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Minimum convex partitions of multidimensional polyhedrons*

Ion Băţ

Abstract

In a normed space $\mathcal{R}^n$ over the field of real numbers $\mathbb{R}$, which is an α-space [26, 29], one derives the formula expressing the minimum number of d-convex pieces into which a geometric n-dimensional polyhedron can be partitioned. The mentioned problem has been kept unsolvable for more than 30 years. The special cases for $\mathcal{R}^2, \mathcal{R}^3$ lead to nontrivial applications [19, 20, 23, 28, 30].

Key Words: geometric n-dimensional polyhedron, d-convexity, point of local non-d-convexity, polyhedral complex, oriented polytope, dividing

Mathematics Subject Classification: 68U05, 52A30, 57Q05

1. Introduction

Let (X, d) be a metric space, and let $x_1, x_2 \in X$ be two arbitrary points of (X, d). By analogy with the classical definition of convex sets one introduces the notion of metric convexity depending on d [4, 6, 16, 26]. The set of points, denoted by $\langle x_1, x_2 \rangle$ and defined by

$$\langle x_1, x_2 \rangle = \{x : d(x_1, x_2) = d(x_1, x) + d(x, x_2)\},$$

is called a *metric segment* joining the points x_1 and x_2. A set $M \subset X$ is said to be *d-convex* if for any two points $x_1, x_2 \in M$ the metric segment $\langle x_1, x_2 \rangle \subset M$. It is easy to see that the intersection of two d-convex

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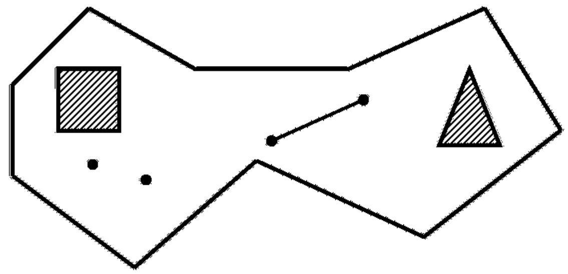

Figure 1.1.

sets is a d-convex set. For a given set $M \subset X$, the d-convex hull of the set M, denoted by *d-conv* M, is defined as the intersection of all d-convex sets containing M. In case that (X, d) is a normed space $\mathcal{R}^n$ over the field of real numbers $\mathbb{R}$ with $d(x_1, x_2) = \|x_1 - x_2\|$ every d-convex set is also a convex set, but not always conversely. Convexity and d-convexity in $\mathcal{R}^n$ coincide if and only if the closed unit ball of $\mathcal{R}^n$ is strictly convex [4, 6, 16, 26]. Thus this notions coincide in the Euclidean space $\mathbb{E}^n$. For a bounded set $N \subset \mathcal{R}^n$ it can happen that *d-conv* $N = \mathcal{R}^n$. We will only consider those normed spaces such that *d-conv* N is bounded, that is, so-called α-spaces [26, 29].

In the papers [5, 19, 20, 23, 28, 30] it is given sufficient information of solving the following problem.

Let $\mathcal{R}^2$ be a normed plane, and let P^2 be an open polygon (see Figure 1.1) with g holes of dimension $d \in \{0, 1, 2\}$ all of whose edges are d-convex.

In the paper [3] $\mathcal{R}^2$ coincide with the Euclidean plane $\mathbb{E}^2$, and the edges of the polygon P^2 are parallel only to two perpendicular directions while the all holes are of dimension 2. In this case it is shown that the minimum number $q(P^2)$ of rectangles partitioning the polygon P^2 is

$$q\left(P^2\right) = \frac{s}{2\pi} - h, \tag{1.1}$$

where s is the total sum of interior angles of the polygon P^2, measured in radians, and h is the maximum number of mutually disjoint segments that can be drawn within the closure of the polygon P^2, parallel to the edges of P^2, and with the endpoints at the concave vertices. This problem appeared in VLSI engineering [25].

