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About one algorithm of C^2 interpolation using quartic splines

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Abstract

The problem of C^2 interpolation of a discrete set of data on the interval [a,b] representing the function f using quartic splines is investigated. An explicit scheme of interpolation is obtained using different quartic splines on even and odd subintervals of interpolation.

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1 Introduction

Let us suppose that the mesh $\Delta : a = x_0 < x_1 < ... < x_n = b$ is given and $f_i = f(x_i), i = 0(1)n$ are the corresponding data points. The problem of the construction of an interpolation function $S \in C^2[a, b]$ is considered. It is well known (e.g. [1]) that cubic splines may be used in order to solve this problem. In this case you have to solve a tri-diagonal system of linear algebraic equations, which is diagonally dominant one. Well, but in the case of very large set of data you might have problems with capacity of your computer in order to solve this problem. In the case when additional data are available and you have to solve the problem again, it may become critical. In the case of two-dimensional interpolation it is much more difficult to overcome these problems. In what follows quartic splines are considered in order to solve this problem.

2 Algorithm of interpolation using quartic splines

In what follows the next notations are used: $m_i = S'(x_i)$, $M_i = S''(x_i)$, $h_i = x_{i+1} - x_i$, $t = (x - x_i)/h_i$, $\delta_i^{(1)} = (f_{i+1} - f_i)/h_i$.

The following three cases are considered.

a) Let us introduce splines as follows

$$\begin{aligned} S(x) &= f_i + (f_{i+1} - f_i)t + h_i^2 M_i(t^4 - 3t^3 + 3t^2 - t)/6 - \\ &\quad -h_i^2 M_{i+1}(t^4 - 3t^3 + 2t)/6 \end{aligned} \tag{1}$$

For derivatives we have

$$\begin{aligned} S'(x) &= \delta_i^{(1)} + h_i M_i(4t^3 - 9t^2 + 6t - 1)/6 - \\ &\quad -h_i M_{i+1}(4t^3 - 9t^2 + 2)/6 \end{aligned} \tag{2}$$

and

$$S"(x) = M_i(2t^2 - 3t + 1) - M_{i+1}(2t^2 - 3t) \tag{3}$$

From (1) it follows immediately that interpolation conditions are fulfilled. From (3) it follows that the second derivative is the continuous one at the knots of the mesh.

From (2) for the first derivative at the knots of the mesh we obtain

$$S'(x_i+) = \delta_i^{(1)} - h_i M_i/6 - h_i M_{i+1}/3 \tag{4}$$

and

$$S'(x_i-) = \delta_{i-1}^{(1)} + h_{i-1} M_i/2 \tag{5}$$

From the requirement of continuity of the first derivative at the knots of the mesh the following system of equations is obtained:

$$(3h_{i-1} + h_i)M_i/6 + h_i M_{i+1}/3 = \delta_i^{(2)}, i = 1(1)n-1,$$

where $\delta_i^{(2)} = \delta_i^{(1)} - \delta_{i-1}^{(1)}$, $i = 1(1)n-1$.

As it can be seen, the system presented above is the undetermined one. In this case end conditions are required. But, it should be mentioned, for example, that if we have end conditions $M_0 = f"(a)$ and $M_n = f"(b)$, in the sistem given above the value of M_0 is not present.

If the reprezentation of the spline via the first derivatives is used we have

$$\begin{aligned} S(x) &= f_i + (f_{i+1} - f_i)(2t^4 - 6t^3 + 5t^2) + \\ &\quad + h_i m_i(-t^4 + 3t^3 - 3t^2 + t) + h_i m_{i+1}(-t^4 + 3t^3 - 2t^2), \end{aligned} \tag{6}$$

$$\begin{aligned} S'(x) &= \delta_i^{(1)}(8t^3 - 18t^2 + 10t) + m_i(-4t^3 + 9t^2 - 6t + 1) + \\ &\quad + m_{i+1}(-4t^3 + 9t^2 - 4t), \end{aligned} \tag{7}$$

$$\begin{aligned} S"(x) &= \frac{\delta_i^1}{h_i}(24t^2 - 36t + 10) + \frac{m_i}{h_i}(-12t^2 + 18t - 6) + \\ &\quad + \frac{m_{i+1}}{h_i}(-12t^2 + 18t - 4). \end{aligned} \tag{8}$$

From (6) it follows that interpolation conditions are fullfilled and from (7) it follows that the first derivative is the continuous one at the knots of the mesh.

