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The Pico's formula Generalization

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Abstract

The Pico formula generalizations are obtained for area calculation of a polygon P through the determination of special nodes of the network in which this P is placed. The case of the polygon with rational coordinates of its vertexes is examined, as well as the case of the polygon with holes. In the case of three-dimensional space a formula of volume calculation for some polyhedrons, such as prism and tetrahedron is presented. On the basis of theoretic outcomes an algorithm that can be applied in calculation for areas of plane figure is elaborated.

1 The Pico's formula

The determination problem of some efficient formulas for areas calculation of some plane figures classes presents a certain interest from both theoretical and practical point of view. In general case, the integral calculus come to help that will make this problem quite difficult, both from the point of view of function construction that describes the figure frontier and of subsequent calculations that must be effectuated. Thus the study of some special plane figure classes, in particular polygons, becomes very important.

Let us consider one arbitrary polygon P in the plane the vertexes of which are the nodes of a rectangular network D. Network D is determined by classes of parallel to axes OX and OY lines. The crossing points of lines are called the network nodes.

Depending on the structure of network D the formulas for efficient calculation of area of the polygon P are known. The result obtained by the Austrian mathematician G. Pico in 1899 is considered to be among the first results in this direction. In his work G. Pico studies the network case, determined by classes of parallel to axes OX and OY straight lines in which the distance between any two neighbour straight lines is equal to one. Such a network will be called unitary network.

Theorem 1.1. **[1].** *The area of any polygon P, constructed in a unitary network D is calculated by the following formula:*

$$S(P) = i + \frac{b}{2} - 1, \tag{1}$$

where b represents the number of points of the network situated on the frontier of the polygon, and i represents the number of points of the network that belong to interior of this polygon.

Let's illustrate the formula from Theorem 1.1 for polygons P_1 and P_2 from Figure 1.

Polygon P_1: On the one hand, as it can be observed the polygon P_1 is formed from 4 complete squares and 8 triangles (halves of squares) of the unitary network. Thus $S(P_1) = 4 + 8 \cdot \frac{1}{2} = 8$. Applying the formula (1), taking into consideration that $i = 1$ and $b = 16$, we obtain the same result:

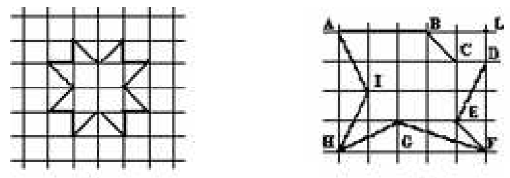
$$S(P_1) = i + \frac{b}{2} - 1 = 1 + \frac{16}{2} - 1 = 8.$$

Polygon P_2: The polygon P_2 can easily be reduced to some simple polygons, trapeziums or triangles, the area of which is easily calculated by the formula:

$$\begin{aligned} S(P_2) &= S(ALFH) - S(AIH) - S(DEF) - S(HGF) - S(BLDC) = \\ &= AH \cdot HF - \frac{AH \cdot h_{AH}}{2} - \frac{HF \cdot h_{FH}}{2} - \frac{BL + CD}{2} \cdot LD = \\ &= 4\dot{5} - \frac{4 \cdot 1}{2} - \frac{5 \cdot 1}{2} - \frac{3 \cdot 1}{2} - \frac{2+1}{2} \cdot 1 = \frac{25}{2}\ . \end{aligned}$$

On the other hand, applying formula (1) we obtain:

$$S(P_2) = i + \frac{b}{2} - 1 = 8 + \frac{11}{2} - 1 = \frac{25}{2}\ .$$



P_1 P_2

Figure 1.

