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Constructing a uniform plane-filling path in the ternary heptagrid of the hyperbolic plane

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Abstract

In this paper, we distinguish two levels for the plane-filling property. We consider a simple and a strong one. In this paper, we give the construction which proves that the simple plane-filling property also holds for the hyperbolic plane. The plane-filling property was established for the Euclidean plane by J. Kari, see [2], in the strong version. We also give an application of the construction to devise a Peano curve in the hyperbolic plane.

Keywords: hyperbolic plane, tilings, tiling problem, plane-filling property, Peano curve.

1 Introduction

Consider a finite set of tiles T based on a regular polygon of the hyperbolic plane. We say that there is a **solution** for tiling the hyperbolic plane with tiles of T, if and only if there is a partition $\mathcal{S}$ of the hyperbolic plane such that the closure of each part of $\mathcal{S}$ is a copy of some tile of T, where a copy of the figure F is an isometric image of F. We may adjoin conditions with colours on the edges: we then require that adjacent tiles always define the same colour on their common edge.

The **simple plane-filling property** consists in finding a finite set of tiles T with the following properties:

(i) for each tile τ of T, exactly two edges of τ are marked; the mid-points of these edges define an arc in τ which we call a **path element**;

(ii) there is a solution of the tiling problem of T such that the path elements are abutted into a single path.

Note that due to the condition (i), the path is not a cycle. Also note that, in the formulation of the problem, the set T does not define an initial tile.

In this case, the path defined by the tiling is called a **uniform plane-filling path**. Note that, both for regular grids of the Euclidean or the hyperbolic planes, it is not difficult to construct paths of the plane which visit each tile exactly once, when starting from a distinguished tile of T.

The **strong** plane-filling property consists in finding a finite set satisfying the simple plane-filling property together with an additional condition:

(iii) for any solution of the tiling problem of T, the path elements are abutted into a single path.

In other words, in case of the strong property, any solution for tiling the plane with T defines a uniform plane-filling path. Note that the plane-filling path defined in this way may then be different from one solution to another.

Our construction defines a uniform plane-filling path in the hyperbolic plane and the generating finite set of tiles almost possesses the strong plane-filling property. All its solutions generate a plane-filling path, expected in one case. In that case, the path elements constitute a countable family of disjoint infinite paths whose union visits each tile exactly once. However, this solution can be seen as a limit case of the other solutions. All these paths, but two ones, π_1 and π_2 can be joined at infinity and the new path and π_1 join π_2 at infinity. In some sense, these paths are the trace of a unique path also visiting the points at infinity.

As mentioned in our abstract, the Euclidean plane satisfies the strong plane-filling property, established by J. Kari, see [2]. Accordingly, the paper shows that the hyperbolic plane also possesses the simple version of the property. Call **ternary heptagrid**, see [3], the tiling of the hyperbolic plane based on the regular heptagon with the angle $\dfrac{2\pi}{3}$. We prove the following property.

Theorem 1 *There is a uniform plane-filling path for the tiling of the ternary heptagrid of the hyperbolic plane.*

We also prove another property:

Theorem 2 *There is a cellular automaton on the ternary heptagrid which constructs a uniform plane-filling path in infinite time.*

The existence of such a path was required for proving a property on cellular automata. Say that a cellular automaton is reversible if and only if its global transition function is bijective and also defined by a cellular automaton. From the plane-filling property which he established, J. Kari proved that it is undecidable to decide whether a cellular automaton on the Euclidean plane is reversible or not, see [2]. The similar question for cellular automata in the hyperbolic plane is open.

Our construction relies on the construction which we defined in [5, 8] in order to establish that the domino problem is undecidable in the hyperbolic plane. We very sketchily remind this construction in section 2, mainly reminding what is needed for the present construction.

In section 3, we indicate the construction of new triangles, the **mauve** triangles which we shall use for guiding the travel of the path. In section 4, we describe the construction of a uniform plane-filling path. We prove that the hyperbolic plane almost possesses the strong property. We also prove that there is a cellular automaton on the tiling $\{7,3\}$ which is able to construct a uniform plane-filling path, of course, in infinite time.

In section 5, we give an application to an algorithmic construction of a Peano curve in the hyperbolic plane. However, we have a simpler construction of such a curve which we shall give in a forthcoming paper.

2 The underlying construction

The tiling which we use is based on the **ternary heptagrid**, the tessellation $\{7,3\}$ of the hyperbolic plane. We remind that it is generated by reflection of a regular heptagon with interior angle $\dfrac{2\pi}{3}$ in its edges and, recursively, of the images in their edges. An illustration of this tiling is given by figure 1, and we refer the reader to [3] for the properties of the tiling used in this paper.

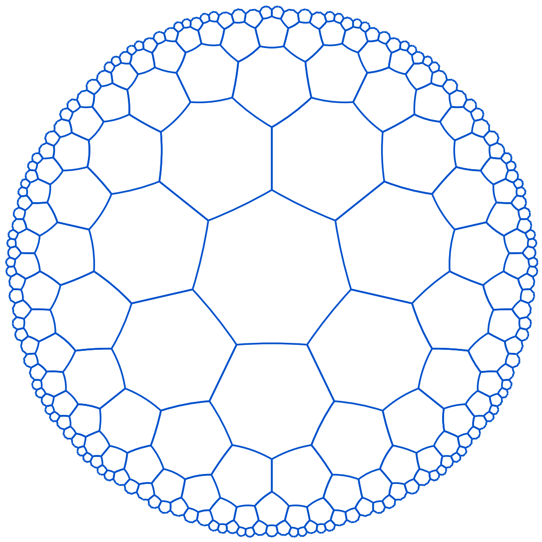


Figure 1 *The tiling $\{7,3\}$ of the hyperbolic plane in the Poincaré's disc model.*

In this tiling, we introduced an auxiliary tiling, the **mantilla**, which was first defined in [4] and which is used in [5, 8] to prove the undecidability of the tiling problem in the hyperbolic plane.

The construction of the mantilla is based in fixing rules to assemble two kinds of tiles: the **centres** and the **petals**, the tiles α and β of figure 2, respectively. By numbering the edges of the centres from 1 up to 7, we prevent the centres to tile the plane by themselves, alone. It is needed to put petals around them.

Now, we can rule the way in which petals are put around a centre, making a figure which we call a **flower**. We define four types of flowers. Figure 3 indicates three types of them: the types F, G and **8**. Now, the type G has two variants, which we call G_ℓ and G_r, respectively, which are in some sense symmetric. The distinction is a consequence of the numbering of the edges which is the same in both cases.

