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Ramanujan-like formulas for $\frac{1}{\pi^2}$ á la Guillera and Zudilin and Calabi-Yau differential equations

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Abstract

Using the PSLQ-algorithm J.Guillera found some formulas for $\frac{1}{\pi^2}$. He proved three of them using WZ-pairs. Then W. Zudilin showed how to produce formulas for $\frac{1}{\pi^2}$ by squaring formulas for $\frac{1}{\pi}$. The success of this depends on facts related to Calabi-Yau differential equations of string theory. Here some examples of this is worked out. Also some formulas containing harmonic numbers are found by differentiating formulas for $\frac{1}{\pi^2}$.

1 Introduction

Ramanujan [10] found several formulas for $\frac{1}{\pi}$ of the following form

$$\sum_{n=0}^{\infty} a_n x_0^n(\alpha+\beta n)=\frac{1}{\pi}$$

where

$$v(x)=\sum_{n=0}^{\infty} a_n x^n$$

satisfies a third order differential equation with polynomial coefficients. J.Guillera [9] found eight (and proved three of them) formulas for $\frac{1}{\pi^2}$ of the form

$$\sum_{n=0}^{\infty} A_n x_0^n(c_0+c_1 n+c_2 n^2)=\frac{1}{\pi^2}$$

where

$$w(x) = \sum_{n=0}^{\infty} A_n x^n$$

satisfies a differential equation of order five. It is quite remarkable that

$$w(x) = x(y_0(x)y_1'(x) - y_0'(x)y_1(x))$$

where y_0, y_1 are solutions of a fourth order differential equation (the pullback) of Calabi-Yau type (see [1] for definitions). With the notation of [2] the w are $\widehat{3}, \widehat{6}, \widehat{7}, \widehat{8}, \widehat{11}, \widehat{12}, A * \beta = \#40, C * \vartheta$. Guillera used the PSLQ-algorithm to find and WZ-pairs to prove his formulas. Also this paper uses modern computer algebra to find the formulas.

In [14] Zudilin showed how to "square" a Ramanujan-like formula for $\frac{1}{\pi}$ to get a formula for $\frac{1}{\pi^2}$. The success of this depends on the fact that $v(x) = u(x)^2$ where $u(x)$ satisfies a second order differential equation. Hence $w(x) = u(x)^4$ which leads to that the Yukawa coupling of the pullback is trivial. This is proved in section 2. In section 1 we give some examples of Zudilin's square. Finally in section 3 we give some examples of formulas containing harmonic numbers obtained by differentiating the formulas for $\frac{1}{\pi^2}$.

2 The square of Ramanujan

In [14] Zudilin has given the recipe for how to obtain a formula for $\frac{1}{\pi^2}$ from a formula for $\frac{1}{\pi}$. The key fact is that for all known formulas

$$\sum_{n=0}^{\infty} B_n x_0^n (\alpha + \beta n) = \frac{1}{\pi}$$

then

$$v = \sum_{n=0}^{\infty} B_n x^n$$

satisfies a third order differential equation

$$v''' + s_2 v'' + s_1 v' + s_0 v = 0$$

which is the symmetric square of a second order differential equation

$$u'' + p_1 u' + p_0 u = 0.$$

This means that $v = u^2$ and in [1] it is shown that it is equivalent to

$$\frac{s_1 s_2}{27} + \frac{s_1'}{2} - \frac{s_2''}{6} - \frac{s_2 s_2'}{3} - s_0 = 0$$

and

$$p_0 = \frac{s_1}{4} - \frac{s_2^2}{18} - \frac{s_2'}{12} \ ,$$

$$p_1 = \frac{s_2}{3} \ .$$

In Zudilin [14] it is shown that squaring the formula for $\frac{1}{\pi}$ one obtains the following formula for $\frac{1}{\pi^2}$:

$$\sum_{n=0}^{\infty} A_n x_0^n (c_0 + c_1 n + c_2 n^2) = \frac{1}{\pi^2}$$

where

$$v^2 = \sum_{n=0}^{\infty} A_n x^n$$

and

$$c_0 = \alpha^2 + \frac{4}{3}\beta^2 x_0^2 p_0(x_0) = \alpha^2 + \frac{1}{27}\beta^2 x_0^2 (9 s_1(x_0) - 2 s_2(x_0)^2 - 3 s_2'(x_0)) \ ,$$

$$c_1 = \alpha\beta + \frac{1}{3}\beta^2 (x_0 p_1(x_0) - 1) = \alpha\beta + \frac{1}{9}\beta^2 (x_0 s_2(x_0) - 3) \ ,$$

$$c_2 = \frac{1}{3}\beta^2 \ .$$

The hypergeometric case.
Assume that

$$v = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (a)_n (1-a)_n}{n!^3} x^n.$$