2 The generalization of Pico's Formula in the rational polygons

Let us consider P a polygon, vertexes of which have rational coordinates. This kind of polygons will be called rational polygon. If the polygon has $k+1$ vertexes, than we form the increasing series of abscissas

$$x_1, \quad x_2, \quad \dots, \quad x_k, \quad x_{k+1}$$

and of ordinates

$$y_1, \quad y_2, \quad \dots, \quad y_k, \quad y_{k+1}$$

of the polygon P vertexes. We mark

$$d_i^x = x_{i+1} - x_i, \quad \forall i = \overline{1,k} \ ,$$

$$d_i^y = y_{i+1} - y_i, \quad \forall i = \overline{1,k} \ .$$

Without losing from the generality we consider that $d_1^x, \quad d_2^x, \quad \dots,$ d_k^x are distinct, and $d_1^y, \quad d_2^y, \quad \dots, \quad d_k^y$ as well. Let us form the multitude $D = \{d_1^x, \quad d_2^x, \quad \dots, \quad d_k^x, \quad d_1^y, \quad d_2^y, \quad \dots, \quad d_k^y\}$. One rational number α will be named *divisor* of the rational number β if $\frac{\beta}{\alpha} \in \mathbb{Z}$ (Here $\mathbb{Z}$ represents the multitude of integer numbers). In the case when $\alpha, \quad \beta, \quad \gamma$ are 3 rational numbers and $\frac{\alpha}{\gamma} \in \mathbb{Z}, \quad \frac{\beta}{\gamma} \in \mathbb{Z}$, than we will say that γ is the common divisor of numbers α and β. It is obviously that for any multitude of rational numbers there exists a common divisor. Let us denote the common divisor of the multitude D of elements by d'.

Let us trace in space $\mathbb{R}^2$ parallel to axes OX and OY straight lines so that the distance between each 2 neighbour straight lines to be equal to d'. As a result we obtain a network marked by $D(d')$. Obviously, the vertexes of the rational polygon P are situated in the nodes of this network.

Theorem 2.1. *If the vertexes of rational polygon P are situated in the nodes of the network $D(d')$, then the area of this polygon is calculated by the formula:*

$$S(P) = \left(i' + \frac{b'}{2} - 1\right) \cdot (d')^2 \ , \tag{2}$$

where i' is the number of nodes of network $D(d')$ which belong to the interior of polygon P and b' - number of nodes on the frontier of P.

Proof: We'll prove the affirmation of the theorem by the mathematical induction on n vertexes of polygon P. When $n = 3$ polygon is a triangle ABC with certain rational coordinates of vertexes: $A = (x_A, \quad y_A)$, $B = (x_B, \quad y_B)$, $C = (x_C, \quad y_C)$. Let us pass to another coordinate system OX^*Y^*, in which $x^* = \frac{x}{d'}, y^* = \frac{y}{d'}$. In this system vertexes of the triangle will have coordinates: $A^* = \left(\frac{x_A}{d'}, \quad \frac{y_A}{d'}\right)$, $B^* = \left(\frac{x_B}{d'}, \quad \frac{y_B}{d'}\right)$ and $C^* = \left(\frac{x_C}{d'}, \quad \frac{y_C}{d'}\right)$. Let us calculate the triangle's area in the coordinate system OX^*Y^*:

$$S_{\triangle A^*B^*C^*} = \left| \frac{1}{2} \cdot \begin{vmatrix} \frac{x_A}{d'} & \frac{y_A}{d'} \\ \frac{x_B}{d'} & \frac{y_B}{d'} \end{vmatrix} + \frac{1}{2} \cdot \begin{vmatrix} \frac{x_B}{d'} & \frac{y_B}{d'} \\ \frac{x_C}{d'} & \frac{y_C}{d'} \end{vmatrix} + \frac{1}{2} \cdot \begin{vmatrix} \frac{x_C}{d'} & \frac{y_C}{d'} \\ \frac{x_A}{d'} & \frac{y_A}{d'} \end{vmatrix} \right| =$$

$$= \left| \frac{1}{2} \cdot \frac{1}{d'} \cdot \frac{1}{d'} \cdot \begin{vmatrix} x_A & y_A \\ x_B & y_B \end{vmatrix} + \frac{1}{2} \cdot \frac{1}{d'} \cdot \frac{1}{d'} \cdot \begin{vmatrix} x_B & y_B \\ x_C & y_C \end{vmatrix} + \frac{1}{2} \cdot \frac{1}{d'} \cdot \frac{1}{d'} \cdot \begin{vmatrix} x_C & y_C \\ x_A & y_A \end{vmatrix} \right| =$$

$$= \frac{1}{2} \cdot \left(\frac{1}{d'}\right)^2 \cdot \left| \begin{vmatrix} x_A & y_A \\ x_B & y_B \end{vmatrix} + \begin{vmatrix} x_B & y_B \\ x_C & y_C \end{vmatrix} + \begin{vmatrix} x_C & y_C \\ x_A & y_A \end{vmatrix} \right| = \frac{1}{(d')^2} \cdot S_{\triangle ABC} \ .$$