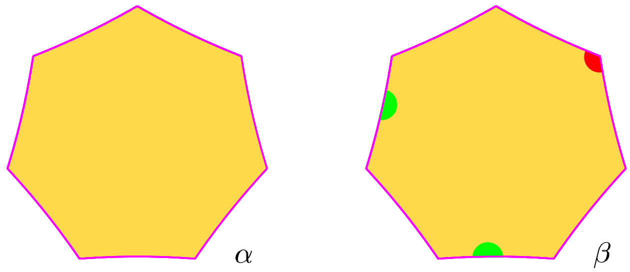


Figure 2 *Left-hand side: the tile for the centres of the flowers. Right-hand side: the tile for the petals.*

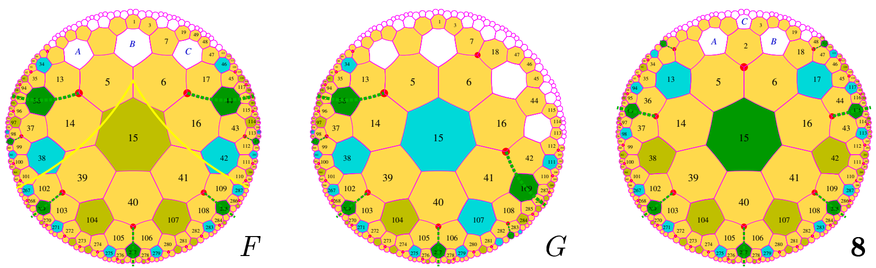


Figure 3 *Splitting of the sectors defined by the flowers. From left to right: an F-sector, G-sector and* **8***-sector.*

These figures also display the way in which each sector determined by a flower is split in such a way that in each sector, the complement of its defining flower can be expressed in F- and G-sectors, with the help of half **8**-sectors. The green rays of the three pictures in figure 3 indicate this splitting. This defines a recursive process to generate the mantilla. The algorithm is deterministic when we proceed downwards and it is non-deterministic when we proceed upwards.

A last ingredient consists in introducing **isoclines**, which play the rôle of horizontals in the Euclidean plane. The levels are illustrated by figure 4, below. They are defined by fixing the **8**-centres as black nodes in the sense given to the nodes of Fibonacci trees, see [3, 5]. Four other cases appear outside those indicated in figure 4, we refer the reader to [5].

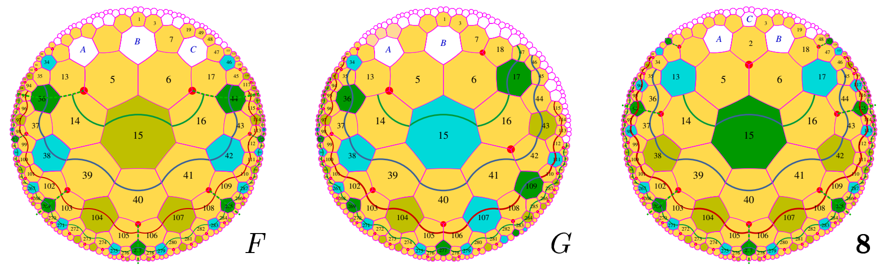


Figure 4 *Splitting of the sectors defined by the flowers. From left to right: an F-sector, G-sector and* **8***-sector.*

These isoclines are very important: they are the basis of the construction of the **interwoven triangles** which we need for proving theorem 1. These triangles were introduced in [5, 8] in order to prove the undecidability of the tiling problem in the hyperbolic plane.

Now, we sketchily indicate how to construct these triangles in the Euclidean plane. First, we have a line of light blue equal triangles, as it can be seen in figure 5. They are isosceles and their main heights are supported by the same line, the **axis**. Note that triangles with thick edges alternate with triangles with thin edges. Following [5, 8], we call **phantoms** the triangles with thin edges. The triangles and phantoms which we just described constitute those of the generation 0. Now, the colours of the generation will alternate between red and blue, which will be a medium blue, and say that red and blue are **opposite** to each other. Assume that we constructed the generation n. We fix a triangle of the generation n and, at the mid-point of its main height, on the axis, we put the vertex of a triangle of the opposite colour, with respect to that of the generation n. Then, we construct an isosceles triangle T whose height is supported by the axis, in such a way that its basis crosses the main height of the next triangle of generation n. For simplicity, we may assume that the legs of T are parallel to the corresponding legs of the triangles of generation n which are all equal and whose legs are also parallel. We replicate T by shifts along the axis in such a way that we obtain an alternation of triangles and phantoms

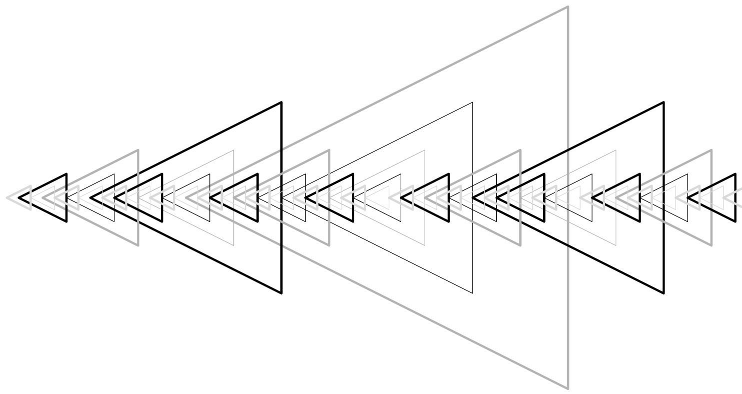


Figure 5 *An illustration for the interwoven triangles. The dark strokes are for red and the light ones for blue.*

of the same colour as T and such that a vertex of a phantom is on the mid-point of a basis of a triangle and a vertex of a triangle is on the mid-point of a basis of a phantom. These triangles and phantoms, T being included, constitute the generation $n+1$. Figure 5 illustrates this point.

Our last step is to implement these triangles in the hyperbolic plane. We are faced with three problems.

First, the choice of the place of the triangles with respect to those of the previous generation generates a continuous number of solutions. Let us call **infinite model** a given way to fix the successive generations.

The second is to define what will be the supports of the triangles and the phantoms and what will be their axis. We have to leave the precise details to the quoted papers, [5, 8], but here, we still give an idea of the situation.

First we define the supports of the triangles and the phantoms. The isoclines which we defined are periodically numbered from 0 up to 19, the increasing numbers going downwards. The number 20 appears for technical reasons which are clearly explained in [5]. Now, the candidates for the support of the triangles are defined by Fibonacci trees, see [3], rooted at the F-son of a G-flower on an isocline 0, 5, 10 or 15. Now, not all the indicated such nodes, call them **seeds**, are allowed to generate a tree, in which case we say that the seed is **active**. However, all seeds on an isocline 0 are active. But, for the others, they are active only if they are inside a tree rooted at an active seed. This induces a tree of the active seeds on the isoclines 5, 10 and 15 which have the same ancestor inside a given tree rooted at an isocline 0. Such branches and their upward continuations are called **threads**. The legs of triangles are supported by the extremal branches of the trees rooted at active seeds and the role of the axis is played by the threads. This induces many problems.

