Computer Science Journal of Moldova, vol.17, no.1(49), 2009

On Covering Approximation Subspaces*

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Abstract

Let $(U';\mathcal{C}')$ be a subspace of a covering approximation space $(U;\mathcal{C})$ and $X \subset U'$. In this paper, we show that $\overline{\mathcal{C}'}(X) = \overline{\mathcal{C}}(X)\bigcap U'$ and $B'(X) \subset B(X)\bigcap U'$. Also, $\underline{\mathcal{C}}(X) = \underline{\mathcal{C}'}(X)\bigcap \underline{\mathcal{C}}(U')$ iff $(U;\mathcal{C})$ has Property Multiplication. Furthermore, some connections between outer (resp. inner) definable subsets in $(U;\mathcal{C})$ and outer (resp. inner) definable subsets in $(U';\mathcal{C}')$ are established. These results answer a question on covering approximation subspace posed by J. Li, and are helpful to obtain further applications of Pawlak rough set theory in pattern recognition and artificial intelligence.

Keywords: Rough set; covering approximation subspace; covering approximation operator; definable; outer definable; inner definable.

1 Introduction

In order to extract useful information hidden in voluminous data, many methods in addition to classical logic have been proposed. Pawlak rough-set theory, which was proposed by Z. Pawlak in [11], plays an important role in applications of these methods. Their usefulness has been demonstrated by many successful applications in pattern recognition and artificial intelligence (see [4, 5, 6, 8, 10, 11, 12, 13, 14, 16, 19, 24, 28], for example). In the past years, Pawlak rough-set theory have been extended from Pawlak approximation spaces to covering approximation spaces (see [1, 2, 3, 7, 17, 18, 20, 21, 22, 23, 25, 26, 27, 28, 29], for example).

* This paper is supported by Natural Science Foundation of P. R. China (No.10571151 and 10671173)

Definition 1.1. *Let U be a finite set (a universe of discourse), $\mathcal{C}$ be a cover of U and $X \subset U$. Put*

$$\underline{\mathcal{C}}(X) = \bigcup\{K : K \in \mathcal{C} \bigwedge K \subset X\};$$

$$\overline{\mathcal{C}}(X) = \bigcup\{K : K \in \mathcal{C} \bigwedge K \bigcap X \neq \emptyset\};$$

$$B(X) = X - \underline{\mathcal{C}}(X).$$

(1) $(U;\mathcal{C})$ is called a covering approximation space.
(2) $\underline{C} : 2^U \longrightarrow 2^U$ is called lower covering approximation operator.
(3) $\overline{C} : 2^U \longrightarrow 2^U$ is called upper covering approximation operator.
(4) $\underline{\mathcal{C}}(X)$ is called lower covering approximation of X.
(5) $\overline{\mathcal{C}}(X)$ is called upper covering approximation of X.
(6) $B(X)$ is called boundary of X.
(7) X is called definable in $(U;\mathcal{C})$ if $\overline{\mathcal{C}}(X) = \underline{\mathcal{C}}(X)$.

However, in many applications of Pawlak rough-set theory, we need to consider the case that a cover $\mathcal{C}$ of a universe of discourse U is restricted on some subset U' of U (see [19], for example). More precisely, we are also interested in subspace $(U';\mathcal{C}')$ of covering approximation space $(U;\mathcal{C})$.

Definition 1.2. *Let $(U;\mathcal{C})$ be a covering approximation space. $(U';\mathcal{C}')$ is called a subspace of $(U;\mathcal{C})$ if $U' \subset U$ and $\mathcal{C}' = \{K \bigcap U' : K \in \mathcal{C}\}$.*

Remark 1.3. *For a subspace $(U';\mathcal{C}')$ of a covering approximation space $(U;\mathcal{C})$ and a subset X of U', it is the same as Definition 1.1 to define lower covering approximation operator $\underline{C'}$, upper covering approximation operator $\overline{C'}$, lower covering approximation $\underline{\mathcal{C}'}(X)$ of X, upper covering approximation $\overline{\mathcal{C}'}(X)$ of X, boundary $B'(X)$ of X and definable subsets in $(U';\mathcal{C}')$. We omit these definitions.*