The network $D(d')$ nodes in the coordinate system OX^*Y^* are the coordinates in integer numbers. According to Pico's formula

$$S_{\triangle A^*B^*C^*} = i' + \frac{b'}{2} - 1,$$

where i' is the number of nodes of the network $D(d')$ which belong to the interior of triangle $A^*B^*C^*$, and b' – is the number of nodes of this network situated on the frontier of P. Thus

$$S_{\triangle A^*B^*C^*} = \frac{1}{(d')^2} \cdot S_{\triangle ABC} \; .$$

Hence

$$S_{\triangle ABC} = S_{\triangle A^*B^*C^*} \cdot (d')^2$$

or

$$S_{\triangle ABC} = \left(i' + \frac{b'}{2} - 1 \right) \cdot (d')^2 \; .$$

Thus the induction base is proved.

Let's admit that the theorem affirmation is true for any polygon P with the number of vertexes smaller than n, $n \geq 4$. Let us analyse a polygon with n vertexes from network $D(d')$. Let $A_1, \quad A_2, \quad \ldots, \quad A_n$ be the vertexes of this polygon, described clockwise (CW). Without losing the generality let us consider that $A_{n-1}, \quad A_n, \quad A_1$ are non-collinear points in plan. Let us connect A_{n-1} with A_1. We obtain the triangle $A_1A_{n-1}A_n$ and $A_1A_2\ldots A_{n-1}$ polygon with $(n-1)$ vertexes. Depending on the position of the initial polygon vertexes in plan we obtain

$$S(P) = S_{A_1 \ldots A_{n-1}} + S_{\triangle A_1 A_{n-1} A_n}$$

or

$$S(P) = S_{A_1 \ldots A_{n-1}} - S_{\triangle A_1 A_{n-1} A_n}$$

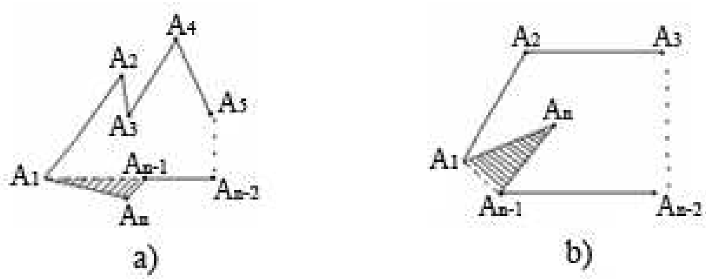


Figure 2.

(see Figure 2, cases a) and b)).

Let i' be the number of nodes of the network $D(d')$ that are the interior points of the polygon $P = A_1A_2\dots A_{n-1}A_n$, $i^{'}_{n-1}$ – the number of nodes of the polygon $A_1A_2\dots A_{n-1}$ with $n-1$ vertexes, and $i^{'}_{\triangle}$ – the number of nodes of the triangle $A_1A_{n-1}A_n$. Let b', $b^{'}_{n-1}$, $b^{'}_{\triangle}$ be the number of nodes of the network $D(d^{'})$ that are situated on the frontier of the mentioned polygons. Let $b^{''}_{A_1A_{n-1}}$ also be the number of nodes of the network, that are interior points of the segment $[A_1A_{n-1}]$, and let $b^{''}_{n-1}$ and $b^{''}_{\triangle}$ be the number of nodes that respectively belong to the broken lines $A_1A_2\dots A_{n-1}$ and $A_{n-1}A_nA_1$. Thus, in the polygon from the Figure 2 a) we have the relations

$$i^{'} = i^{'}_{n-1} + i^{'}_{\triangle} + b^{''}_{A_1A_{n-1}} \ ,$$

$$b^{'} = b^{'}_{n-1} + b^{'}_{\triangle} - 2\cdot b^{''}_{A_1A_{n-1}} - 2 = b^{''}_{n-1} + b^{''}_{\triangle} - 2 \ .$$

In the case of polygon from figure 2 b) we obtain

$$i^{'} = i^{'}_{n-1} - i^{'}_{\triangle} - b^{''}_{\triangle} + 2$$

(number 2 corresponds to the vertexes A_1 and A_{n-1} that are included into the number $b^{''}_{\triangle}$ but aren't interior for the polygon)

$$b^{'} = b^{'}_{n-1} + b^{''}_{\triangle} - b^{''}_{A_1A_{n-1}} - 2 \ .$$

Let us pass to the calculation of areas for the polygons from the Figure 2 a) and b).