It may happen that a thread traverses the hyperbolic plane. If this happens, the corresponding threads do coincide, starting from a certain point. We call such threads **ultra-threads**. All the other threads have the structure of a ray. Now, the existence of ultra-threads or not depends on the particular mantilla which we constructed.

The other point is that we better control the situation if the triangles and the phantoms of the same generation but on different threads have their vertices and bases on the same isoclines. In this case, we say that the triangles are **synchronized**. Synchronizing the triangles, of course, also the phantoms, boils down to consider that each thread implements the same model of interwoven triangles. Now, something must be made more clear. We can realize a whole infinite model along an ultra-thread. But, as a thread is bounded from above, this is not possible for an ordinary thread. In fact, we have to study what happens in an infinite model if we introduce a cut: we fix a line λ, orthogonal to the axis, we erase all triangles whose vertex is on the left-hand side of λ and we keep all of them which are on its right-hand side.

In [5, 8], we proved that by observing these constraints, we can obtain the synchronization of all the implementations of the cuts defined by the threads of the same infinite model. Note that as we have continuously many different realizations of the mantilla and continuously many different realizations of an infinite model, we have in fact continuously many implementations of the interwoven triangles.

At this point, let us note that if we could fix the infinite model followed by the interwoven triangle, we could skip the next section and directly go to the following one, making it much more simple. But one model, which cannot be avoided, requires the solution which we define in the mauve triangles.

Before turning to what is introduced for proving theorem 1, we insist on a particularity of the implementation: a triangle always contains several triangles of the previous generation on the same set of isoclines, which is another aspect of the synchronization.

3 The mauve triangles

Now, we turn to the construction of the regions which control the path which is defined in the next section.

3.1 Construction of the mauve triangles

To this purpose, we keep the red triangles only, but we keep in mind the red phantoms generated by their bases, as they play a role in the construction.

Now, to the red triangles, we superpose new triangles, which we call the **mauve** triangles. Each vertex of a red triangle is also the vertex of a mauve triangle and conversely. The legs of the mauve triangle are supported by the legs of the red triangles, but they go further, on the same extremal branch of the tree which defines the red triangle. The legs are stopped by the next isocline supporting a vertex of a red triangle of the same generation. In some sense, the length of the height of a mauve triangle is twice the length of the height of the red triangle sharing its vertex.

It is not difficult to construct the mauve triangles from the red triangles. Consider a red triangle T. The mauve triangle M defined by T is constructed as follows. The vertex of M is that of T and its legs follow those of T. When a leg of M arrives to the corner of T, this corner sends a **purple** signal along the basis of T in the direction of the other corner: this can easily be determined by the tiles which materialize the corners of T. When this signal reaches the first vertex of a phantom P, necessarily a red one and of the same generation of T, it goes on the leg of the phantom which is on the same side as the corner of T which it has left and it goes down along the leg of P until the corner of P. Then, it goes on its way on the same isocline as the

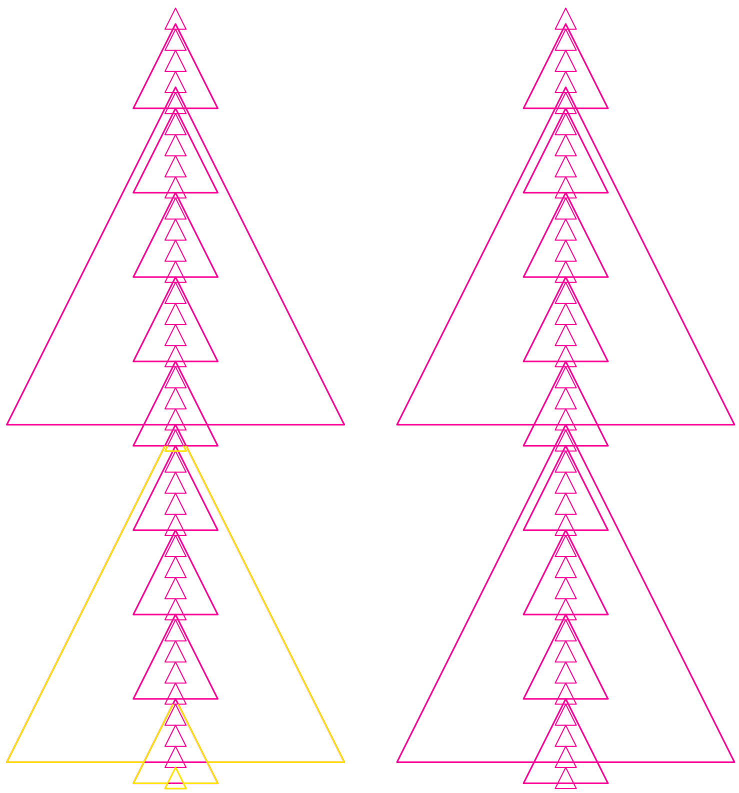


Figure 6 *An illustration for the mauve triangles.*

basis of P but on the direction which goes outside P. The purple signal goes on until it meets the mauve leg which has continued its way on the extremal branch of the tree supporting T, from the corner from which the purple signal originated.

From the point of view of the tiling, it is important to notice that the purple signal cannot be generated by a phantom which would be internal, in this sense that it would have a phantom on each side whose vertices lie on the basis of the same triangle. To realize this, it is enough to give a **laterality** to the purple signals: a purple signal inherits the laterality of the triangle corner from which it is originated. It is enough to forbid a joining tile to prevent the emission of a purple signal by a wrong phantom. Now, as a horizontal purple signal running on the isocline of the phantom and outside it must meet the leg of a mauve triangle, on the inside part of the leg, always clear from the tiles, see [5, 8], if the emitting triangle does not exist, which may happen, even if the phantom exists, then the purple signal will meet a leg of phantom of the opposite laterality: it is easy to rule out this.

Now, the purple signal has only a construction purpose. As it plays no more role, we shall forget it in the representations of the mauve triangles, see figure 6.

Presently, let us indicate the properties of the mauve triangle.

