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Minimum d-convex partition of a multidimensional polyhedron with holes*

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Abstract

In a normed space $\mathcal{R}^n$ over the field of real numbers $\mathbb{R}$, which is an α-space [36, 39], one derives the formula expressing the minimum number of d-convex pieces into which a geometric n-dimensional polyhedron with holes can be partitioned. The problem of partitioning a geometric n-dimensional polyhedron has many theoretical and practical applications in various fields such as computational geometry, image processing, pattern recognition, computer graphics, VLSI engineering, and others [5, 10, 11, 19, 21, 28, 29, 31, 43].

Key Words: geometric n-dimensional polyhedron, d-convexity, CW complex, dividing

Mathematics Subject Classification: 68U05, 52A30, 57Q05

1. Introduction

Let (X, d) be a metric space, and let $x_1, x_2 \in X$ be two arbitrary points of (X, d). By analogy with the classical definition of a convex set one introduces the notion of metric convexity depending on d [6, 8, 20, 36]. The set of points, denoted by $\langle x_1, x_2 \rangle$ and defined by $\langle x_1, x_2 \rangle = \{x \mid d(x_1, x_2) = d(x_1, x) + d(x, x_2)\}$, is called a *metric segment* joining the points x_1 and x_2. A set $M \subset X$ is said to be *d-convex* if for any two points $x_1, x_2 \in M$ the metric segment $\langle x_1, x_2 \rangle \subset M$. It is easy to see that the intersection of two d-convex sets is a d-convex set.

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For a given set $M \subset X$, the d-convex hull of the set M, denoted by $d\text{-}conv\, M$, is defined as the intersection of all d-convex sets containing M. In case that (X, d) is a normed space $\mathcal{R}^n$ over the field of real numbers $\mathbb{R}$ with $d(x_1, x_2) = \|x_1 - x_2\|$ every d-convex set is also a convex set, but not always conversely. Convexity and d-convexity in $\mathcal{R}^n$ coincide if and only if the closed unit ball of $\mathcal{R}^n$ is strictly convex [6, 8, 20, 36]. Thus this notions coincide in the Euclidean space $\mathbb{E}^n$. For a bounded set $N \subset \mathcal{R}^n$ it can happen that $d\text{-}conv\, N = \mathcal{R}^n$ [6, 36]. We will only consider those normed spaces such that $d\text{-}conv\, N$ is bounded, that is, so-called α-spaces [36, 39].

In the papers [1, 7, 24, 26, 32, 38, 40, 42] it is given sufficient information on the problem of partitioning a polygon with holes or a 3-dimensional geometric polyhedron with holes into the minimum number of d-convex pieces.

Let $\mathcal{R}^2$ be a normed plane, and let P^2 be an open polygon with g holes of dimension $d \in \{0, 1, 2\}$ all of whose edges are d-convex.

In the paper [24] $\mathcal{R}^2$ coincides with the Euclidean plane $\mathbb{E}^2$, and the edges of the polygon P^2 are only parallel to two perpendicular directions while all the holes are of dimension 2. In this case it is shown that the minimum number $q\left(P^2\right)$ of rectangles partitioning the polygon P^2 is

$$q\left(P^2\right) = \frac{s}{2\pi} - h, \tag{1.1}$$

where s is the total sum of interior angles of the polygon P^2, measured in radians, and h is the maximum number of mutually disjoint segments that can be drawn within the closure of the polygon P^2, parallel to the edges of P^2, and with the endpoints at the concave vertices. This formula was generalized by Soltan and Gorpinevich [42] for the case of a rectilinear polygonal domain with possible degenerate holes. That problem appeared in VLSI engineering [35].

In the papers [7, 26, 32] the problem to partition the polygon P^2 into a minimum number of d-convex pieces is completely solved. The respective formula is shown to be

$$q\left(P^2\right) = m + 1 - g - h, \tag{1.2}$$

where m, g are the total sum of all measures of local nonconvexity of points of local nonconvexity [7, 26], the number of holes of the polygon P^2, respectively, and h is the number of elements of a maximum concordant system of dividing trees [7, 26]. Considering $\mathcal{R}^2$ with the norm $\|x\| = |x_1| + |x_2|$, it is easy to obtain that q in (1.1) and q in (1.2) are the same number for the case from [25].