Let $(U';\mathcal{C}')$ be a subspace of a covering approximation space $(U;\mathcal{C})$ and $X \subset U'$. It is worthy to give some relations between covering approximations of subsets in $(U;\mathcal{C})$ and covering approximations of subsets in $(U';\mathcal{C}')$ and to establish some connections between definable subsets in $(U;\mathcal{C})$ and definable subsets in $(U';\mathcal{C}')$. It is well-known that if $(U;\mathcal{C})$ is a Pawlak approximation space, i.e., $\mathcal{C}$ is a partition of U, then $(U;\mathcal{C})$ is a topological space with a base $\mathcal{C}$. $\underline{\mathcal{C}}(X)$, $\overline{\mathcal{C}}(X)$ and $B(X)$ are exactly interior of X, closure of X and boundary of X in $(U;\mathcal{C})$, respectively (see [7, 15, 26], for example). Thus, $(U';\mathcal{C}')$ is a topological subspace of $(U;\mathcal{C})$ with a base $\mathcal{C}'$. $\underline{\mathcal{C}'}(X)$, $\overline{\mathcal{C}'}(X)$ and $B'(X)$ are exactly interior of X, closure of X and boundary of X in $(U';\mathcal{C}')$, respectively. So the following results are obtained naturally.

Proposition 1.4. *Let $(U';\mathcal{C}')$ be a subspace of a Pawlak approximation space $(U;\mathcal{C})$, and $X \subset U'$. Then the following hold.*

(1) $\overline{\mathcal{C}'}(X) = \overline{\mathcal{C}}(X) \bigcap U'$.

(2) $\underline{\mathcal{C}}(X) = \underline{\mathcal{C}'}(X) \bigcap \underline{\mathcal{C}}(U')$.

(3) $B'(X) \subset B(X) \bigcap U'$.

(4) If U' is definable in $(U;\mathcal{C})$, then X is definable in $(U;\mathcal{C})$ iff X is definable in $(U';\mathcal{C}')$.

By viewing Proposition 1.4, J. Li raised the following question in [9].

Question 1.5. *If $(U;\mathcal{C})$ is a covering approximation space, does Proposition 1.4 hold?*

In this paper, we investigate and answer Question 1.5. For a subspace $(U';\mathcal{C}')$ of a covering approximation space $(U;\mathcal{C})$ and a subset X of U', we show that $\overline{\mathcal{C}'}(X) = \overline{\mathcal{C}}(X) \bigcap U'$ and $B'(X) \subset B(X) \bigcap U'$. Also, $\underline{\mathcal{C}}(X) = \underline{\mathcal{C}'}(X) \bigcap \underline{\mathcal{C}}(U')$ iff $(U;\mathcal{C})$ has Property Multiplication. Furthermore, we establish some connections between outer (resp. inner) definable subsets in $(U;\mathcal{C})$ and outer (resp. inner) definable subsets in $(U';\mathcal{C}')$. These results are helpful to obtain further applications of Pawlak rough set theory in pattern recognition and artificial intelligence.

2 On Covering Approximations of subsets

The following lemma is known (see [18, 29], for example).

Lemma 2.1. *Let $(U;\mathcal{C})$ be a covering approximation space. Then the following hold.*

(1) If $X \subset U$, then $\underline{\mathcal{C}}(X) \subset X \subset \overline{\mathcal{C}}(X)$.

(2) If $X \subset Y \subset U$, then $\overline{\mathcal{C}}(X) \subset \overline{\mathcal{C}}(Y)$ and $\underline{\mathcal{C}}(X) \subset \underline{\mathcal{C}}(Y)$.

(3) If $X, Y \subset U$, then $\underline{\mathcal{C}}(X \bigcap Y) \subset \underline{\mathcal{C}}(X) \bigcap \underline{\mathcal{C}}(Y)$.

(4) If X is a union of some elements of $\mathcal{C}$, then $\underline{\mathcal{C}}(X) = X$.

(5) $\underline{\mathcal{C}}(U) = \overline{\mathcal{C}}(U) = U$.

Theorem 2.2. *Let $(U';\mathcal{C}')$ be a subspace of a covering approximation space $(U;\mathcal{C})$ and $X \subset U'$. Then the following hold.*

(1) $\overline{\mathcal{C}'}(X) = \overline{\mathcal{C}}(X) \bigcap U'$.

(2) $B'(X) \subset B(X) \bigcap U'$.

