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On stability of a Pareto-optimal solution under perturbations of the parameters for a multicriteria combinatorial partition problem

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Abstract

Abstract. We consider a multicriteria variant for the well-known partition problem. A formula of the stability radius for an efficient solution was obtained.

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Many problems of design, planning and management in technical and organizational systems have a pronounced multicriteria character. Multiobjective models that appeared in these cases are reduced to the choice of the "best" (in a certain sense) values of variable parameters from some discrete aggregate of the given quantities. Therefore recent interest of mathematicians to multicriteria discrete optimization problems keeps very high which is confirmed by the intensive publishing activity (see, e. g., bibliography [1], which contains 234 references). One of the important directions of study such problems is stability analysis of solutions under perturbations of the initial data. Various questions of stability analysis and regularization for incorrect discrete optimization problems generate numerous directions of research. Nowadays owing to the fundamental investigations of academician I. V. Sergienko and his colleagues [2–11], characteristics of stable problems, necessary and sufficient conditions of existence of Pareto-optimal solutions, methods of

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regularization for incorrect problems and many other properties of multicriteria integer problems under uncertainty of initial data are rather well-known.

Nowadays many specialists study quantitative characteristics of stability of both scalar (single criteria) and vector (multicriteria) discrete optimization problems. Not touching on this wide spectrum of questions, we refer the reader to the extensive bibliography [12] as well as to the works [13–18] which contain the most typical recent results.

We continue to research the cycle of works devoted to the quantitative analysis of stability [19–26] for the Pareto-optimal solutions of the combinatorial problems with various types of vector criteria. We consider a multicriteria variant of the well-known partition problem. A formula of the stability radius of an efficient solution in the case of l_∞-metric is obtained.

The partition problem is a classical extremal combinatorial problem. It is stated as follows: it is needed to partition the finite set of numbers into two nonintersecting subsets such that the sums of numbers of these subsets are differed minimally from each other. In the case where the elements of the set are positive this problem is equivalent to the problem of the scheduling theory that consists in distribution of independent works in two identical processors such that to minimize the time when the last work should be finished [27]. In the scheduling theory this problem is marked as $P|\cdot|C_{\max}$.

We consider a multicriteria (vector) variant of the partition problem.

We define a vector function (vector criterion)

$$f(x, C) = (|C_1 x|, |C_2 x|, \dots, |C_m x|) \to \min_{x \in Q^n}$$

on the set of n-vectors $Q^n,\ n \geq 2,\ Q = \{-1, 1\}$, where C_i denotes the i-th row of matrix $C = [c_{ij}]_{m\times n} \in \mathbf{R}^{m\times n},\ m \geq 1$, $x = (x_1, x_2, \dots, x_n)^T$.

Under the multicriteria partition problem $Z^m(C)$ we understand the problem of finding the set of efficient solutions (Pareto set)

$$P^m(C) = \{x \in Q^n : \pi(x, C) = \emptyset\},$$

where

$$\pi(x, C) = \{x' \in Q^n : f(x, C) \geq f(x', C) \ \& \ f(x, C) \neq f(x', C)\}.$$

Under the stability of the efficient solution x^0 we understand the property of preserving Pareto optimality of x^0 under "small" perturbations of the elements of matrix C. We will model such perturbations by adding a "perturbing" matrices to C.

For each number $k \in \mathbf{N}$, we endow the space $\mathbf{R}^k$ with metrics l_1 and l_∞ :

$$||z||_1 = \sum_{j \in N_k} |z_j|, \ ||z||_\infty = \max_{j \in N_k} |z_j|, \ z = (z_1, z_2, \ldots, z_k) \in \mathbf{R}^k,$$

where $N_k = \{1, 2, \ldots, k\}$. Under the norm of matrix we understand the norm of vector composed from all its elements. For any number $\varepsilon > 0$ we define the set of perturbing matrices

$$\Omega(\varepsilon) = \{C' \in \mathbf{R}^{m \times n} : ||C'||_\infty < \varepsilon\}.$$

According to the definitions from [21–26], under the stability radius of $x^0 \in P^m(C)$ we understand the number

$$\rho^m(x^0, C) = \begin{cases} \sup \ \Xi, & \text{if } \Xi \neq \emptyset, \\ 0, & \text{if } \Xi = \emptyset, \end{cases}$$

where

$$\Xi = \{\varepsilon > 0 : \forall \ C' \in \Omega(\varepsilon) \ (x^0 \in P^m(C + C'))\}.$$

Thus, the stability radius is a limit level of independent perturbations of the elements of C, such that the Pareto optimality of the solution is preserved.