From (8) it follows

$$S"(x_i+) = 10\frac{\delta_i^{(1)}}{h_i} - 6\frac{m_i}{h_i} - 4\frac{m_{i+1}}{h_i}$$

and

$$S"(x_i-) = -2\frac{\delta_{i-1}^{(1)}}{h_{i-1}} + 2\frac{m_i}{h_{i-1}}.$$

From the requirement of continuity of the second derivative the following system of equations is obtained:

$$(h_i + 3h_{i-1})m_i + 2h_{i-1}m_{i+1} = 5h_{i-1}\delta_i^{(1)} + h_i\delta_{i-1}^{(1)}, i = \overline{1, n-1},$$

which is the undetermined one and end conditions are required.

b) Let's consider now the splines in the following form:

$$\begin{aligned} S(x) &= f_i + (f_{i+1} - f_i)t + h_i^2 M_i(-t^4 + t^3 + 3t^2 - 3t)/6 + \\ &+ h_i^2 M_{i+1}(t^4 - t^3)/6. \end{aligned} \tag{9}$$

For derivatives we have

$$S'(x) = \delta_i^{(1)} + h_i M_i(-4t^3 + 3t^2 + 6t - 3)/6 + h_i M_{i+1}(4t^3 - 3t^2)/6$$

$$S"(x) = M_i(-2t^2 + t + 1) + M_{i+1}(2t^2 - t).$$

As in the previous case the interpolation conditions are hold and the second derivative is continuous at the knots of the mesh.

In this case at the knots of the mesh for the first derivative we have

$$S'(x_i+) = \delta_i^{(1)} - h_i M_i/2 \tag{10}$$

and

$$S'(x_i-) = \delta_{i-1}^{(1)} + h_{i-1}M_{i-1}/3 + h_{i-1}M_i/6. \tag{11}$$

From (10) and (11) the corresponding system of linear algebraic equations which ensure the continuity of the first derivative of the spline at the knots of the mesh is obtained:

$$h_{i-1}M_{i-1}/3 + (h_{i-1} + 3h_i)M_i/6 = \delta_i^{(2)}, i = 1(1)n-1.$$

As in the previous case, if the representation via the first derivatives of the spline is used, we have

$$\begin{aligned} S(x) &= f_i + (f_{i+1} - f_i)(-2t^4 + 2t^3 + t^2) + \\ &+ h_i m_i(t^4 - t^3 - t^2 + t) + h_i m_{i+1}(t^4 - t^3), \end{aligned} \tag{12}$$

$$\begin{aligned} S'(x) &= \delta_i^{(1)}(-8t^3 + 6t^2 + 2t) + m_i(4t^3 - 3t^2 - 2t + 1) + \\ &+ m_{i+1}(4t^3 - 3t^2), \end{aligned} \tag{13}$$

$$\begin{aligned} S"(x) &= \frac{\delta_i^{(1)}}{h_i}(24t^2+12t+2)+\frac{m_i}{h_i}(12t^2-6t-2)+ \\ &+\frac{m_{i+1}}{h_i}(12t^2-6t). \end{aligned} \tag{14}$$

At the knots of the mesh in this case

$$S"(x_i+) = 2\frac{\delta_i^{(1)}}{h_i} - 2\frac{m_i}{h_i},$$

$$S"(x_i-) = -10\frac{\delta_{i-1}^{(1)}}{h_{i-1}} + 4\frac{m_{i-1}}{h_{i-1}} + 6\frac{m_i}{h_{i-1}}.$$

