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Postoptimal analysis of one lexicographic combinatorial problem with non-linear criteria

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Abstract

In this article we consider a multicriteria combinatorial problem with ordered MINMIN criteria. We obtain necessary and sufficient conditions of that type of stability to the initial data perturbations for which all lexicographic optima of the original problem are preserved and occurrence of the new ones is allowed.

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Vector (multicriteria) discrete optimization problems may arise as a result of formalization of object-oriented behavior of a human being in various fields of human activity such as e.g. technical system design, planning and management, business administration, environmental analysis and etc. As far as accuracy of input data is not-guaranteed, frequently, even in the well formalized problems, the reliability of the results (solutions) may be questionable. The data inaccuracy may happen due to various factors, among them the most typical ones are measurement and calculation errors, mathematical model inadequacy and many other. Therefore, it seems to be very natural to define classes of optimization problems for which small perturbations of input data are not significant. This research continues the series of works devoted to the above-mentioned topic [1–5]. We study different aspects of stability to the initial data perturbations for the lexicographic combinatorial problem with MINMIN criteria. In the paper, we formulate and prove necessary and sufficient conditions of quasi-stability of the problem.

This type of stability characterizes the case where all optimal solutions remain optimal under small changes of input data.

Let us consider n-criteria trajectory problem, i.e. problem is given on a system T of non-empty subsets (trajectories) of the set $N_m = \{1, 2, \dots, m\}$ with sub-criteria of the MINMIN form

$$f_i(t, A) = \min_{j \in t} a_{ij} \to \min_{t \in T}, \ i \in N_n,$$

where $A = [a_{ij}] \in \mathbf{R}^{n \times m}, \ n \geq 1, \ m \geq 2, \ |T| > 1$.

Under n-criterial trajectory problem $Z^n(A)$ we understand the problem of finding the lexicographic set (the set of lexicographic optimal trajectories):

$$L^n(A) = \{t \in T : \ \forall t' \in T \ \ (t \underset{A}{\not\succ} t')\},$$

where $\underset{A}{\not\succ}$ as usual is a negation of the binary lexicographic relation $\underset{A}{\succ}$ defined on the set of trajectories $T \subseteq 2^{N_m}$ by the formula:

$$t \underset{A}{\succ} t' \ \Leftrightarrow \ \exists p \in N_n \ (f_p(t, A) > f_p(t', A) \ \& \ p =$$

$$= \min\{k \in N_n : f_k(t, A) \neq f_k(t', A)\}).$$

It is easy to see that the set $L^n(A)$ is non-empty for any matrix $A \in \mathbf{R}^{n \times m}$ as the subset of the Pareto set.

Note, that many classical combinatorial extreme problems on graphs (traveling salesman problem, spanning tree problem, matching problem, etc.), various problems of scheduling theory and boolean programming problems [6–8] are included into the scheme of the scalar (singlecriterion) problems (with linear, bottleneck, $\sum$-MINMAX, and $\sum$-MINMIN criteria).

By definition, put $\overline{L^n}(A) = T \setminus L^n(A)$.

The following properties are obvious.

Corollary 1. *If* $t \underset{A}{\succ} t'$, *then* $t \in \overline{L^n}(A)$.

Corollary 2. *If* $t \underset{A}{\succ} t'$, *then* $t' \underset{A}{\not\succ} t$.

It is also known (see, e.g., [9]) that the lexicographic set $L^n(A)$ may be defined as a result of solving the sequence of n scalar problems

$$L_i^n(A) = \operatorname{Arg\,min}\{f_i(t,A) : \ t \in L_{i-1}^n(A)\}, \qquad i \in N_n, \tag{1}$$

where $L_0^n(A) = T$, $\operatorname{Arg\,min}\{\cdot\}$ is the set of all optimal trajectories for corresponding minimization problem. Hence, the following inclusions

$$T \supseteq L_1^n(A) \supseteq L_2^n(A) \supseteq \ldots \supseteq L_n^n(A) = L^n(A) \tag{2}$$

are true.