We will use the following implication

$$\exists q \in Q \ \ \forall q' \in Q \ \ (qz > q'z') \Rightarrow |z| > |z'|, \tag{1}$$

which holds for any numbers $z, z' \in \mathbf{R}$.

Suppose

$$\text{sg } z = \begin{cases} 1, & \text{if } z \geq 0, \\ -1, & \text{if } z < 0, \end{cases}$$

$$K(x^0, x) = \{i \in N_m : \ |C_i x^0| \leq |C_i x|\},$$

$$\alpha_i(x^0, x) = \min\{\beta_i(x^0, x, q) : \ q \in Q\},$$

$$\beta_i(x^0, x, q) = \frac{|C_i(qx^0 + x)|}{||qx^0 + x||_1}.$$

It is evident, that $K(x^0, x) \neq \emptyset$ if $x^0 \in P^m(C)$.

Theorem. *Stability radius of an efficient solution x^0 of the problem $Z^m(C)$, $m \geq 1$ is expressed by the formula*

$$\rho^m(x^0, C) = \min_{x \in Q^n \setminus \{x^0, -x^0\}} \max_{i \in K(x^0, x)} \alpha_i(x^0, x). \tag{2}$$

Proof. Denote by φ the right side of (2). It is easy to see, that $\varphi \geq 0$.

At first we will prove the inequality $\rho^m(x^0, C) \geq \varphi$. Suppose $\varphi > 0$ (otherwise the inequality $\rho^m(x^0, C) \geq \varphi$ is evident). Let $C' \in \Omega(\varphi)$. Then by the definition of φ for any $x \in Q^n \setminus \{x^0, -x^0\}$ there exists $k \in K(x^0, x)$ such that

$$||C'||_\infty < \varphi \leq \alpha_k(x^0, x). \tag{3}$$

Taking into account $\alpha_k(x^0, x) > 0$, we have

$$|C_k x^0| < |C_k x|.$$

From this, assuming

$$\sigma_k = \text{sg } C_k x,$$

we obtain

$$C_k(qx^0 + \sigma_k x) = |C_k(\sigma_k q x^0 + x)|, \ q \in Q.$$

Therefore, using (3), we derive

$$(C_k + C'_k)(qx^0 + \sigma_k x) = |C_k(\sigma_k q x^0 + x)| + C'_k \sigma_k(\sigma_k q x^0 + x) \geq$$
$$\geq |C_k(\sigma_k q x^0 + x)| - ||C'||_\infty \cdot ||\sigma_k q x^0 + x||_1 >$$

$$> |C_k(\sigma_k q x^0 + x)| - \beta_k(x^0, x, \sigma_k q)||\sigma_k q x^0 + x||_1 = 0.$$

Thus, we have

$$(C_k + C'_k)\sigma_k x > (C_k + C'_k) q x^0, \ q \in Q.$$

Taking into account (1) for any $x \in Q^n \setminus \{x^0, -x^0\}$ we obtain

$$|(C_k + C'_k)x| > |(C_k + C'_k)x^0|,$$

and for $x = \pm x^0$ we have

$$|(C + C')x| = |(C + C')x^0|,$$

which imply $x^0 \in P^m(C + C')$.

Resuming the said above, we conclude that for any $C' \in \Omega(\varphi)$ the inclusion $x^0 \in P^m(C + C')$ holds. Hence $\rho^m(x^0, C) \geq \varphi$.

It remains to prove the inequality $\rho^m(x^0, C) \leq \varphi$. By the definition of φ, there exists $x^* \in Q^n \backslash \{x^0, -x^0\}$, such that for any $i \in K(x^0, x^*)$ the following inequalities hold:

$$0 \leq \alpha_i(x^0, x^*) \leq \varphi. \tag{4}$$

Let $\varepsilon > \varphi$. We will prove that there exists $C' \in \Omega(\varepsilon)$ with condition $x^0 \notin P^m(C + C')$.