Let P^3 be an open polyhedron in the Euclidean space $\mathbb{E}^3$ with polyhedral holes and the edges parallel to the coordinate axes of $\mathbb{E}^3$, where the holes can be of dimension $3, 2, 1, 0$. A formula expressing the minimum number of parallelepipeds $q\left(P^3\right)$ into which the polyhedron P^3 can be partitioned is proposed in the paper [1]. The researches done in this paper led to the fact that the minimum estimation of $q\left(P^3\right)$ required the methods of algebraic topology to be applied, as it would see below.

2. Auxiliary elements

In a normed space $\mathcal{R}^n$ it is possible to define the notion of a geometric n-dimensional polyhedron in a simpler or more complicated way. We will introduce a more natural notion of a polyhedron as the (simple) polygon [14] in $\mathcal{R}^2$ and the (simple) geometric polyhedron [14] in $\mathcal{R}^3$ are defined.

By analogy with [9] we propose

Definition 2.1. *A compact n-dimensional PL manifold* [15, 18, 27, 34] *with boundary in the normed space $\mathcal{R}^n$ which admits a decomposition into q handles* [16, 34] *of index* 1 *is said to be a* ***geometric n-dimensional polyhedron of genus q*** *in $\mathcal{R}^n$. It is denoted by P_q^n.*

For a geometric polyhedron P_q^n as a topological subspace of the space $\mathcal{R}^n$, we will denote by $bd\, P_q^n, int\, P_q^n, \overline{P_q^n}$ the boundary, the interior and the closure of P_q^n, respectively. By $B^n(x, \varepsilon) \subset \mathcal{R}^n$ we denote the closed ball with center at x and radius ε. We will use B^n and S^{n-1} as notations for the closed unit ball and the unit sphere of $\mathcal{R}^n$, respectively.

Definition 2.2. *Let x be a boundary point of a geometric polyhedron P_q^n. We will denote by* $aff\left(x, P_q^n\right)$ *the union of x and all lines l in $\mathcal{R}^n$ through the point x such that the intersection $l \bigcap bd\, P_q^n$ contains an open line segment which includes x. The connection component of* $aff\left(x, P_q^n\right) \bigcap bd\, P_q^n$ *which contains the point x is called a* ***face*** *of x of P_q^n. By convention, $\emptyset$ and P_q^n are called* ***improper faces*** *of P_q^n.*

Definition 2.3. *By a* ***m-dimensional face*** *F^m of P_q^n we mean a face of dimension m (the dimension of the affine hull of F^m). We call F^m*

1) a ***vertex*** *of P_q^n, if $m = 0$;*

2) an ***edge*** *of P_q^n, if $m = 1$;*

3) a ***facet*** *of P_q^n, if $m = n - 1$.*

The dimension of the empty face is set to -1 by convention.

Definition 2.4. [36, 39] *A normed space $\mathcal{R}^n$ is called an* ***α-space*** *if for every bounded set $N \subset \mathcal{R}^n$ the d-convex hull of N is bounded.*

Let X be a Hausdorff space.

Definition 2.5. *A subset e of the space X is called an* ***m-dimensional open polyhedral cell****, or* ***m-dimensional polyhedral cell*** *in X if it is PL homeomorphic with an m-dimensional open convex polytope* [13, 25, 33] *in the space $\mathcal{R}^n$.*

Definition 2.6. *A subset e of the space X is called an* ***m-dimensional closed polyhedral cell*** *in X if it is PL homeomorphic with an m-dimensional convex polytope in the space $\mathcal{R}^n$.*

Definition 2.7. *A subset e of the space X is called an* ***m-dimensional open cell****, or* ***m-dimensional cell*** *in X if it is homeomorphic to the open unit ball of the space $\mathcal{R}^m$. Let $\overline{e}$ be the closure of e in X, and let $\dot{e} = \overline{e} \setminus e$.*

Definition 2.8. [12, 18, 22] *A set $\mathcal{E} = \{e_\lambda \mid \lambda \in \Lambda\}$ of cells in the Hausdorff space X is called a* ***cellular decomposition*** *of X if the following three conditions are satisfied:*