In the papers [5, 20, 23] the problem to partition the polygon P^2 (see Figure 1.1) into a minimum number of d-convex pieces is completely solved. The respective formula is shown to be

$$q\left(P^2\right) = m + 1 - g - h, \tag{1.2}$$

where m, g are the total sum of all measures of local nonconvexity of points of local nonconvexity [5, 20], the number of holes of the polygon P^2, respectively, and h is the number of elements of a maximum concordant system of dividing trees [5, 20]. Considering $\mathcal{R}^2$ with the norm $\|x\| = |x_1| + |x_2|$, it is easy to obtain that q in (1.1) and q in (1.2) are the same number for the case from [19].

Let P^3 be an open polyhedron P^3 in the Euclidean space $\mathbb{E}^3$ with the edges parallel to the coordinate axes of $\mathbb{E}^3$. An approximate formula expressing the minimum number of parallelepipeds $q\left(P^3\right)$ into which the polyhedron P^3 can be partitioned is proposed in the paper [28]. Moreover, it is constructed an instance of a polyhedron such that the approximate computed number of parallelepipeds is too large with respect to the minimum number. This polyhedron has the shape indicated in Figure 1.2. The additional researches led to the fact that the minimum estimation of $q\left(P^3\right)$ required the methods of algebraic topology to be applied, as it would be seen below.

2. Auxiliary elements

In a normed space $\mathcal{R}^n$ it is possible to define the notion of a geometric n-dimensional polyhedron in a simpler or more complicated way. We will introduce a more natural notion of a polyhedron as the geometric polyhedron in $\mathcal{R}^3$ [11] is defined. For a given set $N \subset \mathcal{R}^n$, we will denote by $bd\, N, int\, N, \overline{N}$ the boundary, the interior and the closure of N, respectively. By $\Sigma^n(x, \varepsilon) \subset \mathcal{R}^n$ we denote the closed ball with center at x and radius ε.

By analogy with [8] we propose

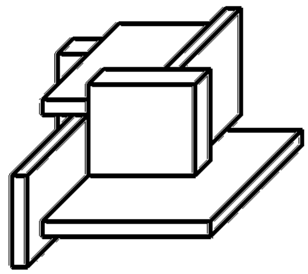

Figure 1.2.

Definition 2.1. *A closed n-dimensional PL manifold* [13, 15, 24] *which admits a decomposition into q handles of index* 1 [24] *is said to be a* ***geometric n-dimensional polyhedron of genus q*** *in the normed space $\mathcal{R}^n$. It is denoted by P_q^n.*

Definition 2.2. [26, 29] *A normed space $\mathcal{R}^n$ is called an* ***α-space*** *if for every bounded set $N \subset \mathcal{R}^n$ the d-convex hull of N is bounded.*

The necessary and sufficient condition for $\mathcal{R}^n$ to admit the mentioned situation consists in the fact that there exist n d-convex $(n-1)$-dimensional subspaces $L_1, \ldots, L_n \subset \mathcal{R}^n$ such that $\bigcap\limits_{k=1}^{n} L_k = 0$ [26, 29].

Let $\mathcal{R}^n$ be an α-space, and let $P^n \subset \mathcal{R}^n$ be a geometric polyhedron.

Definition 2.3. *The point $x \in bd\, P^n$ is called a* ***point of local non-d-convexity*** [4, 23, 30] *of P^n if for any sufficiently small $\varepsilon > 0$ the intersection d-conv $\Sigma^n(x, \varepsilon) \bigcap P^n$ is a non-d-convex set.*

Definition 2.4. *A m-dimensional face F^m* [7, 24, 26] *of the polyhedron P^n, $m = 0, \ldots, n-1$, is called a* ***face of local non-d-convexity*** *if any point in F^m is local non-d-convex.*

Definition 2.5. [12, 17, 24] *A finite set K of convex polytopes in $\mathcal{R}^n$ is called a* ***polyhedral complex*** *in $\mathcal{R}^n$ if K satisfies the following two conditions:*

1. *if $Q_1 \in K$ and Q_2 is a face of Q_1, then $Q_2 \in K$;*

2. *for every* $Q_1, Q_2 \in K$ *the intersection* $Q_1 \bigcap Q_2$ *is a common face of* Q_1 *and* Q_2.