Then $v = u^2$ where

$$u = \sum_{n=0}^{\infty} \frac{(\frac{a}{2})_n (\frac{1-a}{2})_n}{n!^2} x^n$$

by Clausen's identity and u satisfies

$$u'' + \frac{2-3x}{2x(1-x)} u' - \frac{a(1-a)}{4x(1-x)} u = 0$$

which gives

$$c_0 = \alpha^2 - \frac{1}{3}\beta^2 a(1-a) \frac{x_0}{1-x_0} \ ,$$

$$c_1 = \alpha\beta - \frac{1}{6}\beta^2 \frac{x_0}{1-x_0} \ ,$$

$$c_2 = \frac{1}{3}\beta^2 \ .$$

Case a=1/2.
Here

$$v = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{n!^3} x^n$$

and

$$A_n = \sum_{k=0}^{n} \frac{(\frac{1}{2})_k^3 (\frac{1}{2})_{n-k}^3}{k!^3 (n-k)!^3}$$

with

$$w = \sum_{n=0}^{\infty} A_n x^n$$

satisfying the 5-th order differential equation ($\theta = x\frac{d}{dx}$)

$$8\theta^5 - x(2\theta+1)(8\theta^4 + 16\theta^3 + 17\theta^2 + 9\theta + 2) + 8x^2(\theta+1)^5 \ .$$

Example (Ramanujan [10])

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{n!^3} (5 + 42n) \frac{1}{64^n} = \frac{64}{\pi}$$

Here $x_0 = \frac{1}{64}$, $\alpha = \frac{5}{16}$, $\beta = \frac{42}{16}$ which gives $c_0 = \frac{17}{192}$, $c_1 = \frac{77}{96}$, $c_2 = \frac{147}{64}$ and we find

$$\sum_{n=0}^{\infty} A_n \frac{17 + 154n + 441n^2}{192} \frac{1}{64^n} = \frac{1}{\pi^2} \ .$$

Case a=1/3.

Here

$$v = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{3})_n(\frac{2}{3})_n}{n!^3} x^n$$

and

$$A_n = 108^{-n} \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2n-2k}{n-k}^2 \binom{3k}{k} \binom{3n-3k}{n-k}$$

with

$$w = \sum_{n=0}^{\infty} A_n x^n$$

satisfying the 5-th order differential equation

$$324\theta^5 - 18x(2\theta + 1)(18\theta^4 + 36\theta^3 + 37\theta^2 + 19\theta + 4) +$$
$$+x^2(\theta + 1)(3\theta + 2)(3\theta + 4)(6\theta + 5)(6\theta + 7) \ .$$

Example (Chan-Liaw-Tan [7])

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{3})_n(\frac{2}{3})_n}{n!^3} \frac{(-1)^n}{500^{2n}} (827 + 14151n) = \frac{1500\sqrt{3}}{\pi}$$

Here $x_0 = -\frac{1}{500^2}$, $\alpha = \frac{827}{1500\sqrt{3}}$, $\beta = \frac{14151}{1500\sqrt{3}}$, which gives $c_0 = \frac{410393}{4050000}$, $c_1 = \frac{2600669}{1500000}$, $c_2 = \frac{22250089}{2250000}$ and we find

$$\sum_{n=0}^{\infty} A_n \frac{4103930 + 70218063n + 400501602n^2}{40500000} \frac{(-1)^n}{500^{2n}} = \frac{1}{\pi^2} \ .$$

Case a=1/4.

Here

$$v = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{n!^3} x^n$$

and

$$A_n = 256^{-n} \sum_{k=0}^{n} \frac{(4k)!}{k!^4} \frac{(4n-4k)!}{(n-k)!^4}$$

with

$$w := \sum_{n=0}^{\infty} A_n x^n$$

satisfying the differential equation

$$64\theta^5 - 2x(2\theta+1)(32\theta^4 + 64\theta^3 + 63\theta^2 + 31\theta + 6) +$$
$$+x^2(\theta+1)(2\theta+1)(2\theta+3)(4\theta+3)(4\theta+5) \ .$$

Example. (J.Borwein-P.Borwein [4])

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{n!^3} \frac{(-1)^n}{882^{2n}} (1123 + 21460n) = \frac{3528}{\pi}$$

Here $x_0 = -\frac{1}{882^2}$, $\alpha = \frac{1123}{3528}$, $\beta = \frac{21460}{3528}$ which gives $c_0 = \frac{630583}{6223392}$, $c_1 = \frac{18074759}{9335088}$, $c_2 = \frac{28783225}{2333772}$ and we find

$$\sum_{n=0}^{\infty} A_n \frac{1891749 + 36149518n + 230265800n^2}{18670176} \frac{(-1)^n}{882^{2n}} = \frac{1}{\pi^2} \ .$$