1) $e_\lambda \bigcap e_\mu$ *is empty if* $\lambda \neq \mu$*;*

2) $X = \bigcup\limits_{\lambda \in \Lambda} e_\lambda$*;*

3) for every m*-dimensional cell* $e_\lambda \in \mathcal{E}$ *there is a continuous mapping*

$$\varphi\colon \left(B^m, S^{m-1}\right) \to \left(X^{m-1} \bigcup e_\lambda, X^{m-1}\right)$$

such that $\varphi\left(B^m \setminus S^{m-1}\right) \to e_\lambda$ *is a homeomorphism, where* X^{m-1} *is the union of all the cells* $e_\mu \in \mathcal{E}$*, whose dimensions are not greater than* $m-1$*.*

Definition 2.9. [12, 18, 22] *A Hausdorff space* X *together with its cellular decomposition* $\mathcal{E} = \{e_\lambda \mid \lambda \in \Lambda\}$ *is called a* ***cell complex****. A cell complex* X *is said to be* ***finite*** *if the set* $\mathcal{E}$ *is finite.*

Definition 2.10. *The* ***dimension*** $\dim X$ *of a cell complex* X *is* m *if* X *contains an* m*-dimensional cell but no* $(m+1)$*-dimensional cell, and* ∞ *if* X *contains* m*-dimensional cells for all* $m \geqslant 0$*.*

Definition 2.11. [12, 18, 22] *A cell complex* X *is said to be* ***closure finite*** *if the closure of each cell meets only finitely many other cells, and* X *is said to have the* ***weak topology*** *if a subset* $A \subset X$ *is closed iff* $A \bigcap \overline{e}$ *is closed in* $\overline{e}$ *for each cell* e *of* X*.*

Definition 2.12. *We call a cell complex a* ***CW complex*** *if it is closure finite and has the weak topology.*

It is easy to see that a finite cell complex is a CW complex. A geometric polyhedron is also a finite CW complex.

Theorem 2.1. [12, 18, 22] *If* X *is a* CW *complex, then the* m*th integral cellular homology group of* X *is isomorphic to the* m*th integral singular homology group of* X*.*

Definition 2.13. *If* X *is a finite* CW *complex and* β_m *denotes the rank of the* m*th integral singular homology group of* X*, then the number* $\chi(X) = \sum\limits_{m=0}^{\dim X} (-1)^m \beta_m$ *is called the* ***Euler-Poincaré characteristic*** [12, 17, 18, 22] *of* X*.*

Theorem 2.2 (Euler-Poincaré). [12, 18, 22, 23, 30, 44] *For a finite CW complex X it holds that*

$$\chi(X) = \sum_{m=0}^{dim\,X} (-1)^m \beta_m = \sum_{m=0}^{dim\,X} (-1)^m \alpha_m,$$

where α_m denotes the number of m-dimensional cells of X.

It is evident from the definition of the integral singular homology group and the theorems above that the Euler-Poincaré characteristic $\chi(X)$ is an integer topological invariant for a finite CW complex X. Moreover, $\chi(X)$ depends only on the homotopy type of X. In particular, given any CW decomposition of X, we always will get the same integer $\chi(X)$.

In that follows, a subscript in the name of mathematical objects denotes their dimension.

3. Main theorem

Let $\mathcal{R}^n$ be an α-space, and let $P_q^n \subset \mathcal{R}^n$ be a geometric n-polyhedron of genus q all of whose facets belong to the d-convex hyperplanes in $\mathcal{R}^n$. Consider this polyhedron P_q^n having also a finite number of holes, mutually disjoint open geometric polyhedrons of genus 0 of dimension $n, n-1, \ldots, 0$, the facets of which belong to the d-convex linear manifolds in $\mathcal{R}^n$. Suppose also that $\mathcal{R}^n \setminus U$ is an n-cell in $\mathcal{R}^n$, where U is the unbounded connection component of the complement $\mathcal{R}^n \setminus int\, P_q^n$. When speaking of a face of P_q^n, we will mean either a face of the polyhedron P_q^n without holes or a face of a certain hole of P_q^n. Since P_q^n is compact, the set of faces of P_q^n is finite.