So, the next system of equations results in

$$2h_i m_{i-1} + (3h_i + h_{i-1})m_i = 5h_i\delta_{i-1} + h_{i-1}\delta_i, i = \overline{1, n-1.}$$

c) Let us consider now a scheme of interpolation, when splines (1) and (6) are used alternatively, namely on odd subintervals the splines (1) are used and on even subintervals the splines (6), respectively. As a result, at the odd knots of the mesh from (5) and (7) the following condition of continuity of the first derivative is obtained:

$$M_i = 2\delta_i^{(2)}/(h_{i-1}+h_i) \tag{15}$$

and for even knots of the mesh we get

$$2h_{i-1}M_{i-1} + (h_{i-1}+h_i)M_i + 2h_i M_{i+1} = 6\delta_i^{(2)}. \tag{16}$$

Substituting in (16) expressions which follow from (14) for M_{i-1} and M_{i+1} we get the next formulae for the second derivative at the even knots of the mesh

$$\begin{aligned} M_i &= 6(\delta_i^{(2)} - 2h_{i-1}\delta_{i-1}^{(2)}/(3(h_{i-2}+h_{i-1})) - \\ &-2h_i\delta_{i+1}^{(2)}/(3(h_i+h_{i+1})))/(h_{i-1}+h_i). \end{aligned} \tag{17}$$

So, we obtain an explicit scheme of interpolation.

If the representation via the first derivatives is used we have

$$S\text{"}(x_i-) = -2\frac{\delta^{(1)}_{i-1}}{h_{i-1}} + 2\frac{m_i}{h_i}, \tag{18}$$

$$S\text{"}(x_i+) = 2\frac{\delta^{(1)}_{i}}{h_{i}} - 2\frac{m_i}{h_i}. \tag{19}$$

From requirement of continuity of the second derivative of the spline at the knots of the mesh it follows:

$$m_i = \frac{h_{i-1}\delta^{(1)}_i}{h_{i-1}+h_i} + \frac{h_i\delta^{(1)}_{i-1}}{h_{i-1}+h_i}. \tag{20}$$

Let's consider the knots $i+1$. In this case

$$S\text{"}(x_{i+1}-) = -10\frac{\delta^{(1)}_{i}}{h_{i}} + 4\frac{m_i}{h_i} + 6\frac{m_{i+1}}{h_i}, \tag{21}$$

$$S\text{"}(x_{i+1}+) = 10\frac{\delta^{(1)}_{i+1}}{h_{i+1}} - 6\frac{m_{i+1}}{h_{i+1}} - 4\frac{m_{i+1}}{h_{i+1}}. \tag{22}$$

Then we have the following:

$$2h_{i+1}m_i + 3(h_i+h_{i+1})m_{i+1} + 2h_im_{i+2} = 5h_i\delta^{(1)}_{i+1} + 5h_{i+1}\delta^{(1)}_i. \tag{23}$$

Substituting formulae for m_i and m_{i+2} which are obtained from (20) in (23) we get

$$m_{i+1} = \frac{1}{3(h_i+h_{i+1})}\left[-\frac{2h_ih_{i+1}}{h_{i+1}+h_{i+2}}\delta^{(1)}_{i+2} + (5h_i - \frac{2h_ih_{i+2}}{h_{i+1}+h_{i+2}})\delta^{(1)}_{i+1} + \right.$$
$$\left. +(5h_{i+1} - \frac{2h_{i-1}h_{i+1}}{h_{i-1}+h_i})\delta^{(1)}_i - \frac{2h_{i-1}h_{i+1}}{h_{i-1}+h_i}\delta^{(1)}_{i-1}\right] \tag{24}$$

and an explicit scheme of interpolation is obtained when representation of spline via the first derivative is used.