Example. (D.V.Chudnovsky-G.V.Chudnovsky [8])

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{n!^3} \frac{1}{7^{4n}} (3 + 40n) = \frac{49\sqrt{3}}{9\pi}$$

Here $x_0 = \frac{1}{7^4}$, $\alpha = \frac{27}{49\sqrt{3}}$, $\beta = \frac{360}{49\sqrt{3}}$, which gives $c_0 = \frac{1935}{19208}$, $c_1 = \frac{3237}{2401}$, $c_2 = \frac{14400}{2401}$ and we find

$$\sum_{n=0}^{\infty} A_n \frac{1935 + 25896n + 115200n^2}{19208} \frac{1}{7^{4n}} = \frac{1}{\pi^2} .$$

Case a=1/6.
Here

$$v = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{1}{6})_n (\frac{5}{6})_n}{n!^3} x^n$$

and

$$A_n = 1728^{-n} \binom{2n}{n} \binom{3n}{n} \sum_{n=0}^{\infty} 16^{-k} \binom{2k}{k}^3 \binom{2n-2k}{n-k}$$

with

$$w = \sum_{n=0}^{\infty} A_n x^n$$

satisfies

$$648\theta^5 - 9x(2\theta+1)(3\theta+1)(3\theta+2)(8\theta^2+8\theta+5) +$$
$$+8x^2(\theta+1)(3\theta+1)(3\theta+2)(3\theta+4)(3\theta+5)$$

(this is the Hadamard product $B * \vartheta$ which is deleted from the big table since its fourth order pullback has trivial Yukawa coupling)

Example (J.Borwein-P.Borwein [4])

$$\frac{1}{2E\sqrt{3}} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{1}{6})_n (\frac{5}{6})_n}{n!^3} (A + Bn) \frac{1}{E^{2n}} = \frac{1}{\pi}$$

where

$$A = 1657145277365 + 212175710912\sqrt{61} ,$$
$$B = 107578229802750 + 13773980892672\sqrt{61} ,$$
$$E = 4752926464 + 608549875\sqrt{61} .$$

We obtain

$$\sum_{n=0}^{\infty} A_n(c_0 + c_1 n + c_2 n^2)\frac{1}{E^{2n}} = \frac{1}{\pi^2}$$

where

$$c_0 = \frac{1}{1320^3 E^2 F^3}(111664689387605862532927043117398929778009833 4$$

$$+142971984278650150521031407984764718461880160\sqrt{61}) \ ,$$

$$c_1 = \frac{1}{1320^3 E^2 F^3}(724902625002743108064601030279445645635786815 03$$

$$+928142703605173341663106184983443065374812048 0\sqrt{61}) \ ,$$

$$c_2 = \frac{1}{1320^3 E^2 F^3}\times$$

$$\times(156863618098594521579736421566231685382598194990 3$$

$$+200843282363293945697228573609629579498429824000\sqrt{61})$$

where

$$F = 236674 + 30303\sqrt{61} \ .$$

Sporadic formulas

Example. (H.H.Chan-S.H.Chan-Z.G.Liu [6])

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k}(1+5n)\frac{(-1)^n}{64^n} = \frac{8}{\pi\sqrt{3}} \ .$$

Here (case (α))

$$v = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k}x^n$$

satisfies

$$v''' + \frac{3(128x^2-30x+1)}{x(64x^2-20x+1)}v'' + \frac{448x^2-68x+1}{x^2(64x^2-20x+1)}v' + \frac{4}{x^2(4x-1)}v = 0 \ .$$

We get

$$A_n = \sum_{k=0}^{n} \sum_i \sum_j \binom{k}{i}^2 \binom{2i}{i} \binom{2k-2i}{k-i} \binom{n-k}{j}^2 \binom{2j}{j} \binom{2n-2k-2j}{n-k-j}$$

and

$$w = \sum_{n=0}^{\infty} A_n x^n$$

satisfies

$$\theta^5 - 2x(2\theta+1)(10\theta^4+20\theta^3+25\theta^2+15\theta+4)$$
$$+2^2x^2(\theta+1)(132\theta^4+528\theta^3+947\theta^2+838\theta+312)$$
$$-2^7x^3(2\theta+3)(10\theta^4+60\theta^3+145\theta^2+165\theta+74)$$
$$+2^{12}x^4(\theta+2)^5 \ .$$

Then we have $x_0 = -\frac{1}{64}$, $\alpha = \frac{\sqrt{3}}{8}$, $\beta = \frac{5\sqrt{3}}{8}$ which gives $c_0 = -\frac{1}{72}$, $c_1 = \frac{5}{32}$, $c_2 = \frac{25}{64}$ and

$$\sum_{n=0}^{\infty} A_n \frac{-8+90n+225n^2}{576} \frac{(-1)^n}{64^n} = \frac{1}{\pi^2} \ .$$

Example.