Definition 2.6. [12, 17] *The set of all points belonging to convex polytopes in the polyhedral complex* K*, endowed with the induced topology of* $\mathcal{R}^n$*, will be called the* ***underlying space*** *of* K *and will be denoted by* $|K|$.

Definition 2.7. *The* ***dimension*** *of* K *is the maximum of the dimensions of its polytopes. We will denote a* n*-dimensional polyhedral complex by* K^n.

In that follows, a subscript in the name of mathematical objects denotes their dimension. To avoid overusing the word "convex" we adopt the convention that polytopes are always assumed to be convex unless otherwise stated.

It is known that any n-dimensional vector space $\mathbb{V}^n$ has two possible orientations, the orientation of the space being determined by the choice of a basis in this space [10, 14]. Let's remark, that the concept of orientation is essentially connected by that the base is considered as the ordered system of vectors.

Definition 2.8. *By an* ***orientation*** *of a* m*-polytope* Q^m *in* $\mathcal{R}^n$*,* $m = 1, \dots, n$*, we mean an orientation of the parallel subspace of the affine hull of* Q^m*. A polytope with one of two possible orientations is called an* ***oriented polytope****.*

By definition we consider that a 0-polytope has also two orientations. If Q^m is an oriented polytope, then $-Q^m$ will mean the polytope with the other orientation. We will say that Q^m and $-Q^m$ have opposite orientations. The orientation of the polytope Q^m can be given by the vectors $e_1, \dots, e_m$ of some basis for the parallel subspace of the affine hull of Q^m. Let $L(e_1, \dots, e_m)$ denote the m-dimensional vector space spanned by $e_1, \dots, e_m$.

Given an oriented m-polytope Q^m in $\mathcal{R}^n$, let Q^{m-1} be an oriented $(m-1)$-face of the polytope Q^m, and let $e_1, \dots, e_m$ and $e'_1, \dots, e'_{m-1}$ be the bases, respectively. Let's expand the system of vectors $e'_1, \dots, e'_{m-1}$

by adding to it a vector e'_m in $L(e_1, \ldots, e_m)$ directed to the open half space containing $int\, Q^m$ and determined by the affine hull of Q^{m-1}. It is obvious that the system of vectors $e'_1, \ldots, e'_{m-1}, e'_m$ forms a basis for the vector space $L(e_1, \ldots, e_m)$.

Definition 2.9. *The polytopes Q^m and Q^{m-1} in $\mathcal{R}^n$ are said to be* ***coherently oriented*** [12] *if the bases $e_1, \ldots, e_m$ and $e'_1, \ldots, e'_{m-1}, e'_m$ for the vector space $L(e_1, \ldots, e_m)$ are equally oriented, otherwise they are* ***noncoherently oriented****.*

The above definition is correct since the relation for two bases to be equally oriented is an equivalence relation [10]. The orientation of an $(m-1)$-face of the polytope Q^m coherent with the orientation of Q^m is also called an *orientation induced by* the orientation of the polytope Q^m. We call a polyhedral complex with all its polytopes oriented an *oriented polyhedral complex*.

Definition 2.10. [12, 18] *Let K^n be an oriented polyhedral complex. For each pair of polytopes Q_i^m, Q_j^{m-1}, the* ***incidence number*** $\left[Q_i^m : Q_j^{m-1}\right]$ *is defined as follows:*

$$\left[Q_i^m : Q_j^{m-1}\right] = \begin{cases} 0, & \text{if } Q_j^{m-1} \text{ is not a face of the polytope } Q_i^m; \\ +1, & \text{if } Q_j^{m-1} \text{ is a face of the polytope } Q_i^m \\ & \text{coherently oriented with } Q_i^m; \\ -1, & \text{if } Q_j^{m-1} \text{ is a face of the polytope } Q_i^m \\ & \text{noncoherently oriented with } Q_i^m. \end{cases}$$

Let K^n be an oriented polyhedral complex. We denote by $\mathcal{L}^m$ and α_m, $0 \le m \le n$, the set of all m-polytopes in K^n and the cardinality of this set, respectively. $\mathbb{Z}$ denotes the group of integral numbers.