Definition 3.1. *A point $x \in bd\, P_q^n$ is called a* ***point of local non-d-convexity*** [6, 7, 26, 32, 40] *of P_q^n if, for any sufficiently small $\varepsilon > 0$, there exists at least one non-d-convex connection component of the intersection d-conv $B^n(x, \varepsilon) \bigcap int\, P_q^n$.*

Let R be the set of all points of local non-d-convexity of the polyhedron P_q^n. We will always assume that the set R is not empty.

Definition 3.2. *The geometric polyhedron P_q^n is called* ***partitioned into d-convex pieces*** *$Q_1, Q_2, \dots, Q_r$ if*

1) $int\, Q_i \neq \emptyset,\ i = 1, 2, \dots, r;$

2) $\bigcup\limits_{i=1}^{r} int\, Q_i \subset int\, P_q^n \subset \bigcup\limits_{i=1}^{r} Q_i;$

3) $int\, Q_i \bigcap int\, Q_j = \emptyset,\ i \neq j.$

It is obvious that P_q^n has at least one d-convex partition. We will denote by $p\left(P_q^n\right)$ the minimum number of d-convex pieces into which P_q^n can be partitioned. Since the number of d-convex pieces of a d-convex partition $Q_1, Q_2, \dots, Q_r$ of the geometric polyhedron P_q^n is equal to the number of d-convex pieces of the d-convex partition $\overline{Q_1}, \overline{Q_2}, \dots, \overline{Q_r}$, without losing generality, we will assume that the d-convex pieces into which P_q^n can be partitioned are closed.

By D^{n-1} and $\left|D^{n-1}\right|$ we will denote a finite set of polyhedral cells of dimension $\leq n-1$ in the α-space $\mathcal{R}^n$, belonging to the interior of P_q^n, and the set of all points of the cells, respectively.

Definition 3.3. *A set D^{n-1} is called a* ***dividing*** [1 – 4, 37] *of the polyhedron P_q^n if D^{n-1} satisfies the following condition: for every $x \in R \bigcup \overline{|D^{n-1}|}$ there exists an $\varepsilon > 0$ such that the intersection $\left(int\, P_q^n \setminus \overline{|D^{n-1}|}\right) \bigcap d\text{-}conv\, B^n(x, \varepsilon)$ consists only of d-convex connection components;*

The set $dvz\, P_q^n$ of all dividings of the polyhedron P_q^n is not empty. This assertion relies on the existence of a d-convex partition of P_q^n. Inverse, any d-convex partition of P_q^n can be obtained by a certain dividing of P_q^n.

Definition 3.4. *The number $\chi\left(D^{n-1}\right) = \sum\limits_{i=0}^{n-1} (-1)^i \alpha_i$ will be called the* ***Euler-Poincaré characteristic of the dividing*** *D^{n-1}, where α_i is the number of polyhedral cells of dimension i of D^{n-1}.*

The Euler-Poincaré characteristic is an integer topological invariant for $\left|D^{n-1}\right|$. This fact results from the definition of the dividing. The notion of a dividing is not other than a generalization of the notion of a concordant system of dividing trees $[6, 26]$ for a polygonal domain.

Theorem 3.1. *The Euler-Poincaré characteristic of a polyhedron* $P_q^n \subset \mathcal{R}^n$, $n \geq 3$, *satisfies the property:*

$$\chi\left(bd\, P_q^n\right) - \chi\left(\overline{P_q^n}\right) = (-1)^{n-1}(1-q) + g, \tag{3.1}$$

where g *is the number of bounded connection components of the complement* $\mathcal{R}^n \setminus int\, P_q^n$.

Proof. First we consider that $q = 0$. We will prove the statement of the theorem by induction on g.

The case $g = 0$ is trivial: $\chi\left(\overline{P_0^n}\right) = \chi\left(bd\, P_0^n\right) + (-1)^n$.