3 Remarks on errors of approximation.

We'll consider the case of uniform mesh with step h. Then the formula (15) has the form

$$M_i = (f_{i-1} - 2f_i + f_{i+1})/h^2 \tag{25}$$

and the formula (17), respectively,

$$M_i = (-f_{i-2} + 5f_{i-1} - 8f_i + 5f_{i+1} - f_{i+2})/h^2. \tag{26}$$

If the reprezentation via the first derivatives of the spline at the knots of the mesh is used we have

$$m_i = \frac{f_{i+1} - f_i}{2h}, \tag{27}$$

which follows from (20) and

$$m_{i+1} = \frac{-f_{i+3} + 5f_{i+2} - 5f_i + f_{i-1}}{6h}, \tag{28}$$

which follows from (24).

The set of functions f, which have absolutely continuous derivatives of order $r-1$ on the interval $[a,b]$ and which have derivatives of order r from $L_\infty[a,b]$, is denoted by $W^r_\infty[a,b]$. The norm in this case is defined as follows:

$$\|f(x)\|_\infty = ess\sup|f(x)|\,, x \in [a,b].$$

Lemma 1 *Let us suppose that $f(x) \in W^3_\infty[a,b]$. Then the following estimates are valid for regular mesh:*

$$|m_i - f'_i| \le \frac{h^2}{6}\left\|f^{(3)}(x)\right\|_\infty \tag{29}$$

at the odd knots and

$$|m_i - f'_i| \le \frac{13h^2}{18}\left\|f^{(3)}(x)\right\|_\infty \tag{30}$$

for the even knots.

Proof. Let's consider the case (29).

We have

$$|m_i - f_i'| = \left| \frac{f_{i+1} - f_{i-1}}{2h} - f_i' \right|.$$

Substituting f_{i+1} and f_{i-1} by the corresponding Taylor series expansions at the point x_i with the remainder term in the integral form, after necessary transformations we get

$$|m_i - f_i'| = \frac{1}{4h} \left| \int\limits_{x_i}^{x_{i+1}} (x_{i+1} - v)^2 f^{(3)}(v)dv - \int\limits_{x_i}^{x_{i-1}} (x_{i-1} - v)^2 f^{(3)}(v)dv \right|.$$

Using the Hölder inequality in the last relation and computing integrals the presented above estimation follows immediately.

Let's consider now the case (30). We have

$$|m_i - f_i'| = \left| \frac{-f_{i+2} + 5f_{i+1} - 5f_{i-1} + f_{i-2}}{6h} - f_i' \right|.$$

Using the corresponding Taylor series expansions for f_{i-2}, f_{i-1}, f_{i+1}, f_{i+2} with the remainder term in the integral form we get

$$|m_i - f_i'| = \left| \frac{1}{12h} \left(- \int\limits_{x_i}^{x_{i+2}} (x_{i+2} - v)^2 f^{(3)}(v)dv + \right.\right.$$

$$+5 \int\limits_{x_i}^{x_{i+1}} (x_{i+1} - v)^2 f^{(3)}(v)dv - 5 \int\limits_{x_i}^{x_{i-1}} (x_{i-1} - v)^2 f^{(3)}(v)dv +$$

$$\left.\left. + \int\limits_{x_i}^{x_{i-2}} (x_{i-2} - v)^2 f^{(3)}(v)dv \right) \right|.$$

From the last relation using the Hölder inequality and computing integrals we get

$$|m_i - f_i'| \leq \frac{13h^2}{18} \left\| f^{(3)}(x) \right\|_\infty.$$

So, the lemma is proved.

Lemma 2 *Let us suppose that $f(x) \in W^3_\infty[a,b]$. Then the following estimates are valid for regular mesh:*

$$|M_i - f"_i| \leq \frac{h}{3} \left\| f^{(3)}(x) \right\|_\infty$$

at the odd knots and

$$|M_i - f"_i| \leq \frac{13h}{3} \left\| f^{(3)}(x) \right\|_\infty$$

at the even knots.

