a,b) = \frac{2ab}{a+b}.$$

The test, proposed in [7], was used for checking the obtained computer program

$$u_0 = \begin{cases} -x\sin(\frac{3}{2}\pi x^2, & -1 \le x < -\frac{1}{3}, \\ |\sin(2\pi x)|, & |x| < \frac{1}{3}, \\ 2x - 1 - \frac{1}{6}\sin(3\pi x), & \frac{1}{3} < x < 1. \end{cases} \quad (8)$$

The solution obtained by TVD scheme with the **vanLeer** limiter (stared line) is presented in Figure 1. The analytical solution (solid line) is included for comparison. The Courant – Friedrich's – Levy number CFL = 1. A close agreement between numerical and analytical solutions should be noted [8].

## 1.3 Equations of gravitational gas dynamics

The solution of equations of gravitational gas dynamics that describes the collapsing star may be written in the following way

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_i}(\rho v_i) = 0, \quad (9)$$

$$\frac{\partial \rho v_i}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i v_j + P\delta_{ij}) = -\rho\frac{\partial \phi}{\partial x_i}, \quad (10)$$

$$\frac{\partial e}{\partial t} + \frac{\partial}{\partial x_i}[(e + P)v_i] = -\rho v_i\frac{\partial \phi}{\partial x_i}. \quad (11)$$

The value of gravitational potential $\phi$ is defined from Poisson equation: $\triangle\phi = 4\pi G\rho$. Equation of state is used in the form of (4). In the equations from above $\rho$ - density, v - field of velocities, P - pressure, $\varepsilon$ - specific internal energy, e - total energy:

$$e = \frac{1}{2}\rho v^2 + \varepsilon \ . \quad (12)$$

TVD scheme testing was accomplished for the Sedov-Taylor test-problem of point explosion. For this purpose computational domain was defined in the form of a cube with 128 cells. The cube domain is filled in with the medium of constant density $\rho_1$ while the pressure is a negligible quantity. A high energy deposition takes place at the moment

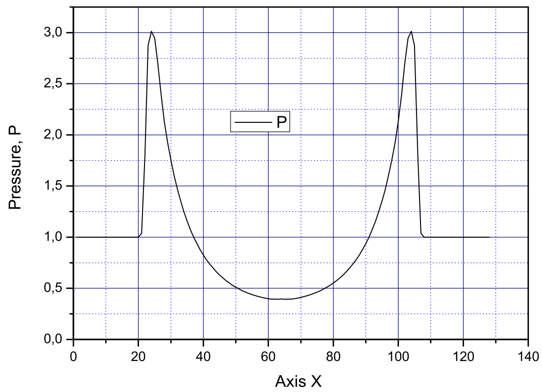


Figure 2. Pressure profile for the Sedov–Taylor test problem

t=0 in the center of the computational domain. Pressure profile is plotted in the Figure 2 at the moment t = t*. A good coincidence of numerical and analytical results has to be mentioned.

## 1.4 The main results

Let us consider the case of interaction of two shocks. Two sources of energy are placed in the center of a cube for this purpose. Instantaneous energy production takes place in the start time and the explosion is of the same yield as in the previous section. The complexity of this test consists in the necessity of an accurate computation of interaction of two shock waves. This test is more often used in astrophysical computations as a basis of supernova explosion simulating.

Pressure profile is plotted in the Figure 3 for the problem (1)-(3). The initial density and energy are respectively: $\rho_0 = 1.0$ and $E = 10^5$. The problem was solved for the case of rectangular coordinate system which is not invariant with respect to the rotation. However, the non

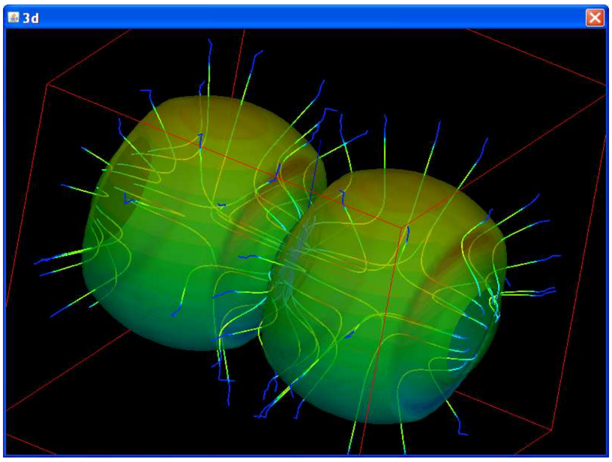


Figure 3. Pressure distribution for the case of two interacting shocks.