Following [1–5], the problem $Z^n(A)$ is quasi-stable if the formula

$$\exists \varepsilon > 0 \ \ \forall A' \in \Omega(\varepsilon) \ \ (L^n(A) \subseteq L^n(A + A'))$$

is valid. Here

$$\Omega(\varepsilon) = \{A' \in \mathbf{R}^{n \times m} \quad : \quad ||A'|| < \varepsilon\}$$

is a set of perturbing matrices

$$||A'|| = \max\{|a'_{ij}| : \ (i,j) \in N_n \times N_m\}, \ A' = [a'_{ij}].$$

Thus, quasi-stability characterizes the case when all trajectories from lexicographic set preserve a property of optimality for sufficiently small initial data perturbations. Therefore, quasi-stability may be interpreted as the discrete analogue of Hausdorff lower semicontinuity [10] at a point A of the many-valued optimal mapping

$$L^n : \ \mathbf{R}^{n \times m} \to 2^T.$$

We define binary relations for any non-empty set $I \subseteq N_n$ on the set of trajectories T for the problem $Z^n(A)$

$$t \underset{I,A}{\geq} t' \ \Leftrightarrow \ \forall i \in I \ (f_i(t,A) \geq f_i(t,A)),$$

$$t \underset{I,A}{>} t' \ \Leftrightarrow \ \forall i \in I \ (f_i(t,A) > f_i(t,A)),$$

$$t \underset{I,A}{\vdash} t' \;\Leftrightarrow\; \forall i \in I \; (N_i(t,A) \supseteq N_i(t',A)),$$

where $N_i(t,A) = \mathrm{Argmin}\{a_{ij} : j \in t\}$, i. e. $N_i(t,A) = \{j \in t : a_{ij} = f_i(t,A)\}$.

The following properties are obvious.

Corollary 3. *If* $t \underset{I,A}{\vdash} t'$, *then there exists a number* $\varepsilon > 0$ *such that for any perturbing matrix* $A' \in \Omega(\varepsilon)$ *the relation*

$$t' \underset{I,A+A'}{\geq} t$$

holds.

Corollary 4. *If* $t \underset{N_n,A}{\geq} t'$, *then* $t' \underset{A}{\overline{\succ}} t$.

Consequently applying the properties 3, 4 and using continuity of the functions $f_i(t,A), i \in N_n$ on the set of parameters $\mathbf{R}^m$, we deduce the following properties.

Corollary 5. *If* $t \underset{N_n,A}{\vdash} t'$,*then*

$$\exists \varepsilon > 0 \;\; \forall A' \in \Omega(\varepsilon) \; (t \underset{A+A'}{\overline{\succ}} t').$$

Corollary 6. *If any of the following conclusions:*

$$(i) \; t \underset{1,A}{>} t',$$

$$(ii) \; \exists k \in N_{n-1} \; (t' \underset{N_k,A}{\vdash} t \;\&\; t \underset{k+1,A}{>} t'),$$

holds for trajectories t *and* t', *then the formula*

$$\exists \varepsilon > 0 \;\; \forall A' \in \Omega(\varepsilon) \; (t \underset{A+A'}{\succ} t')$$

is true.

Denote

$$U^n(A) = \{t \in L^n(A) : \forall i \in N_n \;\; \forall t' \in L_i^n(A) \; (t \underset{i,A}{\vdash} t')\}.$$

Next property follows directly from the previous definition.

Corollary 7. *If $t \in U^n(A)$ and $t' \in L^n(A)$, then $t \underset{N_n,A}{\vdash} t'$.*

In order to prove the quasi-stability criteria we need a series of lemmas.

Lemma 1. *If $t \in U^n(A)$ and $t' \in T$, then*

$$\exists \varepsilon > 0 \quad \forall A' \in \Omega(\varepsilon) \quad (t \underset{A+A'}{\succeq} t') \tag{3}$$

Proof. Let $t \in U^n(A)$. We consider two possible cases for trajectory t'.

Case 1: $t' \in L_1^n(A)$. Suppose that $t' \in L^n(A)$. Then by virtue of the property 7 the relation

$$t \underset{N_n,A}{\vdash} t'$$

holds. Hence, taking into account the property 5, we get (3).

Now let $t' \in L_1^n(A) \setminus L^n(A)$. Thus, there exists an index $k = k(t') \in N_n \setminus \{1\}$, such that $t' \notin L_k^n(A)$ and $t' \in L_i^n(A)$ for $i \in N_{k-1}$. Therefore, we obtain

$$t \underset{N_{k-1},A}{\vdash} t' \quad \text{and} \quad t' \underset{k,A}{>} t.$$

Making use of this facts and property 6, we conclude that the formula

$$\exists \varepsilon > 0 \quad \forall A' \in \Omega(\varepsilon) \quad (t' \underset{A+A'}{\succ} t)$$

is true. Therefore, due to the property 2, we obtain (3).