Suppose

$$N(x^0, x^*) = |\{j \in N_n : \ x_j^0 = 1 \ \& \ x_j^* = -1\}|,$$

$$M(x^0, x^*) = |\{j \in N_n : \ x_j^0 = x_j^*\}|,$$

$$\sigma_i^* = \text{sg } C_i x^*.$$

It is easy to see, that

$$M(x^0, x^*) = M(x^*, x^0),$$

$$2(N(x^0, x^*) + N(x^*, x^0)) = ||x^0 - x^*||_1, \tag{5}$$

$$2M(x^0, x^*) = ||x^0 + x^*||_1. \tag{6}$$

To construct the rows $C'_i,\ i \in N_m$ of the needed matrix C', we consider four possible cases.

Case 1: $i \in K(x^0, x^*),\ \beta_i(x^0, x^*, -1) < \beta_i(x^0, x^*, 1)$. Then under (4) the following inequalities hold:

$$|C_i(x^0 + x^*)| > 0,$$

$$\beta_i(x^0, x^*, -1) \leq \varphi < \varepsilon.$$

Therefore if we consider a perturbing row

$$C'_i = (c'_{i1}, c'_{i2}, \ldots, c'_{in}),$$

obtained by setting

$$c'_{ij} = \begin{cases} \sigma_i^* \delta_i, & \text{if } x_j^0 = 1,\ x_j^* = -1, \\ -\sigma_i^* \delta_i, & \text{if } x_j^0 = -1,\ x_j^* = 1, \\ 0 & \text{in other cases,} \end{cases}$$

$$\varphi < \delta_i < \varepsilon,$$

then we have $||C'_i||_\infty = \delta_i$, and taking into account (5), we derive

$$\sigma_i^*(C_i + C'_i)x^0 - \sigma_i^*(C_i + C'_i)x^* =$$
$$= \sigma_i^* C_i(x^0 - x^*) + 2\delta_i(N(x^*, x^0) + N(x^0, x^*)) \geq$$
$$\geq -|C_i(x^0 - x^*)| + \delta_i ||x^0 - x^*||_1 >$$
$$> -|C_i(x^0 - x^*)| + \beta_i(x^0, x^*, -1)||x^0 - x^*||_1 = 0,$$
$$\sigma_i^*(C_i + C'_i)x^0 + \sigma_i^*(C_i + C'_i)x^* = \sigma_i^* C_i(x^0 + x^*) =$$
$$= |C_i(x^0 + x^*)| > 0.$$

Therefore we obtain

$$\sigma_i^*(C_i + C'_i)x^0 > \sigma_i^*(C_i + C'_i)qx^*,\ q \in Q.$$

From this, using (1), we find

$$|(C_i + C'_i)x^0| > |(C_i + C'_i)x^*|. \tag{7}$$

Note, that the inequality (7) is coordinated with condition $x^* \in Q^n \backslash \{x^0, -x^0\}$.

Case 2: $i \in K(x^0, x^*)$, $\beta_i(x^0, x^*, -1) > \beta_i(x^0, x^*, 1)$. Then under (4) we have

$$|C_i(x^* - x^0)| > 0,$$

$$\beta_i(x^0, x^*, 1) \leq \varphi < \varepsilon.$$

Therefore, constructing the row C_i' by the rule

$$c_{ij}' = \begin{cases} -\sigma_i^* \delta_i, & \text{if } x_j^0 = x_j^* = 1, \\ \sigma_i^* \delta_i, & \text{if } x_j^0 = x_j^* = -1, \\ 0 & \text{in the other cases,} \end{cases}$$

where $\varphi < \delta_i < \varepsilon$, we obtain $||C_i'||_\infty = \delta_i$ and, using (6), we derive

$$-\sigma_i^*(C_i + C_i')x^0 - \sigma_i^*(C_i + C_i')x^* = -\sigma_i^* C_i(x^0 + x^*) + 2\delta_i M(x^0, x^*) >$$

$$> -|C_i(x^0 + x^*)| + \beta_i(x^0, x^*, 1)||x^0 + x^*||_1 = 0,$$

$$-\sigma_i^*(C_i + C_i')x^0 + \sigma_i^*(C_i + C_i')x^* = \sigma_i^* C_i(x^* - x^0) = |C_i(x^* - x^0)| > 0.$$

Thus, the following inequalities hold:

$$-\sigma_i^*(C_i + C_i')x^0 > \sigma_i^*(C_i + C_i')qx^*, \ q \in Q.$$

Therefore, using (1), we obtain (7).

Case 3: $i \in K(x^0, x^*)$, $\beta_i := \beta_i(x^0, x^*, -1) = \beta_i(x^0, x^*, 1) = \alpha_i(x^0, x^*)$.

Consider two possible variants.