For the case $g = 1$, let Q denote the bounded connection component of the complement $\mathcal{R}^n \setminus int\, P_0^n$. Let z be a vertex of Q and let H be a hyperplane through the point z such that $H \bigcap Q = z$. The existence of such a point z results from the facts that the set of holes is finite and the holes are geometric polyhedrons. Consider that the connection component of the intersection $H \bigcap P_0^n$ which contains the point z is a closed polyhedral $(n-1)$-cell in the space $\mathcal{R}^n$. This connection component divides $int\, P_0^n$ into two polyhedral n-cells in $\mathcal{R}^n$. The polyhedron P_0^n can be PL homeomorphically deformed in order to pass such a hyperplane H. Therefore

$$\chi\left(\overline{P_0^n}\right) = \left(\chi\left(bd\, P_0^n\right) - 1\right) + (-1)^{n-1} + (-1)^n \cdot 2.$$

As a result,

$$\chi\left(bd\, P_0^n\right) - \chi\left(\overline{P_0^n}\right) = (-1)^{n-1}(1-0) + 1.$$

Assume the equality (3.1) holds for all polyhedrons M_0^n with at most $l-1$, $l \geqslant 2$, bounded connection components of the complement $\mathcal{R}^n \setminus int\, M_0^n$, and let P_0^n be a polyhedron with l bounded connection components of the complement $\mathcal{R}^n \setminus int\, P_0^n$. Choose a hyperplane H

such that $H \bigcap int\, P_0^n \neq \emptyset$ and regard a connection component C of the intersection $H \bigcap P_0^n$. Suppose that the connection component C is a closed polyhedral $(n-1)$-cell in the space $\mathcal{R}^n$ and $C \bigcap int\, P_0^n$ is an open polyhedral $(n-1)$-cell. The connection component C determines two n-polyhedrons P_1 and P_2. Consider that each of these polyhedrons contains at least one bounded connection component of its complement. The polyhedron P_0^n can be PL homeomorphically deformed in order to pass such a hyperplane H. By inductive assumption,

$$\chi\left(bd\, P_1\right) - \chi\left(\overline{P_1}\right) = (-1)^{n-1}(1-0) + g_1$$

and

$$\chi\left(bd\, P_2\right) - \chi\left(\overline{P_2}\right) = (-1)^{n-1}(1-0) + g_2,$$

where $g_1,\ g_2$ are the bounded connection components of the complements $\mathcal{R}^n \setminus int\, P_1$ and $\mathcal{R}^n \setminus int\, P_2$, respectively. It is easily seen, that $g_1 + g_2 = g$. Since $\chi\left(S^{n-1}\right) = 1 + (-1)^{n-1}$ for the unit sphere S^{n-1} in $\mathcal{R}^n$, we have

$$\begin{aligned} \chi\left(bd\, P_0^n\right) &= \chi\left(bd\, P_1\right) + \chi\left(bd\, P_2\right) - (-1)^{n-1}\cdot 2 - \left(1 + (-1)^{n-2}\right), \\ \chi\left(\overline{P_0^n}\right) &= \chi\left(\overline{P_1}\right) + \chi\left(\overline{P_2}\right) - (-1)^{n-1} - \left(1 + (-1)^{n-2}\right). \end{aligned}$$

Hence

$$\chi\left(bd\, P_0^n\right) - \chi\left(\overline{P_0^n}\right) = (-1)^{n-1}(1-0) + g_1 + (-1)^{n-1}(1-0) + g_2 - (-1)^{n-1}.$$

Whence

$$\chi\left(bd\, P_0^n\right) - \chi\left(\overline{P_0^n}\right) = (-1)^{n-1}(1-0) + g.$$

Now assume that $q > 0$. Choose for the handles in P_q^n two secant balls each. Suppose that each of these secant balls intersects the polyhedron P_q^n and the interior of P_q^n by a closed polyhedral $(n-1)$-cell and by an open polyhedral $(n-1)$-cell in the space $\mathcal{R}^n$, respectively. The polyhedron P_q^n can be PL homeomorphically deformed in order to pass such balls. The balls separate P_q^n into $q+1$ polyhedrons P_i of dimension n of genus 0, $i = 1, 2, \ldots, q+1$. Reasoning as above, we obtain