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2n-2k}{n-k}^2 \frac{1+(3+2\sqrt{3})n}{4} \left(\frac{2-\sqrt{3}}{64}\right)^n = \frac{1}{\pi}$$

Here (case (β))

$$v = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2n-2k}{n-k}^2 x^n$$

satisfies

$$v''' + \frac{3(32x-1)}{x(16x-1)} v'' + \frac{1792x^2-112x+1}{x^2(16x-1)^2} v' + \frac{8(32x-1)}{x^2(16x-1)^2} v = 0 \ .$$

We get

$$A_n = \sum_{k=0}^{n} \sum_i \sum_j \binom{2i}{i}^2 \binom{2k-2i}{k-i}^2 \binom{2j}{j}^2 \binom{2n-2k-2j}{n-k-j}$$

where

$$w = \sum_{n=0}^{\infty} A_n x^n$$

satisfies

$$\begin{gathered}\theta^5 - 2^3 x(2\theta+1)(4\theta^4+8\theta^3+11\theta^2+7\theta+2)\\ +2^9x^2(\theta+1)(3\theta^4+12\theta^3+23\theta^2+22\theta+9)\\ -2^{11}x^3(2\theta+3)(4\theta^4+24\theta^3+59\theta^2+69\theta+32)\\ +2^{16}x^4(\theta+2)^5\ .\end{gathered}$$

Then we have $x_0 = \frac{2-\sqrt{3}}{64}$, $\alpha = \frac{1}{4}$, $\beta = \frac{3+2\sqrt{3}}{4}$ which gives $c_0 = 0$, $c_1 = \frac{1+\sqrt{3}}{8}$, $c_2 = \frac{7+4\sqrt{3}}{16}$ and we get

$$\sum_{n=0}^{\infty} A_n \frac{(2+2\sqrt{3})n + (7+4\sqrt{3})n^2}{16} \left(\frac{2-\sqrt{3}}{64}\right)^n = \frac{1}{\pi^2}\ .$$

Example (T.Sato [12])

$$\sum_{n=0}^{\infty} \sum_k \binom{n}{k}^2 \binom{n+k}{n}^2 (10-3\sqrt{5}+20n) \left(\frac{\sqrt{5}-1}{2}\right)^{12n} = \frac{20\sqrt{3}+9\sqrt{15}}{6\pi}$$

Here (case (γ))

$$v = \sum_{n=0}^{\infty} \sum_k \binom{n}{k}^2 \binom{n+k}{n}^2 x^n$$

satisfies

$$v''' + \frac{3(2x^2-51x+1)}{x(x^2-34x+1)} v'' + \frac{7x^2-112x+1}{x^2(x^2-34x+1)} v' + \frac{x-5}{x^2(x^2-34x+1)} v = 0\,.$$

We get

$$A_n = \sum_{k=0}^{n} \sum_i \sum_j \binom{k}{i}^2 \binom{k+i}{k}^2 \binom{n-k}{j}^2 \binom{n-k+j}{j}^2$$

and

$$w = \sum_{n=0}^{\infty} A_n x^n$$

satisfies

$$\theta^5 - 2x(2\theta+1)(17\theta^4 + 34\theta^3 + 38\theta^2 + 21\theta + 5)$$

$$+2x^2(\theta+1)(579\theta^4 + 2316\theta^3 + 3604\theta^2 + 2576\theta + 714)$$

$$-2x^3(2\theta+3)(17\theta^4 + 102\theta^3 + 242\theta^2 + 267\theta + 115)$$

$$+x^4(\theta+2)^5 \ .$$

We have $x_0 = \left(\frac{\sqrt{5}-1}{2}\right)^{12}$, $\alpha = \frac{6(10-3\sqrt{5})}{20\sqrt{3}+9\sqrt{15}}$, $\beta = \frac{120}{20\sqrt{3}+9\sqrt{15}}$ which gives

$$c_0 = \frac{1473122}{9} - 73200\sqrt{5} \ ,$$

$$c_1 = 183680 - 82144\sqrt{5} \ ,$$

$$c_2 = 51520 - 23040\sqrt{5}$$

and

$$\sum_{n=0}^{\infty} A_n (c_0 + c_1 n + c_2 n^2) \left(\frac{\sqrt{5}-1}{2}\right)^{12n} = \frac{1}{\pi^2}.$$