At first let $\beta_i = 0$. Then

$$C_i x^0 = C_i x^* = 0. \tag{8}$$

It is easy to see, that taking into account $x^* \neq \pm x^0$, we may choose $k, p \in N_n$ such that

$$x_k^* = x_k^0, \quad x_p^* \neq x_p^0.$$

Therefore, if we define the elements of the row $C'_i = (c'_{i1}, c'_{i2}, \dots, c'_{in})$ by

$$c'_{ij} = \begin{cases} x^0_k \delta_i, & \text{if } j = k, \\ x^0_p \delta_i, & \text{if } j = p, \\ 0 & \text{in other cases }, \end{cases}$$

where

$$0 \le \varphi < \delta_i < \varepsilon,$$

then we have, that $||C'_i||_\infty = \delta_i$ and under (8) the inequality (7) holds.

Now let $\beta_i > 0$. Then, repeating all the argumentations from case 1, we obtain (7).

Case 4: $i \in N_m \backslash K(x^0, x^*)$. Then, assuming $C'_i = (0, 0, \dots, 0) \in \mathbf{R}^n$, we have (7).

So we obtain the matrix C' with norm

$$||C'||_\infty = \max\{\delta_i : \ i \in N_m \setminus K(x^0, x^*)\} < \varepsilon.$$

Summarizing what has been proven in four cases we see, that for any $\varepsilon > \varphi$ there exists $C' \in \Omega(\varepsilon)$ such that $x^0 \notin P^m(C + C')$. Hence, $\rho^m(x^0, C) \le \varphi$.

Theorem is proved.

Remark 1. If we impose on $C = [c_{ij}] \in \mathbf{R}^{m \times n}$ the condition of preserving positivity for all its elements during perturbing, then the stability radius of $x^0 \in P^m(C)$ is equal to

$$\min\{\varphi, c_{\min}\},$$

where φ is the right side of (2), $c_{\min} = \min\{c_{ij} : \ (i, j) \in N_m \times N_n\}$.

Efficient solution x^0 is called stable if $\rho^m(x^0, C) > 0$ and special if the following condition holds:

$$\nexists \ x \in Q^n \backslash \{x^0, -x^0\} \ \ (f(x, C) \le f(x^0, C)).$$

The following statement follows directly from the theorem.

Corollary. *Solution $x^0 \in P^m(C)$ is stable if and only if it is special.*

Remark 2. As a rule (see, e. g., [22,26]) the strict efficiency (Smale optimality [28]) of a solution of a multicriteria discrete optimization problem is a sufficient condition of the solution stability. But, it is easy to see, that our problem does not have strictly efficient solutions. Nevertheless efficient solutions can be stable (see corollary).

References

[1] Ehrgott M., Gandibleux X. *A survey and annotated bibliography of multiobjective combinatorial optimization.* OR Spectrum, 2000, **22**, N 4, p. 425–460.

[2] Sergienko I.V. *Mathematical models and decision methods of discrete optimization problems.* Kiev, Naukova dumka, 1988 (in Russian).

[3] Sergienko I.V., Kozeratskaya L.N., Lebedeva T.T. *Stability investigation and parametric analysis of discrete optimization problems.* Kiev, Naukova dumka, 1995 (in Russian).

[4] Sergienko I.V., Shilo V.P. *Discrete optimization problems. Challenges, decision methods, investigations.* Kiev, Naukova dumka, 2003 (in Russian).

[5] Kozeratskaya L.N., Lebedeva T.T., Sergienko T.I. *Integer programming problems with a vector criterion. Parametric analysis and investigation of stability.* Soviet Mathematics Doklady, 1990, **40**, N 1, p. 98–100.

[6] Kozeratskaya L.N., Lebedeva T.T., Sergienko I.V. *Stability of discrete optimization problems.* Cybernetics and Systems Analysis, 1993, **29**, N 3, p. 367–378.

[7] Kozeratskaya L.N., Lebedeva T.T., Sergienko T.I. *Regularization of integer vector optimization problems.* Cybernetics and Systems Analysis, 1993, **29**, N 3, p. 455–458.

[8] Sergienko I.V., Lebedeva T.T., Semenova N.V. *Existence of solutions in vector optimization problems.* Cybernetics and Systems Analysis, 2000, **36**, N 6, p. 823–828.

[9] Lebedeva T.T., Sergienko T.I. *Comparative analysis of different types of stability with respect to constraints of a vector integer-optimization problem.* Cybernetics and Systems Analysis, 2004, **40**, N 1, p. 52–57.