$$\begin{aligned}
\chi\left(bd\,P_i\right)-\chi\left(\overline{P_i}\right) &= (-1)^{n-1}(1-0)+g_i,\ i=1,2,\ldots,q+1,\\
\sum_{i=1}^{q+1} g_i &= g,\\
\chi\left(bd\,P_q^n\right) &= \sum_{i=1}^{q+1}\chi\left(bd\,P_i\right)-(-1)^{n-1}\cdot 4q-\left(1+(-1)^{n-2}\right)\cdot 2q,\\
\chi\left(\overline{P_q^n}\right) &= \sum_{i=1}^{q+1}\chi\left(\overline{P_i}\right)-(-1)^{n-1}\cdot 2q-\left(1+(-1)^{n-2}\right)\cdot 2q,
\end{aligned}$$

where g_i is the bounded connection components of the complement $\mathcal{R}^n\setminus int\,P_i,\ i=1,2,\ldots,q+1$. It follows immediately that

$$\chi\left(bd\,P_q^n\right)-\chi\left(\overline{P_q^n}\right)=\sum_{i=1}^{q+1}\left((-1)^{n-1}(1-0)+g_i\right)-(-1)^{n-1}\cdot 2q.$$

Therefore we get

$$\chi\left(bd\,P_q^n\right)-\chi\left(\overline{P_q^n}\right)=(-1)^{n-1}(1-q)+g.$$

This completes the proof. □

Theorem 3.2 (Main Theorem). *For a polyhedron* $P_q^n\subset\mathcal{R}^n$, $n\geq 3$, *the equality*

$$p\left(P_q^n\right)=(-1)^{n-1}\left(\chi\left(bd\,P_q^n\right)-\chi\left(\overline{P_q^n}\right)\right)+\min_{D^{n-1}\in dvz\,P_q^n}\left|\chi\left(D^{n-1}\right)\right|.$$

holds.

Proof. Let D^{n-1} be a dividing of the polyhedron P_q^n. This dividing determines a finite CW n-complex K^n representing the polyhedron P_q^n and whose n-cells are open d-convex polytopes. Indeed, from the definition of the dividing, the set of all points belonging to the closures of polyhedral cells of the dividing D^{n-1} is a finite CW $(n-1)$-complex

M^{n-1}. Moreover, the set of the cells of M^{n-1}, each of which belongs to the boundary of P_q^n, determines a decomposition of the boundary into polyhedral cells in the space $\mathcal{R}^n$. Denote by L^{n-1} the CW $(n-1)$-complex formed by this decomposition. The finite CW complex $M^{n-1}\bigcup L^{n-1}$ together with the connection components C_i of the set $int\, P_q^n \setminus \left(M^{n-1}\bigcup L^{n-1}\right)$ forms the required cell decomposition. The connection components C_i are open, local d-convex, so they are also d-convex [41]. The closures of these connection components partition P_q^n into d-convex pieces. We have $\chi\left(\overline{P_q^n}\right) = \chi\left(K^n\right)$ and $\chi\left(bd\, P_q^n\right) = \chi\left(L^{n-1}\right)$. From the theorem 2.2, it is clear that

$$\chi\left(K^n\right) = \sum_{i=0}^{n}(-1)^i\alpha_i, \tag{3.2}$$

where α_n is the number of d-convex pieces into which P_q^n is partitioned, and α_i represents the number of polyhedral i-cells of K^n, $i = 0,1,\ldots,n-1$. Rewrite (3.2) as follows

$$(-1)^n\alpha_n = \chi\left(K^n\right) - \sum_{i=0}^{n-1}(-1)^i\alpha_i.$$

Whence

$$(-1)^n\alpha_n = \chi\left(K^n\right) - \sum_{i=0}^{n-1}(-1)^i\alpha_i' - \sum_{i=0}^{n-1}(-1)^i\alpha_i'',$$

where α_i' is the number of polyhedral i-cells belonging to the boundary of P_q^n, and α_i'' is the number of polyhedral i-cells belonging to the dividing D^{n-1}. Therefore we get

$$(-1)^n\alpha_n = \chi\left(K^n\right) - \chi\left(L^{n-1}\right) - \chi\left(D^{n-1}\right). \tag{3.3}$$