Example (H.H.Chan-H.Verrill [5])

$$\sum_{n=0}^{\infty} \sum_k (-1)^{n+k} 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3} (1+4n) \frac{1}{81^n} = \frac{3\sqrt{3}}{2\pi}$$

Here (case (δ))

$$v = \sum_{n=0}^{\infty} \sum_{k} (-1)^{n+k} 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3} x^n$$

satisfies

$$v''' + \frac{3(162x^2+21x+1)}{x(81x^2+14x+1)} v'' + \frac{567x^2+48x+1}{x^2(81x^2+14x+1)} v' +$$

$$+ \frac{3(27x+1)}{x^2(81x^2+14x+1)} v = 0 \ .$$

We get

$$A_n = \sum_{k=0}^{n} \sum_{i} \sum_{j} (-1)^{n+i+j} 3^{n-3i-3j} \binom{k}{3i} \binom{n-k}{3j} \binom{k+i}{k} \times$$

$$\times \binom{n-k+j}{j} \frac{(3i)!}{i!^3} \frac{(3j)!}{j!^3}$$

and

$$w := \sum_{n=0}^{\infty} A_n x^n$$

satisfies

$$\begin{gathered} \theta^5 + 2x(2\theta+1)(7\theta^4+14\theta^3+18\theta^2+11\theta+3) \\ +2x^2(\theta+1)(179\theta^4+716\theta^3+1364\theta^2+1296\theta+522) \\ +2\cdot 3^4 x^3 (2\theta+3)(7\theta^4+42\theta^3+102\theta^2+117\theta+53) \\ +3^8 x^4 (\theta+2)^5 \ . \end{gathered}$$

We have $x_0 = \frac{1}{81}$, $\alpha = \frac{2}{3\sqrt{3}}$, $\beta = \frac{8}{3\sqrt{3}}$, which gives $c_0 = \frac{50}{243}$, $c_1 = \frac{160}{243}$, $c_2 = \frac{64}{81}$ and we get

$$\sum_{n=0}^{\infty} A_n \frac{50+160n+192n^2}{243} \frac{1}{81^n} = \frac{1}{\pi^2} \ .$$

Example (Yifan Yang [13])

$$\sum_{n=0}^{\infty}\sum_k \binom{n}{k}^4 (1+4n)\frac{1}{36^n} = \frac{18}{\pi\sqrt{15}}$$

We have

$$v = \sum_{n=0}^{\infty}\sum_k \binom{n}{k}^4 x^n$$

which satisfies

$$v''' + \frac{3(128x^2+18x-1)}{x(64x^2+12x-1)}v'' + \frac{444x^2+40x-1}{x^2(64x^2+12x-1)}v' +$$

$$+\frac{2(30x+1)}{x^2(64x^2+12x-1)}v = 0 \ .$$

We have

$$A_n = \sum_{k=0}^{n}\sum_i\sum_j \binom{k}{i}^4 \binom{n-k}{j}^4$$

and

$$w := \sum_{n=0}^{\infty} A_n x^n$$

satisfies

$$\theta^5 - 4x(2\theta+1)(\theta^2+\theta+1)(3\theta^2+3\theta+1)$$

$$+16x^2(\theta+1)(\theta^4+4\theta^3-9\theta^2-26\theta-17)$$

$$+8x^3(2\theta+3)(96\theta^4+576\theta^3+1361\theta^2+1491\theta+634)$$

$$+64x^4(\theta+2)(2\theta+3)(2\theta+5)(4\theta+7)(4\theta+9) \ .$$

We have $x_0 = \frac{1}{36}$, $\alpha = \frac{\sqrt{15}}{18}$, $\beta = \frac{2\sqrt{15}}{9}$ which gives $c_0 = -\frac{1}{60}$, $c_1 = \frac{8}{81}$, $c_2 = \frac{20}{81}$ and

$$\sum_{n=0}^{\infty} A_n \frac{-27+160n+400n^2}{1620}\frac{1}{36^n} = \frac{1}{\pi^2}$$

Example (Rogers [11])

$$\sum_{n=0}^{\infty}(-1)^n\binom{2n}{n}\sum_k\binom{n}{k}^2\binom{2k}{k}(159-48\sqrt{3}+520n)(\frac{8-5\sqrt{3}}{22})^{2n}=$$

$$=\frac{2(64+29\sqrt{3})}{\pi}$$

We have

$$v=\sum_{n=0}^{\infty}\binom{2n}{n}\sum_k\binom{n}{k}^2\binom{2k}{k}x^n$$

which satisfies

$$v'''+\frac{3(1-60x+288x^2)}{x(1-40x+144x^2)}v''+\frac{11-132x+972x^2}{x^2(1-40x+144x^2)}v'+$$

$$+\frac{6(-1+18x)}{x^2(1-40x+144x^2)}v=0\ .$$

We have

$$A_n=\sum_{k=0}^{\infty}\binom{2k}{k}\binom{2n-2k}{n-k}\sum_{i,j}\binom{k}{i}^2\binom{2i}{i}\binom{n-k}{j}^2\binom{2j}{j}\ .$$