Proof. (1) Let $x \in \overline{\mathcal{C}'}(X)$, then there exists $K \in \mathcal{C}$ such that $x \in K \bigcap U'$ and $(K \bigcap U') \bigcap X \neq \emptyset$, so $x \in K$ and $K \bigcap X \neq \emptyset$. Thus $x \in \overline{\mathcal{C}}(X)$ and $x \in U'$, i.e., $x \in \overline{\mathcal{C}}(X) \bigcap U'$. On the other hand, let $x \in \overline{\mathcal{C}}(X) \bigcap U'$, then there exists $K \in \mathcal{C}$ such that $x \in K$ and $K \bigcap X \neq \emptyset$. Since $X \subset U'$, $(K \bigcap U') \bigcap X = K \bigcap X \neq \emptyset$. Note that $x \in K \bigcap U'$ and $K \bigcap U' \in \mathcal{C}'$. So $x \in \overline{\mathcal{C}'}(X)$.

(2) Since $B'(x) = U' - \underline{\mathcal{C}'}(X)$ and $B(X) \bigcap U' = (U - \underline{\mathcal{C}}(X)) \bigcap U' = U' - (\underline{\mathcal{C}}(X) \bigcap U')$, it suffices to prove that $\underline{\mathcal{C}}(X) \bigcap U' \subset \underline{\mathcal{C}'}(X)$. Let $x \in \underline{\mathcal{C}}(X) \bigcap U'$, then $x \in U'$ and there exists $K \in \mathcal{C}$ such that $x \in K \subset X$, So $x \in K \bigcap U' \subset X$. Note that $K \bigcap U' \in \mathcal{C}'$, so $x \in \underline{\mathcal{C}'}(X)$. This proves that $\underline{\mathcal{C}}(X) \bigcap U' \subset \underline{\mathcal{C}'}(X)$. □

Remark 2.3. *The following example shows that "$\subset$" in Theorem 2.2(2) can not be replaced by "$=$".*

Example 2.4. *There exist a subspace $(U';\mathcal{C}')$ of a covering approximation space $(U;\mathcal{C})$ and a subset X of U' such that $B'(X) \neq B(X) \bigcap U'$.*

Proof. Let $U = \{a, b, c\}$, $\mathcal{C} = \{\{a, b\}, \{b, c\}\}$, $U' = \{a, b\}$ and $\mathcal{C}' = \{\{a, b\}, \{b\}\}$, then $(U';\mathcal{C}')$ is a subspace $(U;\mathcal{C})$. Put $X = \{b\}$, then $X \subset U'$.

(1) $\underline{\mathcal{C}}(X) = \emptyset$, so $B(X) = X - \underline{\mathcal{C}}(X) = X$.

(2) $\underline{\mathcal{C}'}(X) = X$, so $B'(X) = X - \underline{\mathcal{C}'}(X) = \emptyset$.

Consequently, $B'(X) \neq B(X) \bigcap U'$. □

In general, Proposition 1.4(2) does not hold for covering approximation spaces (see [18, 29], for example). We give a sufficient and necessary condition such that it holds.

Definition 2.5. *Let $(U;\mathcal{C})$ be a covering approximation space. $(U;\mathcal{C})$ is called to have Property Multiplication (Property (M), in brief), if $\underline{C}(X\bigcap Y)=\underline{C}(X)\bigcap\underline{C}(Y)$ for any $X,Y\subset U$.*

Remark 2.6. *Every Pawlak approximation space has Property (M). In general, covering approximation spaces have not Property (M) (see [26, Proposition 4].*

The following lemma comes from [26, Theorem 1].

Lemma 2.7. *Let $(U;\mathcal{C})$ be a covering approximation space. Then the following are equivalent.*

(1) $(U;\mathcal{C})$ has Property (M).

(2) If $K_1,K_2\in\mathcal{C}$ and $x\in K_1\bigcap K_2$, then there exists $K\in\mathcal{C}$ such that $x\in K\subset K_1\bigcap K_2$.

Theorem 2.8. *Let $(U;\mathcal{C})$ be a covering approximation space. Then the following are equivalent.*

(1) $(U;\mathcal{C})$ has Property (M).

(2) If $(U';\mathcal{C}')$ is a subspace of $(U;\mathcal{C})$ and $X\subset U'$, then $\underline{\mathcal{C}}(X)=\underline{\mathcal{C}'}(X)\bigcap\underline{\mathcal{C}}(U')$.