3.2 Properties of the mauve triangles

Using the terminology of the interwoven triangles, see [5, 8], we say that the set of isoclines crossed by a mauve triangle, the basis and the vertex being included, defines the **latitude** of the mauve triangle. Also, we know that red triangles have an odd index in the generations of the interwoven triangles. We shall say that a mauve triangle associated to a red triangles of generation $2n+1$ is of generation n. The first property is very important for the following:

Lemma 1 *Let τ be a tile of the tiling. Then for any non-negative n, there is a mauve latitude Λ of this generation such that $\tau \in \Lambda$. And then: either τ falls within a mauve triangle of generation n in this latitude or τ falls outside two consecutive mauve triangles of generation n and of the latitude Λ and in between them.*

This property follows immediately from the fact that the latitude of a mauve triangle exactly covers that of the corresponding red triangle and the following latitude of red phantoms.

However, there is a price to pay to this: the red triangles are either disjoint or embedded. Mauve triangles do intersect from one generation to another. Fortunately, this intersection is not that big and we characterize it in the following statement.

Lemma 2 *A mauve triangle T of positive generation n intersects mauve triangles of generation $n-1$, and it possibly intersects one mauve triangle of generation $n+h+1$, with $h \geq 0$. When the intersection occurs, i the legs of T cut the basis of the mauve triangle of the higher generation at a point which is on the mid-distance line of the phantoms of generation $2(n+h+1)+1$ which share their basis with that of T. Call* **low point** *this point on the legs of T. The basis of T is cut by the legs of mauve triangles of generation $n-1$ at their low points.*

The proof is easy and it comes from the relations of red triangles of consecutive generations. Representing the first three generations, figure 6 illustrates this property.

3.3 Determination of the low points

As we shall see in the next sections, the low points of a mauve triangle play an important role. Let us show that they can be determined from the tiles themselves.

Consider a mauve triangle T. Let R be the red triangle which shares its vertex with that of T and let P denote both, the leftmost and rightmost phantoms generated by the basis of T. From the above definition, we know that the low points of T lie on the isocline which supports the mid-distance line of P. Now, the leg of P which is on the same side as the closest mauve leg of T are covered by the purple signal with the laterality of the signal coinciding with that of the leg. On the part of this signal which runs on the leg of P, the mid-point is easily found. Accordingly, the purple signal sends a **dark purple signal** on the corresponding isocline, outside P. In between the leg of P and the leg of T, the dark purple signal will meet mauve legs of triangles of smaller generations, see figure 6. To avoid problems connected with possible nestings of such triangles, the purple dark signal looks at the laterality of the first mauve leg it meets: if it is of the same laterality as his own one, he found the appropriate leg. If not, it climbs along the mauve triangle until it meets its vertex and goes down on the other side: by induction, it is assumed that the low points of mauve triangles of previous generations have been determined. Assume that this is the case. Then the dark purple signal goes on along the right isocline, avoiding smaller mauve triangles possibly contained on those he jumped over. Now, it will meet a first mauve leg of its laterality which will be the expected one and so, it will determine the low point of this leg. And so, as the low point for the mauve generation 0 is easy to determine because it contains no mauve triangle, this process works and it can easily be implemented with finitely many tiles. Note that this process is similar to the one which we used in [5, 8], in order to synchronize horizontal signals travelling on certain isoclines.

Now, we can turn to the construction of the path.

4 A uniform plane-filling path

Until the last sub-section, we assume that all the mauve triangles of the tiling are **finite**. In the last section, we shall see what happens when this is no more the case.

The construction of the path is based upon two basic patterns which we now define.

4.1 The guidelines

The idea is to look at things globally, at the level of latitudes of mauve triangles. We have to check that we can construct the path as the result of an algorithmic process, infinite in time, punctuated by times t_k in such a way that at time t_{k+1} we fill up more space in the latitudes already visited up to time t_k and that at time t_{k+1}, we access to latitudes of higher generation with respect to those accessed up to time t_k.

We shall look at the tiling by making a rather rough approximation which will turn out to be good enough. This consists in looking at the mauve triangles as if there were no overlapping between different generations: we can look at latitudes as over disjoint or embedded. We shall call this the **first approximation**.

At the level of the latitudes, in first approximation, we have two figures: the triangle and the trapeze, see figure 7. The triangle is a mauve triangle whose height is that of the latitude and the trapeze is the part of a latitude which lies in between two consecutive mauve triangles of the latitude and of its generation, also delimited by the isoclines going through the vertices and the bases of these triangles.

There are two **accesses** of the path into the triangle or into the trapeze. They will be called **entries** or **exits**, depending on the way we look at the path which, by construction, is not oriented. In fact, looking at the figures from the left to the right, we can define two displays for the access. There is the **ascending** one: an access on the lower left-hand side corner and the other access at the vertex, for the triangle, looking to the right, and on the upper right-hand side corner for the trapeze. There is a **descending** display: an access on the lower right-hand side corner and the other access at the vertex, for the triangle, looking to the left, and on the upper left-hand side corner for the trapeze.

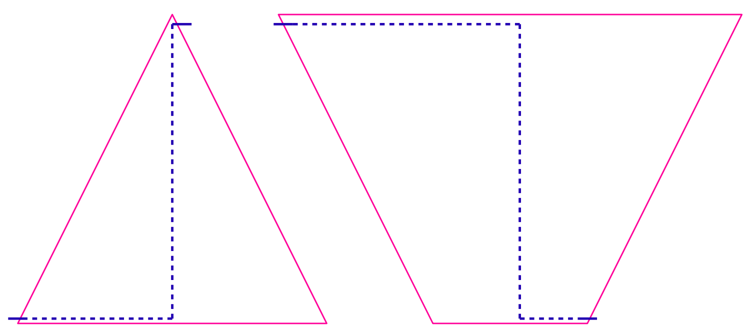


Figure 7 *The basic figures: triangle and trapeze within a latitude.*

Note that the descending figures of one kind match with the ascending ones of the other kind, also provided that they exactly fit within the same latitudes.

Now, the path cannot remain for ever within a given latitude of the mauve triangles. This is a consequence of lemma 1. Now, the trapezes and triangles which we define are bigger and bigger. If we fix a tile τ_0 once for all, then after a time t_{k_0}, the path completely contains a ball of radius n around τ_0 at time t_{k+n}. It is enough to define the t_k's by this condition which is satisfied by lemma 1: note that a trapeze is much much bigger than a triangle of the same latitude.

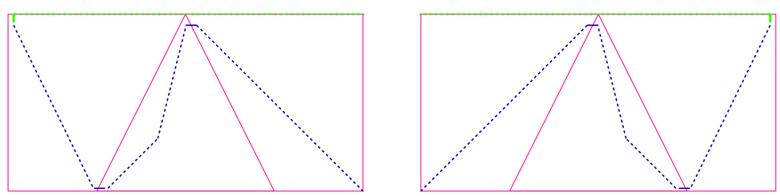
Now, we have to define the internal structure of the triangles and trapezes above defined. For this, we notice that inside a triangle, there are four latitudes of mauve triangles of the previous generation, and, in first approximation, these latitudes are disjoint.