We have $x_0=-(\frac{8-5\sqrt{3}}{22})^2$, $\alpha=\frac{159-48\sqrt{3}}{2(64+29\sqrt{3})}$, $\beta=\frac{260}{64+29\sqrt{3}}$ which gives $c_0=\frac{1084121-624832\sqrt{3}}{11^4}$, $c_1=\frac{10(581611-333800\sqrt{3}}{3\cdot 11^4}$ $c_2=\frac{100}{3\cdot 11^4}(26476-14848\sqrt{3})$ and

$$\sum_{n=0}^{\infty}A_n(c_0+c_1n+c_2n^2)(-1)^n(\frac{8-5\sqrt{3}}{22})^{2n}=\frac{1}{\pi^2}\ .$$

3 Symmetric squares of third order differential equations

For a fourth order differential equation there are six 2×2-wronskians of the four solutions. In general they satisfy a differential equation of order 6. But there is an interesting exception, the Calabi-Yau equations, for which the wronskians satisfy a fifth order equation. Dually, there are six symmetric squares of the three solutions to a third order differential equation. When do these satisfy a fifth order equation? The answer is:

Theorem:
Consider the differential equation

$$v''' + s_2 v'' + s_1 v' + s_0 v = 0 \ .$$

Then $w = v^2$ satisfies a fifth order equation if and only if

$$\frac{1}{3}s_1 s_2 - \frac{2}{27}s_2^3 + \frac{1}{2}s_1' - \frac{1}{6}s_2'' - \frac{1}{3}s_2 s_2' - s_0 = 0 \ .$$

This means that already v is a square and w is a fourth power of a solution to a second order differential equation.

Proof:
Differentiating $w = v^2$ five times and eliminating $vv', vv'', v'v''$ we get

$$\begin{aligned} w^{(5)} + \frac{10}{3}s_2 w^{(4)} + 5(s_1 + \frac{5}{9}s_2^2 + \frac{1}{3}s_2')w''' + \\ +(11s_0 + 2s_1' + s_2'' + \frac{19}{3}s_1 s_2 + \frac{4}{9}s_1 s_2^2 + 2s_2 s_2')w'' \\ +(7s_0' + s_1'' + 4s_1^2 + \frac{32}{3}s_0 s_2 + \frac{4}{9}s_1 s_2^2 - \frac{1}{3}s_1 s_2' + \frac{7}{3}s_1' s_2)w' + \\ +(2s_0'' + 8s_0 s_1 + \frac{8}{9}s_0 s_2^2 - \frac{2}{3}s_0 s_2' + \frac{14}{3}s_0' s_2)w \\ = -12v'^2(\frac{1}{3}s_1 s_2 - \frac{2}{27}s_2^3 + \frac{1}{2}s_1' - \frac{1}{6}s_2'' - \frac{1}{3}s_2 s_2' - s_0) \end{aligned}$$

The right hand side is zero if and only if v is the square of a solution to a second order equation.

Corollary.

The fifth order equation in the Theorem is Calabi-Yau (but its fourth order pullback has trivial Yukawa coupling)

Proof: The C-Y2 condition for

$$w^{(5)} + b_4 w^{(4)} + b_3 w''' + b_2 w'' + b_1 w' + b_0 w = 0$$

is (see [3])

$$-b_2 + \frac{3}{2}b_3' + \frac{3}{5}b_3 b_4 - b_4'' - \frac{6}{5}b_4 b_4' - \frac{4}{25}b_4^3 = 0$$

and we compute the left hand side

$$-b_2 + \frac{3}{2}b_3' + \frac{3}{5}b_3 b_4 - b_4'' - \frac{6}{5}b_4 b_4' - \frac{4}{25}b_4^3 =$$

$$= 11(\frac{1}{3}s_1 s_2 - \frac{2}{27}s_2^3 + \frac{1}{2}s_1' - \frac{1}{6}s_2'' - \frac{1}{3}s_2 s_2' - s_0) = 0 \ .$$