Case 2: $t' \in T \setminus L_1^n(A)$. Thus,

$$t' \underset{1,A}{>} t.$$

Therefore, in view of the properties 2 and 6 the formula (3) is true.

Lemma 1 is thus proved.

Lemma 2. *If $t \in L^n(A) \setminus U^n(A)$, then the formula*

$$\exists t^0 \in T \quad \forall \varepsilon > 0 \quad \exists A^0 \in \Omega(\varepsilon) \ (t \underset{A+A^0}{\succ} t^0) \tag{4}$$

is true.

Proof. Since $t \notin U^n(A)$, then there exist $k \in N_n$ and $t^0 \in L_k^n(A)$ such that $N_k(t, A) \not\supseteq N_k(t^0, A)$ and $t \in L_k^n(A)$ (by virtue of $t \in L^n(A)$). Hence $f_k(t, A) = f_k(t^0, A) = a_{kp}$, if $p \in N_k(t^0, A) \setminus N_k(t, A)$. Therefore, let us assume $\varepsilon > 0$ and construct elements of a perturbing matrix $A^0 = [a_{ij}^0] \in \mathbf{R}^{n\times m}$ according to the rule

$$a_{ij}^0 = \begin{cases} -\alpha, & \text{if } i = k,\ j = p, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < \alpha < \varepsilon$, in view of $p \in N_k(t^0, A) \setminus N_k(t, A)$ we conclude that the relations

$$f_k(t^0, A + A^0) = \min\{a_{kj} + a_{kj}^0 : j \in t^0\} = a_{kp} - \alpha < a_{kp} =$$

$$= f_k(t, A) = f_k(t, A + A^0),$$

$$f_i(t^0, A + A_0) = f_i(t^0, A) = f_i(t, A) = f_i(t, A + A_0),\ i \in N_{k-1}$$

hold true. Hence,

$$t \underset{A+A^0}{\succ} t^0,$$

i.e. formula (4) is true.

Lemma 2 is thus proved.

Now let us formulate quasi-stability criterion for the concerned problem.

Theorem. *The vector problem $Z^n(A)$, $n \geq 1$, is quasi-stable if and only if the formula*

$$\forall t \in L^n(A) \ \ \forall i \in N_n \ \ \forall t' \in L_i^n(A) \ \ (t \underset{i,A}{\vdash} t') \tag{5}$$

is true.

Proof. Sufficiency. Let the formula (5) holds true and $t \in L^n(A)$. Then $t \in U^n(A)$ and, therefore, due to Lemma 1 we find that

$$\forall t' \in T \ \ \exists \varepsilon(t') > 0 \ \ \forall A' \in \Omega(\varepsilon(t')) \ (t \underset{A+A'}{\succeq} t').$$

Hence, by putting $\varepsilon(t) = \min\{\varepsilon(t') : t' \in T\}$, it is easy to see that for any trajectory $t \in L^n(A)$ and for any perturbing matrix $A' \in \Omega(\varepsilon(t))$

the inclusion $t \in L^n(A + A')$ is true. Therefore, if $\varepsilon^* = \min\{\varepsilon(t) : t \in L^n(A)\}$, we obtain

$$\exists \varepsilon^* > 0 \ \ \forall A' \in \Omega(\varepsilon^*) \ \ (L^n(A) \subseteq L^n(A + A')).$$

Thus, the problem $Z^n(A)$ is quasi-stable.

Necessity. We assume that, on the contrary, the problem $Z^n(A)$ is quasi-stable, but the formula (5) is not true. Then there exists trajectory $t \in L^n(A) \setminus U^n(A)$, for which on account of Lemma 2 and property 1 the formula

$$\forall \varepsilon > 0 \ \ \exists A^0 \in \Omega(\varepsilon) \ (t \in \overline{L^n}(A + A^0))$$

is true. Hence, we conclude

$$\forall \varepsilon > 0 \ \ \exists A^0 \in \Omega(\varepsilon) \ (L^n(A) \not\subseteq L^n(A + A^0)),$$

This is contradiction to the quasi-stability of the problem $Z^n(A)$.

Theorem is proved.

Let us give two examples which illustrate stated result.