We shall look at the way we can fulfill the requirement posed upon the triangles and the trapezes. For this, we shall introduce an intermediate picture which we call the **quadrangle**, as the word *rectangle* would be misleading in this context. The advantage of this figure is that we can look at it as the trace of a latitude either inside a triangle or inside a trapeze, see figure 8. We have two kinds of such quadrangles, corresponding to the main motion of the path in filling up this region. One version is the descending one, see picture (d) in figure 8. The second version is the ascending one, see picture (a) in the same figure. In the decomposition of a quadrangle, we again find triangles and trapezes, but of the previous generation, which are smaller. The decomposition is repeated until the generation 0 is reached.

Three precisions must be given about the quadrangles, see figure 8.

First, a quadrangle occurs both in a trapeze and a triangle. Note that the lateral sides of a quadrangle determine where it is in a given latitude. If the lateralities of these sides are identical, we are inside a triangle and outside a smaller one. If the lateralities are different, we have to look at whether they correspond to the position of these borders with respect to the region which they delimit. If the lateralities of the borders define their position, we are inside a triangle of the considered latitude. If the lateralities of the borders are opposite to their actual position, we are in a trapeze of the considered latitude.

The second point is about the dotted lines in figures 7, 8 and 9. In these figures, the blue dotted lines represent zig-zags going from one vertical border to the other while following an isocline. The zig-zag runs over all the isoclines of the considered areas, one after the other. In figure 8, the green dotted line represents a path along an isocline. The blue dotted-line is a zig-zag which is bordered by the green dotted line and by the concerned borders: they are legs of triangles, but this time, the place of the region bordered by the legs is defined by their lateralities.



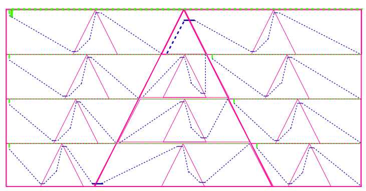
(*d*) (*a*)

Figure 8 *The splitting of the slices.*
On the left-hand side: the descending case. On the right-hand side: the ascending case.

Now, these slices can again be split into four horizontal **slices** defined by the four latitudes of mauve triangles of the just previous generation which are contained in the latitude of the slice which we consider. Figure 9 illustrates the result of this splitting for a slice.

The third point is that the slices almost follow the same pattern, possibly after reflection in a vertical axis. We simply notice that on one side, the lowest slice gives access to the triangle instead of putting the path on the next isocline. On the other side, the topmost slice also gives access to the triangle instead of going to the next isocline. Also, we have to note that the topmost region inside a triangle, see figure 9 is never a triangle. The representation of the figure is due to the distortion introduced by the Euclidean representation. The upmost slice inside a triangle is again a slice, even if the part outside the triangle seems to be much smaller. At last, we also notice that the green line of a slice is also the green line of the topmost slice of the previous generation which is just below the isocline of the line. As we assume that all mauve triangles are finite, such a line will always meet a leg of a triangle, not at a corner of the leg.

Also, a last point to notice is that inside a slice there are several triangles of the same generation within the latitude of the slice. In the figures, we represented a single one due to the Euclidean constraints. But in the hyperbolic plane there are a lot of them. Hower, from the figure itself, it is not difficult to see that the same pattern is followed inside the region delimited by two consecutive triangles: it is simply a trapeze.


(d)

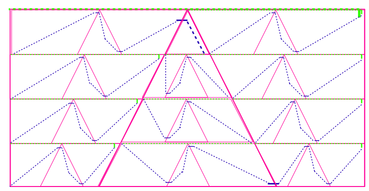

(a)

Figure 9 *The splitting of the slices: second generation. On the left-hand side: the descending case. On the right-hand side: the ascending case.*

At last, note that we have to determine the isoclines which delimit the different slices. The latitude of the slice is that of the mauve triangles of a certain generation. We proceed as follows. The first slice goes from the vertex of the mauve triangle to the mid-point of its supporting red triangle R. The second slice goes from this mid-point to the basis of R: again something which is easy to determine on the mauve leg. The third and the last slices are clearly determined by the low point and the corners of the mauve triangle. In sub-section 3.3 we have seen how to determine the low point on the leg of a mauve triangle.

It is not difficult to see that in this way, provided that the path meets legs of bigger triangles, which is the case from lemma 1 and from our assumption that there is no infinite mauve triangle, it will go further on the latitudes which it already visited and that it enters new latitudes. Thus, we can define times t_k satisfying the above conditions. Accordingly, the path fills up the plane and there is no initial tile.

Now, we have to look at the schema much closer as mauve triangles overlap, which leads us to precisely define the slices. We turn to this point in the next sub-section.

4.2 The tuning of the slices

If we look at the yellow frames of figure 6, we can see that the border of a triangle can be altered in different ways. First, its basis is crossed by many mauve triangles of smaller generations. Next, it is also possibly crossed by a mauve triangle of a bigger generation.

Assume that the mauve triangle T we consider is not crossed by a mauve triangle of a bigger generation. Then, we define the new slices by simply following the border of the mauve triangles which intersect the basis of T and, recursively, for the basis of each one of such triangles. This process terminates on a mauve-1 triangle, see figure 6. In between two triangles of the same generation as T, the lower border of a trapeze is also determined in a similar way: it recursively follows the lower part of mauve triangles crossing the isocline determined by the corner of the mauve triangle. Also, in this case, the lower exit/entry of T is at its left-hand side corner.

Now, consider a mauve triangle T_1 which is of the generation n, where $n{+}1$ is the number of the generation of T, and assume that the legs of T_1 cut the basis of T. In T_1, this intersection occurs along the isocline which passes through the low point of its legs. This isocline determines the upper border of the lowest slice of T up to the recursive detours caused by mauve triangles of smaller generations. Now, as the basis of T_1 is below the basis of T, placing the entry to T_1 at its left-hand side corner will force the path to cut itself. And so, to avoid this point, we place the entry of T_1 at the intersection of the left-hand side leg of T_1 with the basis of T: it is the left-hand side low point of T_1. Now, it is not difficult to adapt the schemes of the figure 7, 8 and 9 to the new situation. We repeat this new definition of the entry each time when a basis of a mauve triangle crosses the low point. Now, it is easy to determine whether the entry of a triangle is at its left-hand side low point or at its corner. The signal of the entry will be present at the low point if a basis is present. If a basis is not present, the low point sends a signal to its corner, below, in order to trigger the signal of the entry.