Definition 2.11. [12, 18] *By an* ***m-dimensional chain*** *of the complex K^n we mean a mapping $c^m : \mathcal{L}^m \to \mathbb{Z}$. For any m-polytope $Q_i^m \in \mathcal{L}^m$, let $c^m(Q_i^m) = g_i$.*

Definition 2.12. *The set of all m-chains of the complex K^n, denoted by $C^m(K^n)$, forms an abelian group with respect to the following addition*

$$(c_1^m + c_2^m)(Q_i^m) = c_1^m(Q_i^m) + c_2^m(Q_i^m), \quad c_1^m, c_2^m \in C^m(K^n),\ Q_i^m \in \mathcal{L}^m,$$

*called the **mth chain group** of the complex K^n.*

Henceforward, because the group $C^m(K^n)$ is isomorphic to the free abelian group [15, 31] generated by the oriented m-polytopes in K^n, for simplicity, we will use the notation:

$$c^m = g_1 Q_1^m + g_2 Q_2^m + \ldots + g_{\alpha_m} Q_{\alpha_m}^m.$$

Definition 2.13. [18] *For a polytope $Q_i^m \in K^n$, the expression*

$$\sum_{Q_j^{m-1} \in K^n} \left[Q_i^m : Q_j^{m-1}\right] Q_j^{m-1}$$

*is called the **algebraic boundary** of the polytope Q^m and is denoted by $\partial^m Q_i^m$. If $m = 0$, we define $\partial^0 Q_i^0 = 0$.*

Definition 2.14. *The algebraic boundary determines the homomorphism*

$$\partial^m : C^m(K^n) \to C^{m-1}(K^n),$$

namely,

$$\partial^m \left(\sum_{i=1}^{\alpha_m} g_i Q_i^m \right) = \sum_{i=1}^{\alpha_m} g_i \left(\partial^m Q_i^m \right) = \sum_{j=1}^{\alpha_{m-1}} \left(\sum_{i=1}^{\alpha_m} g_i \left[Q_i^m : Q_j^{m-1} \right] \right) Q_j^{m-1},$$

*called the **boundary operator**.*

Theorem 2.1. *For every m-chain of the complex K^n,*

$$\partial^{m-1} \partial^m (c^m) = 0.$$

Definition 2.15. *The kernel of the homomorphism* ∂^m *is denoted by* $Z^m(K^n)$, *and the image of the homomorphism* ∂^{m+1} *will be denoted by* $B^m(K^n)$. *The group* $Z^m(K^n)$ *is called the* ***mth cycle group*** *of the complex* K^n, *and its elements are called* ***m-dimensional cycles***. *The group* $B^m(K^n)$ *is called the* ***mth boundary group*** *of the complex* K^n, *and its elements are called* ***m-dimensional boundaries***.

For example, in Figure 1.2, the boundary of the polyhedron P^3 regarded as the underlying space of a polyhedral complex consisting of rectangles contains a 2-cycle.

Theorem 2.2. *The groups* $Z^m(K^n)$ *and* $B^m(K^n)$ *are free abelian normal subgroups of the group* $C^m(K^n)$. *The group* $B^m(K^n)$ *is a normal subgroup of the group* $Z^m(K^n)$.

Definition 2.16. *The quotient group* $H^m(K^n) = Z^m(K^n)/B^m(K^n)$ *is the* ***mth direct homology group*** *of the polyhedral complex* K^n *with coefficients in the group of integral numbers.*

Definition 2.17. *The number* $\chi(K^n) = \sum\limits_{m=0}^{n}(-1)^m\alpha_m$ *is called the* ***Euler-Poincaré characteristic*** *of the polyhedral complex* K^n.

Theorem 2.3 (Euler-Poincaré). [18, 21] *Let* β_m *be the rank of the mth direct homology group of the polyhedral complex* K^n *with coefficients in* $\mathbb{Z}$. *Then it holds that*

$$\chi(K^n) = \sum_{m=0}^{n}(-1)^m\alpha_m = \sum_{m=0}^{n}(-1)^m\beta_m.$$

3. Main theorem

Let $\mathcal{R}^n$ be an α-space, and let $P_q^n \subset \mathcal{R}^n$ be a geometric n-polyhedron of genus q all of whose 1-faces belong to the d-convex lines of $\mathcal{R}^n$. We will denote by $p\left(P_q^n\right)$ the minimum number of d-convex pieces into which P_q^n can be partitioned.