Proof. (1) $\Longrightarrow$ (2): Let $(U';\mathcal{C}')$ be a subspace of $(U;\mathcal{C})$ and $X\subset U'$. If $x\in\underline{\mathcal{C}}(X)$, then there exists $K\in\mathcal{C}$ such that $x\in K\subset X\subset U'$, so $x\in\underline{\mathcal{C}}(U')$. Note that $K\bigcap U'=K$, so $K\in\mathcal{C}'$, thus $x\in\underline{\mathcal{C}'}(X)$. Consequently, $x\in\underline{\mathcal{C}'}(X)\bigcap\underline{\mathcal{C}}(U')$. On the other hand, if $x\in\underline{\mathcal{C}'}(X)\bigcap\underline{\mathcal{C}}(U')$, then there exist $K_1,K_2\in\mathcal{C}$ such that $x\in K_1\bigcap U'\subset X$ and $x\in K_2\subset U'$. Since $(U;\mathcal{C})$ has Property (M), by Lemma 2.7, there exists $K\in\mathcal{C}$ such that $x\in K\subset K_1\bigcap K_2$. So $x\in K\subset K_1\bigcap K_2\subset K_1\bigcap U'\subset X$. Thus $x\in\underline{\mathcal{C}}(X)$. This proves that $\underline{\mathcal{C}}(X)=\underline{\mathcal{C}'}(X)\bigcap\underline{\mathcal{C}}(U')$.

(2) $\Longrightarrow$ (1): Let $K_1,K_2\in\mathcal{C}$ and $x\in K_1\bigcap K_2$. Put $U'=K_1$ and $\mathcal{C}'=\{K\bigcap U':K\in\mathcal{C}\}$, then $(U';\mathcal{C}')$ is a subspace of $(U;\mathcal{C})$. Put $X=K_1\bigcap K_2=K_2\bigcap U'$, then $x\in X\in\mathcal{C}'$. So $x\in X=\underline{\mathcal{C}'}(X)$ from

Lemma 2.1(4). On the other hand, $x \in K_1 = \underline{\mathcal{C}}(K_1) = \underline{\mathcal{C}}(U')$ from Lemma 2.1(4). Thus $x \in \underline{\mathcal{C}'}(X) \bigcap \underline{\mathcal{C}}(U')$. Since $\underline{\mathcal{C}}(X) = \underline{\mathcal{C}'}(X) \bigcap \underline{\mathcal{C}}(U')$, $x \in \underline{\mathcal{C}}(X)$, and so there exists $K \in \mathcal{C}$ such that $x \in K \subset X = K_1 \bigcap K_2$. By Lemma 2.7, $(U;\mathcal{C})$ has Property (M). $\square$

Remark 2.9. *In the proof of Theorem 2.8(1) $\Longrightarrow$ (2), we can see that $\underline{\mathcal{C}}(X) \subset \underline{\mathcal{C}'}(X) \bigcap \underline{\mathcal{C}}(U')$ without requiring Property (M), and so $\underline{\mathcal{C}}(X) \subset \underline{\mathcal{C}'}(X)$ without requiring Property (M).*

3 On Outer and Inner Definable Subsets

As some applications of Theorem 2.2 and Theorem 2.8, we investigate definable subsets in covering approximation subspaces. The following definitions come from [15]

Definition 3.1. *Let $(U;\mathcal{C})$ be a covering approximation space and $X \subset U$.*

(1) X is called outer definable in $(U;\mathcal{C})$ if $\overline{\mathcal{C}}(X) = X$.

(2) X is called inner definable in $(U;\mathcal{C})$ if $\underline{\mathcal{C}}(X) = X$.

Remark 3.2. *It is easy to see that X is definable in $(U,\mathcal{C})$ iff it is both outer definable and inner definable in $(U;\mathcal{C})$.*

Lemma 3.3. *Let $(U;\mathcal{C})$ be a covering approximation space and $X \subset U$. Consider the following conditions.*

(1) X is definable in $(U,\mathcal{C})$.

(2) X is outer definable in $(U,\mathcal{C})$.

(3) X is inner definable in $(U,\mathcal{C})$.

Then (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3).

Proof. By Remark 3.2, (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3). It suffices to prove (2) $\Longrightarrow$ (1).

Let X be outer definable in $(U;\mathcal{C})$, i.e., $\overline{\mathcal{C}}(X) = X$. Let $x \in X$, then there is $K \in \mathcal{C}$ such that $x \in K$. So $K \bigcap X \neq \emptyset$, and hence $K \subset \overline{\mathcal{C}}(X) = X$. It follows that $x \in K \subset \underline{\mathcal{C}}(X)$. This proves that $X \subset \underline{\mathcal{C}}(X)$. By Lemma 2.1(1), $\underline{\mathcal{C}}(X) \subset X$, so $\underline{\mathcal{C}}(X) = X$. Consequently, X is inner definable in $(U,\mathcal{C})$. $\square$

Remark 3.4. *(1) In Lemma 3.3, (3) $\not\Longrightarrow$ (2) (see Example 3.5).*

(2) If $(U;\mathcal{C})$ is a Pawlak approximation space and $X \subset U$, then (1), (2) and (3) in Lemma 3.3 are equivalent ([15]).