Now, it is easy to see that the same new definition of the lower entry applies when T is crossed by the basis of a bigger generation: it is namely crossed at its low points and so the left-hand side one becomes the entry of T. The lower border of the slice associated to T is the basis of the mauve triangle of a bigger generation which cuts T. Also note that, from the lower border of the slice which is just above T, we notice that the end of T around its vertex is cut by possibly a mauve triangle of a smaller generation, involving the same kind of 'embroidery' as in the lower border of the slice of T.

As a last point, we have to indicate that the vertex of a mauve triangle can no more be use for the exit from the triangle, unless it falls within a slice of the generation 0. In the other cases, the exit of triangle T of positive generation is determined by the lower border of the upper slice which cuts its legs. The exit occurs at the point which is on the leg, just below the border. The border is determined by the basis of a mauve triangle of the previous generation which cuts T.

As the lower border of a slice is also the upper border of the next slice, below, we completed the change which we had to introduce in order to take into account the overlappings between mauve triangles.

With these modifications, we can see that the travel of the path is globally the same as what we described in the previous sub-section.

4.3 About the tiles

To conclude the proof of theorem 1, we have to give a few details about the implementation of the just described algorithm in a finite set of tiles.

We remember that the tiles are heptagons on which various signals run, defining the colours of the edges of the tiles.

We have already all the signals inherited by the mantilla and by the construction of the interwoven triangles. To this, we append the signals for the construction of the mauve triangles and the determination of their low points.

The last point is to describe how the path finds its way from which the tiles can easily be deduced.

For this purpose, we remember that most often, the path performs zig-zags in between vertical borders, call them **walls**. In between two walls, the path runs along an isocline. It runs in one direction on one isocline and in the opposite direction on the next isocline. We can have the same colour for both directions. However, we shall have two colours: one for an ascending path and the other for a descending one. Note that in different realizations of the tiling, the colours may be interpreted in the opposite way: what is important is that we have two colours.

Also, to facilitate the implementation, we have to prevent going from one isocline to another, when the path is not at a wall. The wall is always materialized by a mauve leg. Now, the path meets the wall on both sides: when it is inside the corresponding mauve triangle and when it is outside. We decide that the legs of the mauve triangles are always reached by a path which is inside the triangle. As the leg must stop outside portions of the path going to it, in fact the tiles of a mauve leg have a mark on the side of the tile which is in contact with the outside of the triangle and which is on the isocline. Remember that in [5, 8], we assign a local numbering to the sides of a tile. We number the side shared with the father by 1 and the other sides are increasingly numbered when counter-clockwise turning around the tile. As any tile has a father, this fixes the local numbering of the tiles everywhere. Accordingly, on the left-hand side border, the mark for outside parts of the path is on the side 3 of a tile. On the right-hand side border, the mark is on the side 7.

We also have to note that the exact definition of the slices entail a distortion of the path. What is represented by horizontal lines in figures 8 and 9 is not always along an isocline. A few portions of the path go along a wall, stopped by the leg and we have to take this into account: this does not raise big difficulties, especially when we are outside the leg, but we have to be careful when we are inside. This point is masked by the Euclidean representation. We have to remember that inside a triangle, the number of tiles on an isocline from one leg to the other is divided by at least 2 when we go from an isocline to the previous one, in their numbering. There is a shrinking of the space which entails distortions. This requires to propagate the information of the presence of the wall as long as it is needed. This is not very difficult to realize: we have to take into account that, on each border, if the path goes to one tile from the leg on the isocline i, it cannot go closer than the fourth tile from the leg on the isocline $i-1$. The required distortion, in order that the path visits all tile is not difficult to realize: the 3 tiles left on the isocline $i-1$ have to be visited by the path on the isocline i, which is easy to realize. It is important to indicate that only one portion of the path goes in this 'parallel' way which follows a leg: the other parts are performed by zig-zags to which we apply the just indicated constraint, as long as the zig-zag do not again meet the leg, directly.

At last, we notice that the number of isoclines from a slice to another is always even, as this is the case for mauve-1 triangles. From this, there is no problem to apply the following scheme: the leaving part is always on the right isocline.

All the marks needed by the previous indications are easy to implement and require a finite number of tiles only. As we know, from [6], the number of tiles is huge, already for the construction of the interwoven triangles.

This completes the proof of theorem 1.

4.4 The case of infinite triangles

As indicated in our introduction, from what we proved in the previous sub-sections shows that each time we have a tiling which does not generate infinite mauve triangles, we get a uniform plane-filling path defined by the tiling.

But this is no more true if there is an infinite mauve triangle. Once the path falls inside such a triangle, it is trapped: later, it can never go outside the triangle.

Now, such a situation is possible. From [5, 8], we know that in the case of a realization of the butterfly model, there are four possible cases for the line 0 of the model: it may be accompanied by a blue basis, which brings no harm to our construction. It may be accompanied by a red basis. If it is a red basis of a triangle, it is an infinite red triangle, but the corresponding infinite mauve triangle is removed to infinity: it has no trace in the hyperbolic plane and so, this situation is also handled by our construction. If the line 0 is accompanied by a basis of a red phantom, this basis gives rise to red infinite triangles and, consequently, the red vertices of these triangles generate mauve triangles which are also infinite. And so, in this last case, we have infinitely many infinite mauve triangles.

Accordingly, the path is broken into infinitely many components. However, each component is a fully filling path of the region which is delimited by the component. Also, the infinite red basis of a phantom is also the basis of an infinite mauve triangle. Accordingly, as this basis never meet the leg of a mauve triangle, except at its corner, the green part of the path always runs on this isocline without any possibility to leave it. Above this basis, our construction provides us with a single path which visits all tiles exactly once. Denote by π_1 the path which runs along the infinite mauve basis β and by π_2 the path which visits all tiles above β exactly once. Then, we can say that π_1 joins π_2 at infinity. Similarly, the paths defined by each infinite mauve triangle and the corresponding infinite trapezes join each other at infinity. We have that all these paths are pairwise disjoint and that any tile of the plane is visited by exactly one of them and once. Moreover, none of the paths is a cycle.

5 A cellular automaton to implement a uniform plane-filling path

Now, we can prove that there is a cellular automaton which implements a uniform plane-filling path.

The proof is simple: it is enough to construct the automaton in such a way that it controls the construction of the interwoven triangles in such a way that there is no infinite red triangle. It is not difficult to see that there are infinite mauve triangles if and only if there are infinite red ones. Now, to avoid infinite red triangles, it is enough to avoid a realization of the butterfly model.

The idea for that is the following.