isotropic dispersion is not large. One can observe that the numerical solution has spherically symmetric form. Shock wave resolution is of two space cells dimension. Numerical scheme testing convinced us that TVD scheme may be used in solving supernova explosion problems.

Adaptive mesh techniques are currently being used for the improving of the accuracy of numerical calculations as well as of the algorithm efficiency. The methods of this type allow to reduce the computing time and to narrow the volume of employed memory. These techniques are especially effective in solving the problems of gas dynamics characterized by apparition of compression waves, shock waves and contact discontinuities. The use of adaptive meshes makes it possible to investigate the processes with a desirable degree of accuracy in complex geometry domains or steep gradients. AMR method allows to decrease the number of cells and therefore the time of computing. AMR technology is based on use of cells hierarchical structure. In this case every level of the hierarchy is referred to its level of spacial and time resolution. The possibility to add cells to a fixed place of computational domain locally and dynamically is the characteristic property of AMR methods. An algorithm for the refinement of the mesh on several levels with consecutively diminishing space steps is proposed in this work.

Nested meshes were used for solving three-dimensional Poisson equation, for example in [10, 11]. The density of collapsing star varies in many degrees. The density on the surface is not large but in the center the order of density increases up to $10^{14} g/cm^3$. Nested and refined meshes were built in order to take account on such enormous variation of density. In the center of computational domain a cube with the size of cells in $2^3$ times smaller than the initial size of the cells was extracted. In the center of specified cube another cube was constructed with smaller dimension of cells. Dimensions of the nested cube were equal to $M^3$, here the value of M is varying from 64 up to 1024 cells. The solution of Poisson equation was found with the help of successive over-relaxation method. The density profile and the particles paths for two interacting shock waves are plotted in the Figure 4. Calculations were carried out on the 1024x1024x1024 mesh.

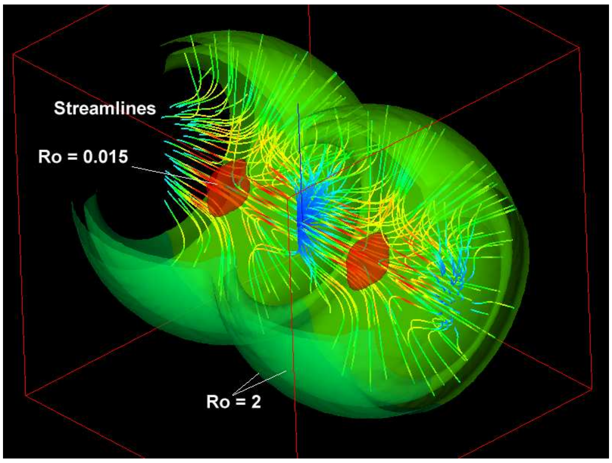


Figure 4. Distribution of the density and the particles paths for the 1024x1024x1024 mesh.

A parallel algorithm for solving Poisson hydrodynamic equations was constructed [11]. The algorithm efficiency is the highest one for 8-12 processors but for the greater number of processors the loss of efficiency is observed. Calculations have been performed on the high performance computing cluster of the Institute of Mathematics and Computer Science of the Academy of Sciences of Moldova.

### 1.5 Summary

A parallel algorithm and a code for three-dimensional gravitational gas dynamic equations were provided in this work. For this purpose a TVD scheme possessing high resolution in the regions of shock fronts and steep gradients was used. Numerical calculations obtained on the sequence of nested meshes have been presented. Calculations were implemented on the meshes from 64x64x64 to 1024x1024x1024 nodes up to 5 nesting levels. It was demonstrated that the algorithm is quite efficient for 8 – 12 processors.

The results of computer modeling obtained in this work were visualized with the help of HDVIS program [12].

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