Example 3.5. *There exist a covering approximation space $(U;\mathcal{C})$ and a subset X of U such that X is inner definable in $(U;\mathcal{C})$, but X is not outer definable in $(U;\mathcal{C})$.*

Proof. Let $U = \{a, b, c\}$, $\mathcal{C} = \{\{a, b\}, \{b, c\}\}$, $X = \{a, b\}$.

(1) Since $X \in \mathcal{C}$, $\underline{\mathcal{C}}(X) = X$ from Lemma 2.1(4), so X is inner definable in $(U;\mathcal{C})$.

(2) It is easy to see that, $\overline{\mathcal{C}}(X) = U \neq X$, so X is not outer definable in $(U;\mathcal{C})$. □

By Lemma 3.3, "outer definable" can be replaced by "definable" throughout the following.

Theorem 3.6. *Let $(U';\mathcal{C}')$ be a subspace of a covering approximation space $(U;\mathcal{C})$ and $X \subset U$. Then the following hold.*

(1) If X is outer definable in $(U;\mathcal{C})$, then $X \bigcap U'$ is outer definable in $(U';\mathcal{C}')$.

(2) If X is inner definable in $(U;\mathcal{C})$, then $X \bigcap U'$ is inner definable in $(U';\mathcal{C}')$.

Proof. (1) Let X is outer definable in $(U;\mathcal{C})$, i.e., $\overline{\mathcal{C}}(X) = X$. By Theorem 2.2(1), Lemma 2.1(3) and Lemma 2.1(1), $\overline{\mathcal{C}'}(X \bigcap U') = \overline{\mathcal{C}}(X \bigcap U') \bigcap U' \subset \overline{\mathcal{C}}(X) \bigcap \overline{\mathcal{C}}(U') \bigcap U' = X \bigcap U'$. On the other hand, $X \bigcap U' \subset \overline{\mathcal{C}'}(X \bigcap U')$ from Lemma 2.1(1). Thus $\overline{\mathcal{C}'}(X \bigcap U') = X \bigcap U'$, so $X \bigcap U'$ is outer definable in $(U';\mathcal{C}')$.

(2) Let X is inner definable in $(U;\mathcal{C})$, i.e., $\underline{\mathcal{C}}(X) = X$. Then $X \bigcap U' = \underline{\mathcal{C}}(X) \bigcap U' = (\bigcup\{K : K \in \mathcal{C} \bigwedge K \subset X\}) \bigcap U' = \bigcup\{K \bigcap U' : K \in \mathcal{C} \bigwedge K \subset X\} \subset \bigcup\{K \bigcap U' : K \in \mathcal{C} \bigwedge K \bigcap U' \subset X \bigcap U'\} = \underline{\mathcal{C}'}(X \bigcap U')$. On the other hand, $\underline{\mathcal{C}'}(X \bigcap U') \subset X \bigcap U'$ from Lemma 2.1(1). Thus $\underline{\mathcal{C}'}(X \bigcap U') = X \bigcap U'$, so $X \bigcap U'$ is inner definable in $(U';\mathcal{C}')$. □

Remark 3.7. *The following example shows that both (1) and (2) in Theorem 3.6 can not be reversed even if $(U;\mathcal{C})$ is a Pawlak approximation space.*

Example 3.8. *There exist a subspace $(U';\mathcal{C}')$ of a Pawlak approximation space $(U;\mathcal{C})$ and a subset X of U, where U' is outer definable in $(U;\mathcal{C})$, such that $X\bigcap U'$ is outer definable in $(U';\mathcal{C}')$, but X is not inner definable in $(U;\mathcal{C})$.*

Proof. Let $U=\{a,b,c,d\}$, $\mathcal{C}=\{\{a,b\},\{c,d\}\}$, then $(U;\mathcal{C})$ is a Pawlak approximation space. Put $U'=\{a,b\}$ and $\mathcal{C}'=\{\{a,b\}\}$, then $(U';\mathcal{C}')$ is a subspace $(U;\mathcal{C})$. Put $X=\{a,b,c\}$.