For the case represented in Figure 2 a), considering mathematical induction method we obtain

$$S(P) = S_{\triangle A_1 A_{n-1} A_n} + S_{A_1 \dots A_{n-1}} =$$

$$= \left(i'_{n-1} + \frac{b'_{\triangle}}{2} - 1 \right) \cdot (d')^2 + \left(i'_{n-1} + \frac{b'_{n-1}}{2} - 1 \right) \cdot (d')^2 =$$

$$= \left(i'_{\triangle} + i'_{n-1} + \frac{(b'_{\triangle} + b'_{n-1} - 2 \cdot b''_{A_1 A_{n-1}} - 2) + 2 \cdot b''_{A_1 A_{n-1}} + 2}{2} - 2 \right) \cdot (d')^2 =$$

$$= \left(i'_{\triangle} + i'_{n-1} + b''_{A_1 A_{n-1}} + \frac{b'_{\triangle} + b'_{n-1} - 2 \cdot b''_{A_1 A_{n-1}} - 2}{2} - 1 \right) \cdot (d')^2 =$$

$$= \left(i' + \frac{b'}{2} - 1 \right) \cdot (d')^2 \ .$$

Analogically, for the polygon represented in Figure 2 b) we obtain

$$S(P) = S_{A_1 \dots A_{n-1}} - S_{\triangle A_1 A_{n-1} A_n} =$$

$$= \left(i'_{n-1} + \frac{b'_{n-1}}{2} - 1 \right) \cdot (d')^2 - \left(i'_{\triangle} + \frac{b'_{\triangle}}{2} - 1 \right) \cdot (d')^2 =$$

$$= \left(i'_{n-1} - i'_{\triangle} + \frac{b'_{n-1} - b'_{\triangle}}{2} - 1 \right) \cdot (d')^2 =$$

$$= \left(\left(\left(\underbrace{i'_{n-1} - i'_{\triangle} - b''_{\triangle}}_{i'} + 2 \right) + b''_{\triangle} - 2 \right) + \right.$$

$$\left. + \frac{\left(\overbrace{b'_{n-1} + b''_{\triangle} - b''_{A_1 A_{n-1}} - 2}^{b'} \right) - b''_{\triangle} + b''_{A_1 A_{n-1}} + 2 - b'_{\triangle}}{2} \right) \cdot$$

$$\cdot (d')^2 = \left(i' + \frac{b'}{2} - 1 + b''_{\triangle} - \frac{b''_{\triangle} + b'_{\triangle} - b''_{A_1 A_{n-1}}}{2} \right) \cdot (d')^2 =$$

$$= \left(i' + \frac{b'}{2} - 1 + b''_{\triangle} - \frac{b''_{\triangle} + b''_{\triangle}}{2} \right) \cdot (d')^2 = \left(i' + \frac{b'}{2} - 1 \right) \cdot (d')^2 \; .$$

Theorem is proved.

Let us illustrate this demonstration for the case of a polygon P with the vertexes $A(\frac{1}{2}, \quad \frac{1}{3})$, $B(\frac{2}{3}, \quad \frac{5}{6})$, $C(\frac{2}{3}, \quad \frac{1}{2})$ and $D(\frac{4}{3}, \quad \frac{2}{3})$. It is easy to calculate $d' = \frac{1}{6}$. Thus the polygon P can be placed in the network $D\left(\frac{1}{6}\right)$ (see Figure 3).