Thus, both sides of the equality (3.3) being multiplied by $(-1)^n$, we obtain

$$\alpha_n = (-1)^n\chi\left(K^n\right) + (-1)^{n-1}\chi\left(L^{n-1}\right) + (-1)^{n-1}\chi\left(D^{n-1}\right).$$

Whence

$$\alpha_n = (-1)^{n-1}\left(\chi\left(bd\,P_q^n\right) - \chi\left(\overline{P_q^n}\right)\right) + (-1)^{n-1}\chi\left(D^{n-1}\right). \qquad (3.4)$$

The Euler-Poincaré characteristic of the dividing D^{n-1} is nonnegative for odd n and is nonpositive for even n in view of the facts that the relations (3.1), (3.4) and the inequality $\alpha_n > g$ hold. Therefore we get $\left|\chi\left(D^{n-1}\right)\right| = (-1)^{n-1}\chi\left(D^{n-1}\right)$. If the dividing D^{n-1} is chosen such that the value of $\left|\chi\left(D^{n-1}\right)\right|$ to be minimum, then α_n is minimum, too. Hence we obtain

$$p\left(P_q^n\right) = (-1)^{n-1}\left(\chi\left(bd\,P_q^n\right) - \chi\left(\overline{P_q^n}\right)\right) + \min_{D^{n-1}\in dvz\,P_q^n}\left|\chi\left(D^{n-1}\right)\right|,$$

and the theorem is proved. □

Corollary 3.1. *Let $P_q^n \subset \mathcal{R}^n$, $n \geq 3$, be a geometric n-polyhedron. Then*

$$p\left(P_q^n\right) = 1 - q + (-1)^{n-1}\cdot g + \min_{D^{n-1}\in dvz\,P_q^n}\left|\chi\left(D^{n-1}\right)\right|,$$

where g is the number of bounded connection components of the complement $\mathcal{R}^n \setminus int\,P_q^n$.

Corollary 3.2. *Let $P_q^2 \subset \mathcal{R}^2$ be a geometric 2-polyhedron (a polygonal domain). Then*

$$p\left(P_q^2\right) = 1 - g + \min_{D^1\in dvz\,P_q^2}\left|\chi\left(D^1\right)\right|,$$

where g is the number of bounded connection components of the complement $\mathcal{R}^2 \setminus int\,P_q^2$.

To prove the corollary, it suffices to analyse the proof of the theorem 3.1.

If the norm of $\mathcal{R}^n$ is defined by $\|x\| = \sum_{i=1}^{n}|x_i|$, then for the polyhedrons P^2 and P^3 in Figure 3.1 and Figure 3.2, respectively, we obtain $p\left(P^2\right) = 1-5+|1-18| = 13$ and $p\left(P^3\right) = 1-0+2+|0-22+34| = 15$.

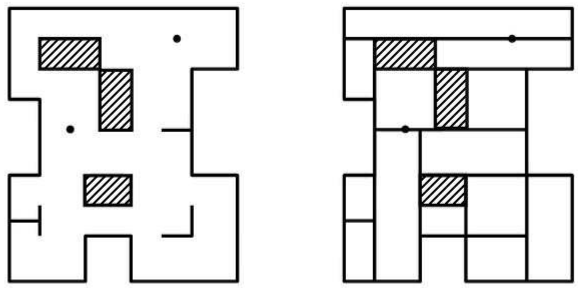

Figure 3.1.

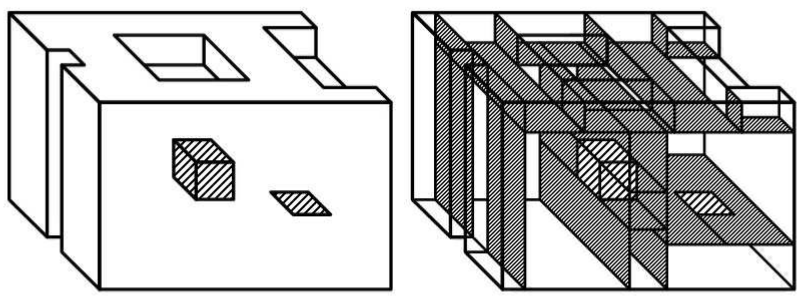

Figure 3.2.

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