We have $w_0 = v_0^2$, $w_1 = v_0 v_1$, $w_2 = v_1^2$. To show that the fourth order pullback has trivial Yukawa coupling we use the identities in [3]

$$x^2 f y_0^2 = \begin{vmatrix} w_0 & w_1 \\ w_0' & w_1' \end{vmatrix} = v_0^2 \begin{vmatrix} v_0 & v_1 \\ v_0' & v_1' \end{vmatrix}$$

$$x^2 f y_0 y_1 = \begin{vmatrix} w_0 & w_1 \\ w_0' & w_1' \end{vmatrix} = 2 v_0 v_1 \begin{vmatrix} v_0 & v_1 \\ v_0' & v_1' \end{vmatrix}$$

$$x^2 f y_0 y_2 = v_1^2 \begin{vmatrix} v_0 & v_1 \\ v_0' & v_1' \end{vmatrix}$$

which implies

$$\frac{y_2}{y_0} = \frac{v_1^2}{v_0^2} = \frac{1}{4}\left(\frac{y_1}{y_0}\right)^2 \ .$$

4 Harmonic sums

The expansions on pp.58-59 of [9] lead to, after differentiation, formulas containing harmonic numbers H_n defined by

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

and $H_0 = 0$. As a curiosity I mention the asymptotic expansion

$$H_n = \log(n) + \gamma - \sum_{k=1}^{\infty} \frac{B_k}{kn^k}$$

which could be a strange definition of the Bernoulli numbers

$$\sum_{n=0}^{\infty} \{120 + 4(9 + 120n)(H_{4n} - H_n)\} \frac{(4n)!}{n!^4} \frac{1}{2^{8n}7^{4n}} =$$

$$= \frac{49(8\log(2) + 4\log(7))}{\pi\sqrt{3}}$$

$$\sum_{n=0}^{\infty} \{52780 + 4(2206 + 52780n)(H_{4n} - H_n)\} \frac{(4n)!}{n!^4} \frac{1}{2^{8n}99^{4n}} =$$

$$= \frac{99^2(8\log(2) + 4\log(99))}{\pi\sqrt{2}}$$

$$\sum_{n=0}^{\infty} (-1)^n \{51 + (7 + 51n)(3H_{3n} + 2H_{2n} - 5H_n)\} \binom{2n}{n}^2 \binom{3n}{n} \frac{1}{2^{4n}108^n} =$$

$$= \frac{36(6\log(2) + 3\log(3))}{\pi\sqrt{3}}$$

$$\sum_{n=0}^{\infty} (-1)^n \{545140134 + (13591409 + 545140134n)(6H_n - 3H_{3n} - 3H_n)\} \times$$

$$\times \frac{(6n)!}{(3n)!n!^3} \frac{1}{12^{3n}53360^{3n}} = \frac{9 \cdot 53360^2(\log(12) + \log(53360))}{2\pi\sqrt{10005}}$$

$$\sum_{n=0}^{\infty}(-1)^n\left\{45+410n+10(\frac{13}{4}+45n+205n^2)(H_{2n}-H_n)\right\}\times$$
$$\times\binom{2n}{n}^5\frac{1}{2^{20n}}=\frac{640\log(2)}{\pi^2}$$

$$\sum_{n=0}^{\infty}(-1)^n\left\{1+5n+10(\frac{1}{8}+n+\frac{5}{2}n^2)(H_{2n}-H_n)\right\}\binom{2n}{n}^5\frac{1}{2^{12n}}=$$
$$=\frac{12\log(2)}{\pi^2}$$

$$\sum_{n=0}^{\infty}\left\{38+480n+(\frac{15}{8}+38n+240n^2)(8H_{8n}-4H_{4n}+2H_{2n}-6H_n)\right\}\times$$
$$\times\binom{2n}{n}\frac{(8n)!}{(4n)!n!^4}\frac{1}{2^{18n}7^{4n}}=\frac{49(18\log(2)+4\log(7)}{\pi^2\sqrt{7}}$$

$$\sum_{n=0}^{\infty}(-1)^n\left\{693+10836n+6(29+693n+5418n^2)(H_{6n}-H_n)\right\}\times$$
$$\times\frac{(6n)!}{n!^6}\frac{1}{2880^{3n}}=\frac{384\sqrt{5}\log(2880)}{\pi^2}$$

$$\sum_{n=0}^{\infty}\sum_{k}\sum_{i}\sum_{j}\left\{160+800n+4(-27+160n+400n^2)(H_{n-k}-H_{n-k-j})\right\}\times$$
$$\times\binom{k}{i}^4\binom{n-k}{j}^4\frac{1}{36^n}=\frac{1620\log(36)}{\pi^2}$$

I have continued the expansions one term longer for a few of Guillera's expansions on p.43 and 46 in [9]