(1) It is clear that U' is outer definable in $(U;\mathcal{C})$.

(2) Since $X\bigcap U'=U'$, $X\bigcap U'$ is outer definable in $(U';\mathcal{C}')$.

(3) It is easy to see that $\underline{\mathcal{C}}(X)=U'\neq X$, so X is not inner definable in $(U;\mathcal{C})$. □

However, we have the following results.

Theorem 3.9. *Let $(U';\mathcal{C}')$ be a subspace of a covering approximation space $(U;\mathcal{C})$ and $X\subset U'$. If U' is outer definable in $(U;\mathcal{C})$, then the following are equivalent.*

(1) X is outer definable in $(U;\mathcal{C})$.

(2) X is outer definable in $(U';\mathcal{C}')$.

Proof. (1) $\Longrightarrow$ (2): It holds from Theorem 3.6(1).

(2) $\Longrightarrow$ (1): Let X be outer definable in $(U';\mathcal{C}')$, i.e., $\overline{\mathcal{C}'}(X)=X$. Since U' is outer definable in $(U;\mathcal{C})$, $\overline{\mathcal{C}}(U')=U'$. $\overline{\mathcal{C}}(X)\subset\overline{\mathcal{C}}(U')=U'$ from Lemma 2.1(2). By Theorem 2.2(1), $\overline{\mathcal{C}}(X)=\overline{\mathcal{C}}(X)\bigcap U'=\overline{\mathcal{C}'}(X)=X$. So X is outer definable in $(U;\mathcal{C})$. □

Remark 3.10. *(1) By Theorem 3.6(1), the condition "U' is outer definable in $(U;\mathcal{C})$" in Theorem 3.9(1) $\Longrightarrow$ (2) can be omitted.*

(2) The condition "U' is outer definable in $(U;\mathcal{C})$" in Theorem 3.9(2) $\Longrightarrow$ (1) can not be relaxed to "U' is inner definable in $(U;\mathcal{C})$" (see Example 3.11).

Example 3.11. *There exist a subspace $(U';\mathcal{C}')$ of a covering approximation space $(U;\mathcal{C})$ and a subset X of U' such that $(U;\mathcal{C})$ has Property (M), U' is inner definable in $(U;\mathcal{C})$, and X is outer definable in $(U';\mathcal{C}')$, but X is not outer definable in $(U;\mathcal{C})$.*

Proof. Let $U = \{a,b,c\}$, $\mathcal{C} = \{\{a,b\},\{b,c\},\{b\}\}$. Put $U' = X = \{a,b\}$ and $\mathcal{C}' = \{\{a,b\},\{b\}\}$, then $(U';\mathcal{C}')$ is a subspace $(U;\mathcal{C})$.

(1) Using Lemma 2.7, it is easy to check that $(U;\mathcal{C})$ has Property (M).

(2) It is clear that U' is inner definable in $(U;\mathcal{C})$.

(3) Since $X \subset \overline{\mathcal{C}'}(X) \subset U' = X$, $\overline{\mathcal{C}'}(X) = X$, so X is outer definable in $(U';\mathcal{C}')$.

(4) $\overline{\mathcal{C}}(X) = U \neq X$, so X is not outer definable in $(U;\mathcal{C})$. □

Theorem 3.12. *Let $(U';\mathcal{C}')$ be a subspace of a covering approximation space $(U;\mathcal{C})$ and $X \subset U'$. If $(U;\mathcal{C})$ has Property (M) and U' is inner definable in $(U;\mathcal{C})$, then the following are equivalent.*

(1) X is inner definable in $(U;\mathcal{C})$.

(2) X is inner definable in $(U';\mathcal{C}')$.

Proof. (1) $\Longrightarrow$ (2): It holds from Theorem 3.6(2).

(2) $\Longrightarrow$ (1): Let X be inner definable in $(U';\mathcal{C}')$, i.e., $\underline{\mathcal{C}'}(X) = X$. Since $(U;\mathcal{C})$ has Property (M), $\underline{\mathcal{C}}(X) = \underline{\mathcal{C}'}(X) \bigcap \underline{\mathcal{C}}(U')$ from Theorem 2.8. Note that $\underline{\mathcal{C}}(U') = U'$ because U' is inner definable in $(U;\mathcal{C})$. Thus $\underline{\mathcal{C}}(X) = \underline{\mathcal{C}'}(X) \bigcap \underline{\mathcal{C}}(U') = X \bigcap \underline{U'} = X$. So X is inner definable in $(U;\mathcal{C})$. □

Remark 3.13. *(1) By Theorem 3.6(2), both condition "$(U;\mathcal{C})$ has Property (M)" and condition "U' is inner definable in $(U;\mathcal{C})$" in Theorem 3.12(1) $\Longrightarrow$ (2) can be omitted.*

(2) The condition "$(U;\mathcal{C})$ has Property (M)" in Theorem 3.12(2) $\Longrightarrow$ (1) can not be omitted (see Example 3.14).