Example 1. Let $n = 2$, $m = 4$, $T = \{t^1, t^2, t^3\}$, $t^1 = \{1, 2, 4\}$, $t^2 = \{1, 4\}$, $t^3 = \{1, 2\}$,

$$A = \begin{pmatrix} 1 & 3 & 1 & 2 \\ 3 & 2 & 2 & 1 \end{pmatrix}.$$

Thus,

$$f_1(t^1, A) = f_1(t^2, A) = f_1(t^3, A) = 1.$$

Therefore $L_1^2(A) = \{t^1, t^2, t^3\} = T$. Moreover, we have

$$f_2(t^1, A) = f_2(t^2, A) = 1, \quad f_2(t^3, A) = 2.$$

Hence, we get lexicographic set $L^2(A) = L_2^2(A) = \{t^1, t^2\}$.

Further we find the sets

$$N_1(t^1, A) = N_1(t^2, A) = N_1(t^3, A) = \{1\},$$

$$N_2(t^1, A) = N_2(t^2, A) = \{4\}.$$

Therefore, the formula

$$\forall t \in L^2(A) \;\; \forall i \in N_2 \;\; \forall t' \in L_i^2(A) \;\; (N_i(t,A) = N_i(t',A))$$

is true. Consequently, in virtue of the theorem the problem $Z^2(A)$ is quasi-stable.

Example 2. Let $n = 2$, $m = 4$, $T = \{t^1, t^2, t^3\}$, $t^1 = \{1,2,4\}$, $t^2 = \{1,2\}$, $t^3 = \{1,3,4\}$,

$$A = \begin{pmatrix} 1 & 3 & 1 & 2 \\ 3 & 2 & 2 & 1 \end{pmatrix}.$$

Then

$$f_1(t^1, A) = f_1(t^2, A) = f_1(t^3, A) = 1.$$

Thus, $L_1^2(A) = \{t^1, t^2, t^3\} = T$. Then, we have

$$f_2(t^1, A) = f_2(t^3, A) = 1, \quad f_2(t^2, A) = 2.$$

Hence we get the lexicographic set $L^2(A) = L_2^2(A) = \{t^1, t^3\}$.

Having found the sets

$$\{1\} = N_1(t^1, A) = N_1(t^2, A) \not\supseteq N_1(t^3, A) = \{1,3\},$$

$$N_2(t^1, A) = N_2(t^3, A) = \{4\},$$

we conclude that conditions of the theorem don't hold. Therefore, the problem $Z^2(A)$ isn't quasi-stable.

Corollary 1. *A sufficient condition for the problem $Z^n(A)$ to be quasi-stable is equality $|L_1^n(A)| = 1$.*

Let us give an example illustrating that the equality $|L_1^n(A)| = 1$ isn't necessary condition for the problem to be quasi-stable.

Example 3. Let $n = 2$, $m = 3$, $T = \{t^1, t^2\}$, $t^1 = \{1,2,3\}$, $t^2 = \{1,2\}$,

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 3 & 5 & 5 \end{pmatrix}.$$

Then

$$f_1(t^1, A) = f_1(t^2, A) = 2.$$

Therefore $L_1^2(A) = \{t^1, t^2\} = T$. Moreover

$$f_2(t^1, A) = f_2(t^2, A) = 3.$$

Hence, the lexicographic set is $L^2(A) = L_2^2(A) = \{t^1, t^2\}$. Thus, $|L_1^2(A)| = 2$.

Further, having found the sets

$$N_1(t^1, A) = N_1(t^2, A) = \{2\},$$

$$N_2(t^1, A) = N_2(t^2, A) = \{1\},$$

we conclude that the formula

$$\forall t \in L^2(A) \;\; \forall i \in N_2 \;\; \forall t' \in L_i^2(A) \;\; (N_i(t, A) = N_i(t', A))$$

is true. Therefore, by theorem, the problem $Z^2(A)$ is quasi-stable but $|L_1^2(A)| > 1$.

Corollary 2. *The formula*

$$\forall t, t' \in L^n(A) \;\; (N_1(t, A) = N_1(t', A)) \tag{6}$$

is necessary condition for the problem $Z^n(A), n \geq 1$ to be quasi-stable.

It is obvious that the formula (6) is simultaneously a sufficient condition for quasi-stability of the problem $Z^1(C)$ in scalar case ($n = 1$).

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