Let X be the set of all local non-d-convex $(n-2)$-faces of the polyhedron P_q^n, and let $|X|$ be the set of all points of the faces. By D^{n-1}

and $\left|D^{n-1}\right|$ we will denote a finite set of cells [9, 12, 15] of dimension $\leq n-1$ in the α-space $\mathcal{R}^n$, belonging to the interior of P_q^n, and the set of all points of the cells, respectively.

Definition 3.1. *A set D^{n-1} is called a* ***dividing*** [1 – 3, 27] *of the polyhedron P_q^n if D^{n-1} satisfies the following two conditions:*

1. *for every $x \in \overline{|D^{n-1}|}$ there exists an $\varepsilon > 0$ such that the intersection $\left(P_q^n \setminus \overline{|D^{n-1}|}\right) \bigcap d\text{-}conv\, \Sigma^n(x, \varepsilon)$ consists only of d-convex connection components;*
2. *$|X| \subset \overline{|D^{n-1}|}$.*

Definition 3.2. *The number $\chi\left(D^{n-1}\right) = \sum\limits_{i=0}^{n-1} (-1)^i \alpha_i$ will be called the* ***Euler-Poincaré characteristic of the dividing*** *D^{n-1}, where α_i is the number of cells of dimension i of D^{n-1}.*

The Euler-Poincaré characteristic is an integer invariant for $\left|D^{n-1}\right|$. This fact results from the definition of the dividing.

By *dvz* P_q^n we denote the set of all dividings of the polyhedron P_q^n.

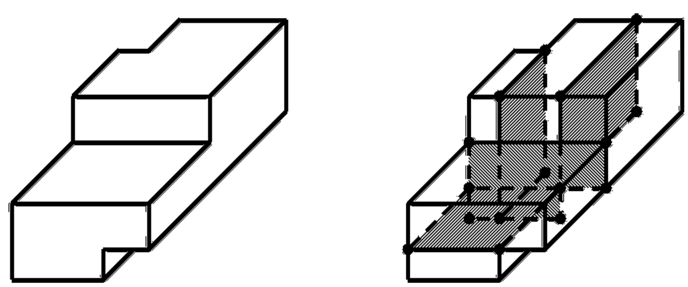

Figure 3.1.

Figure 3.1 displays a polyhedron P_0^3 on the left and a dividing D^2 of the polyhedron on the right (the hatched area). The dividing is composed of the four 2-cells and so $\chi\left(D^2\right) = 0 - 0 + 4 = 4$.

Theorem 3.1. *The Euler-Poincaré characteristic of the polyhedron* P_q^n *satisfies the property:*

$$\chi\left(bd\,P_q^n\right) - \chi\left(P_q^n\right) = (-1)^{n-1}(1-q). \tag{3.1}$$

Proof. From the definition of the Euler-Poincaré characteristic of a finite cell complex [9, 12, 15], representing the given polyhedron and containing among its $(n-1)$-cells open secant balls one for each handle, it follows immediately that

$$\chi\left(P_q^n\right) = \chi\left(bd\,P_q^n\right) + (-1)^{n-1}q + (-1)^n.$$

This completes the proof. □

Theorem 3.2 (Main Theorem). *For the polyhedron* $P_q^n \subset \mathcal{R}^n$ *the equality*

$$p\left(P_q^n\right) = (-1)^{n-1}\left(\chi\left(bd\,P_q^n\right) - \chi\left(P_q^n\right)\right) + \min_{D^{n-1}\in dvz\,P_q^n}\left|\chi\left(D^{n-1}\right)\right|.$$

holds.