In this case we obtain $i' = 1$, $b' = 5$ and, thus

$$S_{ABCD} = \left(1 + \frac{5}{2} - 1\right) \cdot \left(\frac{1}{6}\right)^2 = \frac{5}{72} \; .$$

On the other hand, the polygon $ABCD$, being divided in 2 triangles ABC and ACD has the area

$$S_{\triangle ABC} = \frac{1}{2} \bmod \begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{vmatrix} = \frac{1}{2} \bmod \begin{vmatrix} -\frac{1}{6} & -\frac{1}{6} \\ 0 & \frac{2}{6} \end{vmatrix} = \frac{1}{2} \cdot \frac{2}{36} = \frac{1}{36},$$

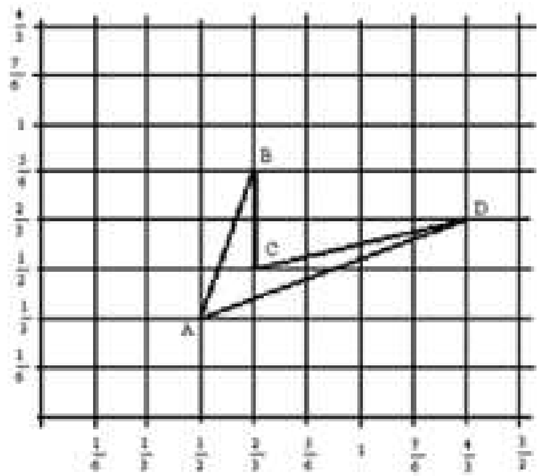


Figure 3.

$$S_{\triangle ACD} = \frac{1}{2}\,\mathrm{mod}\begin{vmatrix} -\frac{5}{6} & -\frac{2}{6} \\ -\frac{4}{6} & -\frac{1}{6} \end{vmatrix} = \frac{1}{2}\cdot\frac{3}{36} = \frac{3}{72}\ .$$

So $S_{ABCD} = S_{\triangle ABC} + S_{\triangle ACD} = \frac{1}{36} + \frac{3}{72} = \frac{5}{72}$, which corresponds to the calculation made in concordance with theorem's formula.

By analogy, we will say that real number β is a divisor of the real number α, if $\frac{\alpha}{\beta} \in \mathbb{Z}$.

Consequence: If P is a polygon, the vertexes of which have real coordinates, and $d^{'}$ is a the common divisor of these coordinates than P can be placed in the network $D(d^{'})$, and its area is calculated by the formula

$$S(P) = \left(i^{'} + \frac{b^{'}}{2} - 1\right)\cdot (d^{'})^2\ ,$$

where i' represents the number of nodes of network $D(d^{'})$ which belongs to the polygon interior, and $b^{'}$ - the number of vertexes situated on the frontier of P.

On the basis of those mentioned above let us elaborate a calculation algorithm of a polygon area according to the studied formula, in the conditions when the common divisor $d^{'}$ exists.

Let us admit that $A_1, \quad A_2, \quad \ldots, \quad A_n$ are the polygon P vertexes, described clockwise.

Description of the algorithm

Step I. Let us draw the network $D(d^{'})$.

Step II. Let a and b represent the smallest and biggest values among the abscissas of the vertexes $A_1, \quad A_2, \quad \ldots, \quad A_n$ of polygon P, and c and d – the smallest and biggest values among the ordinates of these vertexes.

Step III. Let us construct the rectangle D, determinated by the straights $x = a$, $x = b$, $y = c$ and $y = d$. Obviously, in the described conditions, the polygon P will be situated in the interior of the rectangle D.

Step IV. Let us construct the ordinate series $x_1 = a$, $x_2 = x_1 + d^{'}$, $x_3 = x_2 + d^{'}$, $\ldots$, $x_l = b$ where $l = \frac{b-a}{d^{'}} + 1$.

Step V. We trace the straight $X = x_i$ for $\forall i = \overline{1, n}$. We study the knots of the network $D(d^{'})$ which belong to the rectangle D and are situated on the straight $X = x_i$.

Let's consider V one of these knots:

a) If there exists a segment $[A_j, \quad A_{j+1}]$, $j = \overline{1, n}$ (it's considered $A_{n+1} = A_1$) that contains point V, then this, which is a knot on the frontier of P, will be taken up in the calculation of number $b^{'}$.

b) If the condition a) is not realized, then we trace from the point V a semi-straight, parallel to the positive direction of axis OX. We calculate the number L of intersections of this semi-straight with frontier of polygon P. If the intersection is realized in some vertex A_j, $j = \overline{1, n}$ of the polygon, then such intersection will be taken into consideration in the calculation of L in the case when the neighbouring vertexes A_j and A_{j+1} are situated on different sides of the semi-straight. The knot V of the network $D(d^{'})$, belongs to the interior of the polygon P if and only if the number L is odd.