The automaton will operate on two layers: we can see each tile as belonging to two copies of the ternary heptagrid. On one layer, the automaton realizes the tiling and, on the second one, it controls the construction performed on the first layer. Also, the automaton will draw the mantilla and the numbering of the isoclines at the maximal speed, *i.e.* speed 1. The construction of the interwoven triangles is performed at speed at most 1, but there are longer and longer delays so that the construction of the mantilla and the numbering of the isoclines are always in advance. The construction of the mauve triangles and the path will be still slower.

On the second layer, the automaton draws larger and larger circles around a central cell τ, where by **circle**, we mean the border of a ball around τ. In fact, once a new circle is drawn, the old one is erased. The role of the circle is to detect an isocline 15 which is not inside a triangle. Such an isocline will be called **void**. When the isocline 15 falls within a triangle, it is called **covered**.

The cellular automaton will start from a finite configuration: we take a cell which will be, by definition, the place of the first active seed of an isocline 0. This also means that the cell is on the isocline 0. Then the automaton constructs two blue triangles and the corresponding phantoms in between them. With the isocline 0 and the active seed, a line is defined by the cellular automaton: it is defined by the mid-points of the sides 2 and 6 of the active seed, while the isocline 0 crosses its sides 3 and 1. This is a kind of vertical which will be used during the construction, it will be called the **initial vertical**. Then, the cellular automaton proceeds to the construction of the mantilla and the interwoven triangles until the first void isocline which occurs very soon: an isocline 15 lies between the basis of the first blue-0 triangle and the second one which can be constructed a bit further on the next isocline 0. At this time, a cell τ is chosen on the isocline 15 which is defined by the intersection with a vertical issued from the active seed of the first seed, see [5, 8] for the definition of such verticals.

Now, it is decided that red triangles will be generated by the first blue-0 triangle. In this way, the isocline 15 which passes through τ is covered. From this time, the automaton grows a circle around τ on the second layer, until the circle meets the first void isoclines: these isoclines are detected by the fact that nothing exists on them besides the numbering. When it is first realized, the circle meets two such isoclines as it works symmetrically with respect to the isocline 15 passing through τ.

First, the automaton constructs the generations 0 and 1 until it is possible to define a triangle or a phantom of generation 2. Then, in order to cover the void isocline met by the circle above τ, it is needed that a basis of a blue triangle is generated by the red triangle previously determined. Similarly, to cover the other void isocline, the just generated triangle of the generation 2 generates a red triangle, accordingly of the generation 3.

Later, we proceed in this way:

As soon as all void isoclines met by the circle are covered, the construction stops and the circle is grown until a new void isocline appears, below and above τ. The detection proceeds in this way: the circle advances by 20 steps, the distance between two consecutive isoclines 15. Each concerned cell of the circle sends a message to τ. If all isoclines are covered, τ sends a new message to go on by one move by 20 steps. This is repeated until τ receives the message that a void isocline is found in the appropriate direction: at each cycle of the construction, this direction is changed. It is initially upwards and then, it alternates.

When the void isocline is met, the growth of the circle is stopped.

The construction of the interwoven circles is resumed. Note that, during this time, the construction of the mantilla and the numbering of the isoclines never stopped and during the phase which we shall soon describe, it also never stops.

The construction of the interwoven triangles is performed on all generations already constructed until it is possible to construct the first triangles of the next generation which will also cover the presently void isocline. The void isoclines are characterized by the fact that they have only the isocline number, here 15. The basis which accompanies a possible green line is not yet determined. Now, the definition of the next generation is made possible by the construction of at least two consecutive triangles of the same generation and along the same initial vertical. Then, the automaton can easily decide whether the needed triangle has its vertex or its basis in the just previously constructed triangle. Indeed, it is plain that the mid-distance line between the just determined triangle and the next one along the initial vertical and outside the circle is void. There may be void lines before as well but, in any case, the triangle of the next generation which is built on these triangles covers this mid-distance line and, a fortiori any one which would be closer to τ. The alternation of the direction guarantees that the construction will cover the whole hyperbolic plane.

Once the required triangle is constructed, the automaton goes on constructing all triangles of the previous generations which fall within the ball delimited by the circle. Also, at this time, the corresponding mauve triangles are constructed as well as the path.

When this is done, the construction is stopped and a new cycle is performed: first by growing the circle again, and then, by performing the required constructions.

Accordingly, as the automaton covers larger and larger balls around the same tile τ, it constructs the path in infinite time. Also, as the process guarantees that the void isoclines are step by step covered, there is no infinite triangle. And so, the construction of a path satisfying the assumptions of theorem 1 is achieved in infinite time, which proves theorem 2.

6 A Peano curve

Now, a Peano curve can easily be constructed. The topic is not new and has been dealt with in much larger contexts than this one, see [1]. However, we can give a constructive implementation based on the uniform plane-filling path which is obtained in section 5.

The path fixes an order on the tiles so that we can number the tiles with $\mathbb{Z}$, assigning 0 to an arbitrary tile, fixed once for all. For the step 0, we proceed as follows. In each tile, two of its edges, say i and j are marked as ends of the path. Consider the mid-points A and B of the edges i and j respectively. The points A and B determine a segment which is supported by the hyperbolic line passing through A and B. In the considered tile n, the trace of path in n is the segment $[AB]$.

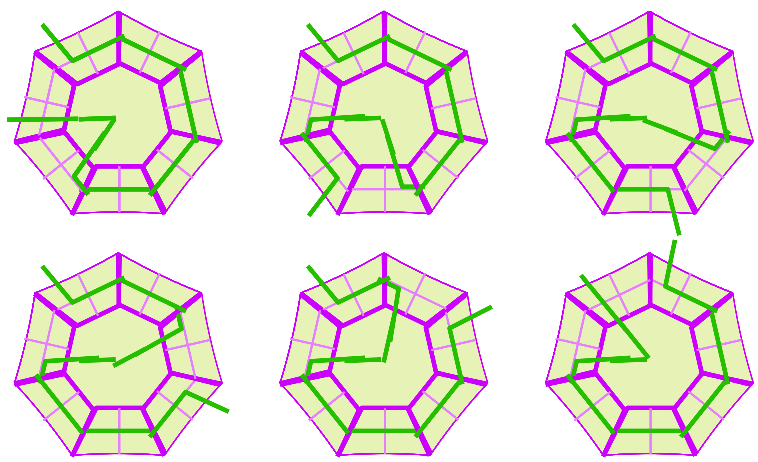


Figure 10 *The first step of the downwards construction.*

For the next step, in each tile n, we replace the segment of the path which crosses n by the appropriate path represented in figure 10. The tiles of the figure represent all the possible cases for an entry on a fixed edge and the exit on another one: there are indeed six possibilities. These basic patterns can be adapted by an appropriate rotation around the centre of the tile so that one of the entries coincide with the side of the tile crossed by the path of step 0. This defines the path of step 1.