(3) The condition "U' is inner definable in $(U;\mathcal{C})$" in Theorem 3.12(2) $\Longrightarrow$ (1) can not be omitted (see Example 3.15).

Example 3.14. *There exist a subspace $(U';\mathcal{C}')$ of a covering approximation space $(U;\mathcal{C})$ and a subset X of U', where U' is inner definable* *in $(U;\mathcal{C})$, such that X is inner definable in $(U';\mathcal{C}')$, but X is not inner definable in $(U;\mathcal{C})$.*

Proof. Let $U = \{a,b,c\}$, $\mathcal{C} = \{\{a,b\},\{b,c\}\}$. Put $U' = \{a,b\}$ and $\mathcal{C}' = \{\{a,b\},\{b\}\}$, then $(U';\mathcal{C}')$ is a subspace $(U;\mathcal{C})$. Put $X = \{b\}$, then $X \subset U'$.

(1) Since $U' \in \mathcal{C}$, $\underline{\mathcal{C}}(U') = U'$ from Lemma 2.1(4), so U' is inner definable in $(U;\mathcal{C})$.

(2) Since $X \in \mathcal{C}'$, $\underline{\mathcal{C}'}(X) = X$ from Lemma 2.1(4), so X is inner definable in $(U';\mathcal{C}')$.

(3) $\underline{\mathcal{C}}(X) = \emptyset \neq X$, so X is not inner definable in $(U;\mathcal{C})$. □

Example 3.15. *There exist a subspace $(U';\mathcal{C}')$ of a covering approximation space $(U;\mathcal{C})$ and a subset X of U', where $(U;\mathcal{C})$ has Property (M), such that X is inner definable in $(U';\mathcal{C}')$, but X is not inner definable in $(U;\mathcal{C})$.*

Proof. Let $U = \{a,b,c\}$, $\mathcal{C} = \{\{a,b\},\{b,c\},\{b\}\}$. Put $U' = X = \{a,c\}$ and $\mathcal{C}' = \{\{a\},\{c\}\}$, then $(U';\mathcal{C}')$ is a subspace $(U;\mathcal{C})$.

(1) $(U;\mathcal{C})$ has Property (M) from Example 3.11.

(2) Since $\underline{\mathcal{C}'}(U') = U'$ from Lemma 2.1(5), $\underline{\mathcal{C}'}(X) = \underline{\mathcal{C}'}(U') = U' = X$, so X is inner definable in $(U';\mathcal{C}')$.

(3) $\underline{\mathcal{C}}(X) = \emptyset \neq X$, so X is not inner definable in $(U;\mathcal{C})$. □

4 Postscript

In this paper, our investigations on covering approximation subspaces are based on lower covering approximation operator $\underline{\mathcal{C}}$ and upper covering approximation operator $\overline{\mathcal{C}}$, which are endowed covering approximation spaces. Because there are also other covering approximation operators (see the following Definition 4.1), It is an interesting work to give some answers of Question 1.5 for these covering approximation operators.

Definition 4.1. *Let $(U;\mathcal{C})$ be a covering approximation space. For each $x \in U$, put*

$$Md(x) = \{K : (x \in K \in \mathcal{C}) \bigwedge (x \in S \in \mathcal{C} \bigwedge S \subset K \Longrightarrow S = K)\};$$

$$N(x) = \bigcap\{K : x \in K \in \mathcal{C}\}.$$

For each $i \in \{1,2,3,4,5,6,7,8,9,10\}$, $\underline{C_i}$ and $\overline{C_i}$ are defined as follows and are called i-th lower covering approximation operator and i-th upper covering approximation operator on $(U;\mathcal{C})$, respectively.

(1) $\underline{\mathcal{C}_1}(X) = \bigcup\{K : K \in \mathcal{C} \bigwedge K \subset X\}$;
$\overline{\mathcal{C}_1}(X) = \underline{\mathcal{C}_1}(X) \bigcup(\bigcup\{\bigcup Md(x) : x \in X - \underline{\mathcal{C}_1}(X)\})$.