[10] Lebedeva T.T., Semenova N.V., Sergienko T.I. *Stability of vector integer optimization problems with quadratic criterion functions.* Theory of Stochastic Processes, 2004, **10(26)**, N 3–4, p. 95–101.

[11] Lebedeva T.T., Semenova N.V., Sergienko T.I. *Stability of vector problems of integer optimization: relationship with the stability of sets of optimal and nonoptimal solutions.* Cybernetics and Systems Analysis, 2005, **41**, N 4, p. 551–558.

[12] Greenberg H.J. *An annotated bibliography for post-solution analysis in mixed integer and combinatorial optimization.* In: D.L. Woodruff, ed. Advances in Computational and Stochastic Optimization, Logic Programming and Heuristic Search, Boston, Kluwer Academic Publishers, 1998, p. 97–148.

[13] Sotskov Yu.N., Leontev V.K., Gordeev E.N. *Some concepts of stability analysis in combinatorial optimization.* Discrete Applied Mathematics. 1995, **58**, N 2, p. 169–190.

[14] Chakravarti N., Wagelmans A. *Calculation of stability radius for combinatorial optimization.* Operations Research Letters, 1998, **23**, N 1, p. 1–7.

[15] Libura M., van der Poort E.S., Sierksma G., van der Veen J.A.A. *Stability aspects of the traveling salesman problem based on k-best solutions.* Discrete Applied Mathematics, 1998, **87**, N 1–3, p. 159–185.

[16] Gordeev E.N. *Stability analysis in optimization problems on matroids in the metric l_1.* Cybernetics and Systems Analysis, 2001, **37**, N. 2, p. 251–259.

[17] Emelichev V.A., Girlich E., Nikulin Yu.V., Podkopaev D.P. *Stability and regularization of vector problems of integer linear programming.* Optimization, 2002, **51**, N 4, p. 645–676.

[18] Libura M., Nikulin Y. *Stability and accuracy functions in multicriteria combinatorial optimization problem with $\sum$–MINMAX and $\sum$–MINMIN partial criteria.* Control and Cybernetics, 2004, **33**, N 3, p. 511–524.

[19] Emelichev V.A., Krichko V.N. *Stability radius of an efficient solution to a vector quadratic problem of Boolean programming.* Computational Mathematics and Mathematical Physics, 2001, **41**, N 2, p. 322–326.

[20] Bukhtoyarov S.E., Emelichev V.A., Stepanishina Y.V. *Stability of discrete vector problems with the parametric principle of optimality.* Cybernetics and Systems Analysis, 2003, **39**, N 4, p. 604–614.

[21] Emelichev V.A., Krichko V.N., Nikulin Y.V. *The stability radius of an efficient solution in minimax Boolean programming problem.* Control and Cybernetics, 2004, **33**, N 1, p. 127–132.

[22] Emelichev V.A., Kuz'min K.G., Leonovich A.M. *Stability in the combinatorial vector optimization problems.* Automation and Remote Control, 2004, **65**, N 2, p. 227–240.

[23] Emelichev V.A., Kuz'min K.G. *Stability analysis of a strictly efficient solution of a vector problem of Boolean programming in the metric l_1.* Discrete Mathematics and Applications, 2004, **14**, N 5, p. 521–526.

[24] Emelichev V.A., Kuz'min K.G. *Stability radius for a strictly effective solution to a vector minimization problem for threshold functions in l_1 metric.* Cybernetics and Systems Analysis, 2004, **40**, N 3, p. 358–362.

[25] Emelichev V.A., Kuz'min K.G. *The stability radius of an efficient solution to a vector problem of Boolean programming in the l_1 metric.* Doklady Mathematics, 2005, **71**, N 2, p. 266–268.

[26] Emelichev V.A., Kuzmin K.G., Nikulin Yu.V. *Stability analysis of the Pareto optimal solution for some vector Boolean optimization problem.* Optimization, 2005, **54**, N 6, p. 545–561.

[27] Lawler E.L., Lenstra J.K., Rinnoy Kan A.H.G., Shmoys D.B. *Sequencing and scheduling: algorithms and complexity.* Handbooks in Operations Research and Management Science. Volume 4. Logistics of Production and Inventory, 1993, p. 445–522.

[28] Smale S. *Global analysis and economics V: Pareto theory with constraints.* Journal of Mathematical Economics, 1974, **1**, N 3, p. 213–221.

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