Note that each pattern of figure 10 can be split into two parts: the **central** one and the **ring** 1. The ring 1 can be viewed as splitted into seven parts. Such a part is a trapeze, defined by an edge of the tile, two radiuses of the tile, from its centre to the vertices of this edge, and the segment joining the mid-points of the radiuses. The mid-points of the radiuses from the centre Ω of the tile and its vertices, are called the points of **order** 1.

Now, inside a tile n, the points of order 1 define a new tile which we denote by n_1. Note that the centre of n_1 is also Ω. Now, the tile n_1 is crossed by a path defined by two rays joining at Ω, the centre of the tile. Note that n_1 is also a regular heptagon, but its angles are not those of n as it is smaller. Remember that in the hyperbolic plane there is no similarity.

Assume that we defined the step k. In each tile n, we define in n_1 the same construction as we defined for n at the step k. If we were in the Euclidean plane, we could simply say that we apply to n_k a dilatation around Ω of amplitude $\frac{1}{2}$. Now, in the hyperbolic plane such dilatations do not exist. However, we can repeat the construction

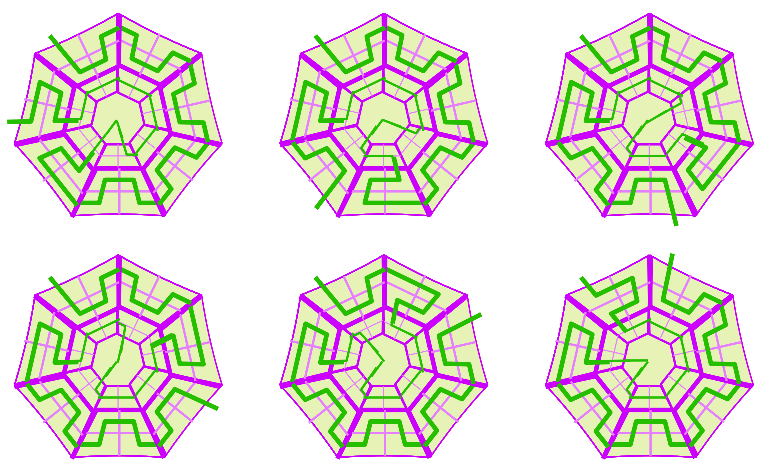


Figure 11 *The second step of the downwards construction.*

which we defined for n to n_1, as it is based on a two dimensional dichotomic process. Now, we have to also define how we transform the ring 1. In each trapeze of the ring, we have cells of order k which are also trapezes. We can see this when we go from the step 1 to the step 2, see figures 10 and 11. Each trapeze of order k has opposite sides and opposite bases, defined at this stage. When $k = 1$, the sides are supported by the radiuses which define the trapezes and the bases are an edge of n and the corresponding edge of n_1. Each trapeze of order k is crossed by a segment from one side to the other or a segment crosses one of the bases and then crosses a side. For the step $k+1$, each trapeze is split into four trapezes of order $k+1$: we join the mid-points of the sides and the mid-points of the bases and this define the four new trapezes. In the new trapezes, what is on a side of the trapeze of order k remains a side and what is a basis remains a basis: this allows to define the sides and the bases of the trapezes of order $k+1$. An edge of a trapeze of order $k+1$ is a side of order $k+1$ if and only if either it is supported by a side of order k, or if it is opposite to a side of order k. Similarly, an edge of a trapeze of order $k+1$ is a basis of order $k+1$ if and only if either it is supported by a basis of order k or it is opposite to a basis of order k.

In fact the pattern of the path inside a trapeze can be described in a precise way which is very close to what is performed in the Euclidean plane, for instance in the construction of the plane-filling path of [2]. However, in the quoted paper, the construction defines growing up structures and here, we go in the opposite direction: the new structures are smaller and smaller. Figure 12 indicates the patterns which are used to go from the step k to the step $k+1$ in the ring 1.

The four trapezes of the figure indicates the four possible paths which can be symbolized by $ABCD$, $BCDA$, $CDBA$ and $DABC$. Now, the connection with a neighbouring trapeze or with the central region is given by paths which are represented by the dotted lines of figure 12. In each picture of the figure, only two points among A, B, C, D give rise to dotted lines. These points are called the **entries**. Now, at each entry, the path goes through one dotted line exactly. And so, each picture gives rise to four paths, depending on the connection with the neighbouring ones.

As k becomes bigger, the situation looks closer and closer to a Euclidean situation. Indeed, as k gets bigger, the size of the trapezes of order k becomes smaller and smaller. Now, it is known that the infinitesimal elements are the same for the Euclidean and for the hyperbolic planes. This means that, as the neighbourhoods of a point get smaller, the hyperbolic situation looks more and more Euclidean.

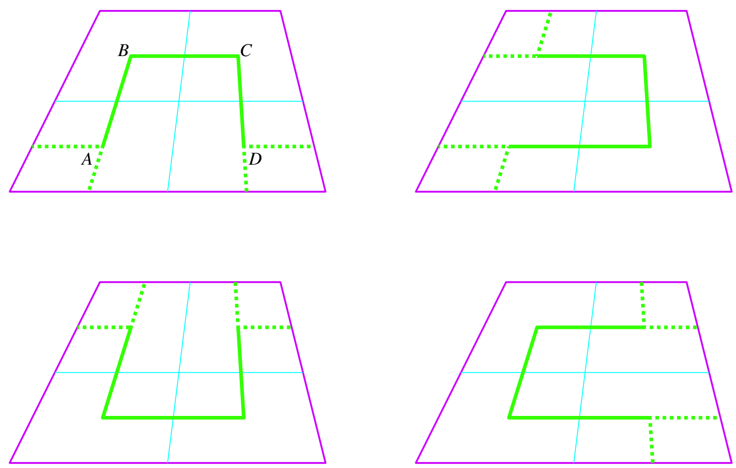


Figure 12 *From the step k to the step $k+1$.*

Now, for each k, the step k defines an infinite curve P_k. We also know that, locally, the Euclidean plane and the hyperbolic one have the same topology. Consequently, we can see that the curves P_k simply converge to a curve P_∞ which goes through each point of the hyperbolic plane.

As indicated in the introduction, it is possible to define a simpler construction of a Peano curve than this one: it will be done in a forthcoming paper. It also makes use of the figures 10, 11 and 12 of this section.

7 Conclusion

I hope that this paper shows the interest of the construction given in [5, 9, 6]. As indicated in [7], there is still much work to do in this domain. In particular, it remains to see whether the strong plane-filling property holds or not in the hyperbolic plane.

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