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{1024^{n+x}}\frac{(\frac{1}{2})_{n+x}^5}{(1)_{n+x}^5}(13+180(n+x)+820(n+x)^2)=$$

$$= \frac{128}{\pi^2} - 320x^2 + \frac{4880}{3}\pi^2 x^4 - 114688\zeta(3)x^5 + O(x^6)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{1024^{n+x}} \frac{(1)_{n+x}^5}{(\frac{3}{2})_{n+x}^5}(13 + 180(n + \frac{1}{2} + x) + 820(n + \frac{1}{2} + x)^2) =$$

$$= 256\zeta(3) + \frac{64}{3}\pi^4 x + O(x^2)$$

$$\sum_{n=0}^{\infty} \frac{1}{64^{n+x}} \frac{(\frac{1}{2})_{n+x}^7}{(1)_{n+x}^7}(1 + 14(n + x) + 76(n + x)^2 + 168(n + x)^3) =$$

$$= \frac{32}{\pi^2}(1 - \pi^2 x^2 + \frac{4}{3}\pi^4 x^4 - \frac{257}{45}\pi^6 x^6 + O(x^7))$$

$$\sum_{n=0}^{\infty} \frac{1}{64^{n+x}} \frac{(1)_{n+x}^7}{(\frac{3}{2})_{n+x}^7}(1+14(n+\frac{1}{2}+x)+76(n+\frac{1}{2}+x)^2+168(n+\frac{1}{2}+x)^3) =$$

$$= \frac{1}{2}\pi^4 - 186\zeta(5)x + O(x^2)$$

Errata: In the thesis on p.58 in [9] $\frac{59\sqrt{3}}{49}$ should be $\frac{9\sqrt{3}}{49}$ (on p.33 formula (1.9) is correct)

References

[1] G.Almkvist and W.Zudilin, *Differential equations, mirror maps and zeta values*, Mirror Symmetry V, N.Yui, S.-T. Yau and J.D. Lewis (eds), Proc. of BIRS workshop on Calabi-Yau Varities and Mirror Symmetry (December 6-11, 2003), AMS/IP Stud. Adv. Math. 38, Amer. Math. Soc. & International Press, Providence, RI (2007), pp. 481–515, math.NT/0402386

[2] G.Almkvist, C. van Enckevort, D. van Straten and W. Zudilin, *Tables of Calabi-Yau equations*, math.AG/ 0507430

[3] G. Almkvist, *Calabi-Yau differential equations of degree 2 and 3 and Yifan Yang's pullback*, math.AG/ 0612215

[4] J.M.Borwein and P.B.Borwein, *Pi and the AGM.* New York, Wiley Interscience, 1987.

[5] H.H.Chan and H.Verrill, *The Apéry numbers, the Almkvist-Zudilin numbers and a new series for* $\frac{1}{\pi}$. Preprint 2007

[6] H.H.Chan, S.H.Chan and Z.Liu, *Domb's numbers and Ramanujan-Sato type series for* $\frac{1}{\pi}$, Adv. Math. 186 (2004), pp. 396–410.

[7] H.H.Chan, W.C.Liaw and V.Tan, *Ramanujan's class invariant* λ_n *and a new class of series for* $\frac{1}{\pi}$. J.London Math. Soc. 64 (2001), pp. 93–106.

[8] D.V.Chudnovsky and G.V.Chudnovsky, *Approximations and complex multiplication according to Ramanujan*, in Ramanujan Revisited: Proceedings of the centennial conference, 1987, edited by G.E.Andrews, B.C.Berndt and R.A.Rankin, pp. 375–472. Academic Press, Boston 1987.

[9] J.Guillera, *Series de Ramanujan: Generalizaciones y conjecturas*, Thesis, Zaragoza 2007, available on http://personal.auna.com/jguillera/

[10] S.Ramanujan, *Modular equations and approximations to* π, Quartily J. of Math. 45 (1914), 350-372.

[11] M.D.Rogers, *New* ${}_5F_4$ *hypergeometric transformations, three-variable Mahler measures, and formulas for* $\frac{1}{\pi}$, math/NT0704-2438

[12] T.Sato, *Apéry numbers and Ramanujan's series for* $\frac{1}{\pi}$, Abstrat of a talk presented at The Annual Meeting of the Mathematical Society of Japan, March 28-31, 2002.

[13] Y.Yang, *On differential equations satisfied by modular forms*, Math. Z. 246 (2004), pp. 1–19.

[14] W.Zudilin, *Quadratic transformations and Guillera's formulae for* $\frac{1}{\pi^2}$, Preprint 2005, http://wain.mi.ras.ru /publications.html

[15] W.Zudilin, *Ramanujan-type formulae for* $\frac{1}{\pi}$ *: A second wind? in Modular forms and duality and string duality*, Fields Institute Communications, AMS, Providence 2008.

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