The proof of the lemma 2 is the analogous one as for lemma 1.
Let us introduce now Hermite splines

$$\begin{aligned} H(x) &= f_i + (f_{i+1} - f_i)(2t^4 - 6t^3 + 5t^2) + \\ &+ h_i f'_i(-t^4 + 3t^3 - 3t^2 + t) + h_i f'_{i+1}(-t^4 + 3t^3 - 2t^2) \end{aligned} \quad (31)$$

and

$$\begin{aligned} H(x) &= f_i + (f_{i+1} - f_i)(-2t^4 + 2t^3 + t^2) + \\ &+ h_i f'_i(t^4 - t^3 - t^2 + t) + h_i f'_{i+1}(t^4 - t^3). \end{aligned} \quad (32)$$

Lemma 3 *Let us suppose that $f(x) \in W^3_\infty[a,b]$. Then for regular mesh:*

$$\left\| H^{(k)}(x) - f^{(k)}(x) \right\|_\infty = O(h^{3-k}), k = 0, 1, 2.$$

Proof. Let's consider the remainder term

$$R(x)=H(x)-f(x).$$

For the case (31), substituting Taylor series expansions for f_i, f_{i+1}, f'_i, f'_{i+1} at the point $x = x_i + th$ with remainder term in the integral form after necessary transformations we obtain

$$R(x) = \int\limits_x^{x_i} \Big[(x_i - v)^2 (\frac{1}{2} - t^4 + 3t^3 - \frac{5t^2}{2}) +$$

$$+h(x_i - v)(-t^4 + 3t^3 - 3t^2 + t)\Big] f^{(3)}(v)dv+$$

$$+ \int\limits_{x}^{x_{i+1}} \Big[(x_{i+1} - v)^2(t^4 - 3t^3 + \frac{5t^2}{2})+$$

$$+h(x_{i+1} - v)(-t^4 + 3t^3 - 2t^2)\Big] f^{(3)}(v)dv.$$

Substituting in the previous relation $v - x_i = \tau h$ we get

$$R(x) = h^3 \int\limits_0^t \psi_1(t,\tau) f^{(3)}(x_i + \tau h)d\tau + h^3 \int\limits_t^1 \psi_2(t,\tau) f^{(3)}(x_i + \tau h)d\tau,$$

where

$$\psi_1(t,\tau) = \tau \left[-t^4 + 3t^3 - 3t^2 + t - \tau(\frac{1}{2} - t^4 + 3t^3 - \frac{5t^2}{2})\right]$$

and

$$\psi_2(t,\tau) = (1-\tau)\left[(1-\tau)(t^4 - 3t^3 + \frac{5t^2}{2}) - t^4 + 3t^3 - 2t^2\right].$$

From the above it follows that $R(x) = O(h^3)$.

Consider the case (32).

$$R(x) = \frac{h^3}{2} \int\limits_0^t \psi_1(t,\tau) f^{(3)}(x_i + \tau h)d\tau + \frac{h^3}{2} \int\limits_t^1 \psi_2(t,\tau) f^{(3)}(x_i + \tau h)d\tau,$$

where

$$\psi_1(t,\tau) = \tau(t^4 - t^3) - \tau^2(1 + 2t^4 - 2t^3 - t^2)$$

and

$$\psi_2(t,\tau) = (1-\tau)^2(-2t^4 + 2t^3 + t^2) - (1-\tau)(t^4 - t^3),$$

from where it follows that $R(x) = O(h^3)$.

Similarly, for derivatives corresponding estimates are obtained.

Now we are in position to state:

Theorem 1 *If $f(x) \in W_\infty^3[a,b]$ then for regular mesh*

$$\left\| S^{(k)}(x) - f^{(k)}(x) \right\|_\infty = O(h^{3-k}), k = 0, 1, 2.$$

The proof of the theorem follows from the identity

$$R(x) = S(x) - H(x) + H(x) - f(x),$$

and from Lemma 1 and Lemma 3.

4 Conclusions.

So, in the presented paper an explicit scheme of interpolation using quartic splines is obtained. The order of approximation by the proposed algorithm is the same as by the one for cubic splines. The presented algorithm can be extended for bidimensional case.

References

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