(2) $\underline{\mathcal{C}_2}(X) = \{x \in U : \forall K \in \mathcal{C}(x \in K \Longrightarrow K \subset X)\}$;
$\overline{\mathcal{C}_2}(X) = \bigcup\{K : K \in \mathcal{C} \bigwedge K \bigcap X \neq \emptyset\}$.

(3) $\underline{\mathcal{C}_3}(X) = \bigcup\{K : K \in \mathcal{C} \bigwedge K \subset X\}$;
$\overline{\mathcal{C}_3}(X) = \bigcup\{\bigcup Md(x) : x \in X\}$.

(4) $\underline{\mathcal{C}_4}(X) = \bigcup\{K : K \in \mathcal{C} \bigwedge K \subset X\}$;
$\overline{\mathcal{C}_4}(X) = \underline{\mathcal{C}_4}(X) \bigcup(\bigcup\{K : K \in \mathcal{C} \bigwedge K \bigcap(X - \underline{\mathcal{C}_4}(X)) \neq \emptyset\})$.

(5) $\underline{\mathcal{C}_5}(X) = \bigcup\{K : K \in \mathcal{C} \bigwedge K \subset X\}$;
$\overline{\mathcal{C}_5}(X) = \underline{\mathcal{C}_5}(X) \bigcup(\bigcup\{N(x) : x \in X - \underline{\mathcal{C}_5}(X)\})$.

(6) $\underline{\mathcal{C}_6}(X) = \{x \in U : N(x) \subset X\}$;
$\overline{\mathcal{C}_6}(X) = \{x \in U : N(x) \bigcap X \neq \emptyset\}$.

(7) $\underline{\mathcal{C}_7}(X) = \bigcup\{K : K \in \mathcal{C} \bigwedge K \subset X\}$;
$\overline{\mathcal{C}_7}(X) = U - \underline{\mathcal{C}_7}(U - X)$.

(8) $\underline{\mathcal{C}_8}(X) = \{x \in U : \exists u(u \in N(x) \bigwedge N(u) \subset X)\}$;
$\overline{\mathcal{C}_8}(X) = \{x \in U : \forall u(u \in N(x) \Longrightarrow N(u) \bigcap X \neq \emptyset)\}$.

(9) $\underline{\mathcal{C}_9}(X) = \{x \in U : \forall u(x \in N(u) \Longrightarrow N(u) \subset X)\}$;
$\overline{\mathcal{C}_9}(X) = \bigcup\{N(x) : x \in U \bigwedge N(x) \bigcap X \neq \emptyset\}$.

(10) $\underline{\mathcal{C}_{10}}(X) = \{x \in U : \forall u(x \in N(u) \Longrightarrow u \in X)\}$;
$\overline{\mathcal{C}_{10}}(X) = \bigcup\{N(x) : x \in X\}$.

Remark 4.2. *$\underline{C_i}$ and $\overline{C_i}$ (i=1,3) come from [29]; $\underline{C_2}$ and $\overline{C_2}$ come from [17]; $\underline{C_4}$ and $\overline{C_4}$ come from [26]; $\underline{C_5}$ and $\overline{C_5}$ come from [27]; $\underline{C_6}$ and $\overline{C_6}$ come from [18, 28]; $\underline{C_i}$ and $\overline{C_i}$ (i=7,8,9,10) come from [18].*

Thus, we have the following question, which is still worthy to be considered in subsequent research.

Question 4.3. *Let* $(U';\mathcal{C}')$ *be a subspace of a covering approximation space* $(U;\mathcal{C})$*,* $X \subset U'$ *and* $i = 1,2,3,4,5,6,7,8,9,10$*. Do following hold?*

(1) $\overline{\mathcal{C}'_i}(X) = \overline{\mathcal{C}_i}(X) \bigcap U'$*.*

(2) $\underline{\mathcal{C}_i}(X) = \underline{\mathcal{C}'_i}(X) \bigcap \underline{\mathcal{C}_i}(U')$*.*

(3) $B'_i(X) \subset B_i(X) \bigcap U'$*.*

(4) If U' *is definable in* $(U;\mathcal{C})$*, then* X *is definable in* $(U;\mathcal{C})$ *iff* X *is definable in* $(U';\mathcal{C}')$*.*

Acknowledgments. The author would like to thank the referee for his/her valuable amendments and corrections.

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Received March, 18, 2009

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