Proof. Let D^{n-1} be a dividing of the polyhedron P_q^n. This dividing determines a finite cell n-complex K^n representing the polyhedron P_q^n and whose n-cells are open d-convex polytopes. Indeed, from the definition of the dividing, the set of all points belonging to the closures of cells of the dividing D^{n-1} is the underlying space of a polyhedral $(n-1)$-complex M^{n-1}. Moreover, the set of the faces of M^{n-1}, each of which belongs to the boundary of P_q^n, determines a division of the boundary into d-convex polytopes. Denote by L^{n-1} the polyhedral $(n-1)$-complex formed by this division (a subdivision of this division, preserving the faces of M^{n-1}, if it is necessary). The open d-convex polytopes of the polyhedral complex $M^{n-1}\bigcup L^{n-1}$ together with the connection components C_i of the set $int\,P_q^n \setminus \left|M^{n-1}\bigcup L^{n-1}\right|$ forms the required cell decomposition. The connection components C_i are open, local d-convex, so they are also d-convex. Thus we have $\chi\left(P_q^n\right) = \chi\left(K^n\right)$ and

$\chi\left(bd\,P_q^n\right) = \chi\left(L^{n-1}\right)$. From the Euler-Poincaré theorem, it is clear that

$$\chi\left(K^n\right) = \sum_{i=0}^{n}(-1)^i\alpha_i, \tag{3.2}$$

where α_n is the number of d-convex pieces into which P_q^n is partitioned, and α_i represents the number of open d-convex i-polytopes of K^n, $i = 0, 1, \ldots, n-1$. Rewrite (3.2) as follows

$$(-1)^n\alpha_n = \chi\left(K^n\right) - \sum_{i=0}^{n-1}(-1)^i\alpha_i. \tag{3.3}$$

Whence

$$(-1)^n\alpha_n = \chi\left(K^n\right) - \sum_{i=0}^{n-1}(-1)^i\alpha_i' - \sum_{i=0}^{n-1}(-1)^i\alpha_i'', \tag{3.4}$$

where α_i' is the number of open d-convex i-polytopes belonging to the boundary of P_q^n, and α_i'' is the number of open d-convex i-polytopes belonging to the dividing D^{n-1}. Therefore we get

$$(-1)^n\alpha_n = \chi\left(K^n\right) - \chi\left(L^{n-1}\right) - \chi\left(D^{n-1}\right). \tag{3.5}$$

Thus, both sides of the equality (3.5) being multiplied by $(-1)^n$, we obtain

$$\alpha_n = (-1)^n\chi\left(K^n\right) - (-1)^n\chi\left(L^{n-1}\right) - (-1)^n\chi\left(D^{n-1}\right). \tag{3.6}$$

Whence

$$\alpha_n = (-1)^{n-1}\left(\chi\left(bd\,P_q^n\right) - \chi\left(P_q^n\right)\right) + (-1)^{n-1}\chi\left(D^{n-1}\right). \tag{3.7}$$

The Euler-Poincaré characteristic of the dividing D^{n-1} is positive for odd n and is negative for even n in view of the fact that the relations (3.1), (3.7) and the inequality $\alpha_n > 0$ hold. Therefore we get $\chi\left(D^{n-1}\right) = (-1)^{n-1}\left|\chi\left(D^{n-1}\right)\right|$. If the dividing D^{n-1} is chosen such that the value of $\left|\chi\left(D^{n-1}\right)\right|$ to be minimum, then α_n is minimum, too. Hence we obtain

$$p\left(P_q^n\right) = (-1)^{n-1}\left(\chi\left(bd\,P_q^n\right) - \chi\left(P_q^n\right)\right) + \min_{D^{n-1}\in dvz\,P_q^n}\left|\chi\left(D^{n-1}\right)\right|,$$

and the theorem is proved. □

Corollary 3.1. *Let $P_q^n \subset \mathcal{R}^n$ be a geometric n-polyhedron of genus q. Then*

$$p\left(P_q^n\right) = 1 - q + \min_{D^{n-1}\in dvz\,P_q^n}\left|\chi\left(D^{n-1}\right)\right|.$$

Applying the above formula for the polyhedron from Figure 1.2 gives us $p\left(P_0^3\right) = 1 - 0 + 6 = 7$.

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Ion Băţ,

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Faculty of Mathematics and Computer Science
Moldova State University, MD-2009 Chisinau
Republic of Moldova
E–mail: *1i2o1n3b@gmail.com*