As a result of application of items a) and b) for each straight $X = X_i$, $i = \overline{1,n}$ we'll obtain the values $i^{'}$ and $b^{'}$. Thus, applying respective formula we calculate the polygon area.

Analyzing the described algorithm, we make sure of correctness of the following result.

Theorem 2.2 *The area of the polygon P situated in one network $D(d^{'})$ can be calculated in time $O(N^2)$, where N is the number of vertexes of the polygon P.*

3 Polygons with holes

According to those described above, it's fascinating the fact that similarly to the formula exposed in the Theorem 1.1, there is the subtraction formula of the area of polygon P_g with holes, built in a single network

$$S(P_g) = i + \frac{b}{2} - \chi(P_g) + \frac{1}{2}\chi(\delta P_g), \tag{3}$$

where b represents the number of network knots which are situated on the frontier of the polygon P_g, but i represents the number of network knots which belong to the interior of this polygon, $\chi(P_g) = 1 - n$ – Euler formula for the considered polygon with holes (n – the number of polygon holes), but δP_g denotes the frontier of this ($\chi(\delta P_g) = b - M_b$, M_b – the number of edges belonging to the frontier of polygon P_g) [1].

Let's illustrate the formula (1) for the polygon from Figure 4.

The polygon P_g can easily be reduced to some simpler polygons, the area of which is easily determined just so:

$$\begin{aligned} S(Pg) =& S(ADVZ) - S(ABC) - S(DEF) - S(FGV) - S(HIZ) - \\ & - S(ALM) - S(BNEO) - S(PRST) = \end{aligned}$$

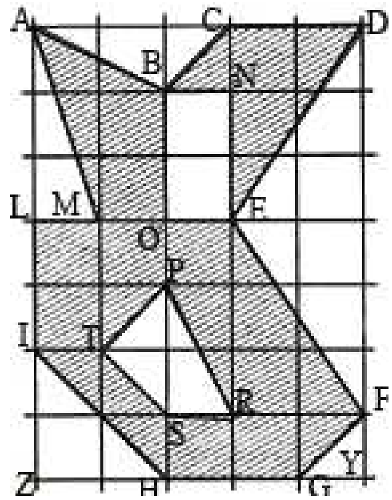


Figure 4.

$$=AZ\cdot ZV-\frac{AC\cdot h_{AC}}{2}-\frac{DF\cdot h_{DF}}{2}-\frac{FV\cdot VG}{2}-\frac{HZ\cdot ZI}{2}-\frac{AL\cdot LM}{2}-$$
$$-BO\cdot OE-\left(\frac{PS\cdot h_{PS}}{2}+\frac{SR\cdot h_{SR}}{2}\right)=$$

$$=7\cdot 5-\frac{3\cdot 1}{2}-\frac{6\cdot 2}{2}-\frac{1\cdot 1}{2}-\frac{2\cdot 2}{2}-\frac{3\cdot 1}{2}-2\cdot 1-\left(\frac{2\cdot 1}{2}+\frac{1\cdot 2}{2}\right)=\frac{39}{2}.$$

Elsewhere, applying formula (1) and taking up that $i=8$, $b=23$, $\chi(P_g)=1-2=-1$ and $\chi(\delta P_g)=b-M_b=23-25=-2$, we obtain the same result:

$$S(P_g)=8+\frac{23}{2}-(-1)+\frac{1}{2}\cdot(-2)=\frac{39}{2}$$

We can easily make sure that, the area of the rational polygon P_g with holes, the vertexes of which belong to the network $D(d')$ (previously described), are calculated by formula

$$S(P_g) = \left(i^{'} + \frac{b^{'}}{2} - \chi(P_g) + \frac{1}{2} \cdot \chi(\delta P_g) \right) \cdot (d^{'})^2 \ ,$$

where $i^{'}$ – number of knots of network $D(d^{'})$ which belong to the interior of polygon P_g with holes, $b^{'}$ – number of knots which are situated on the frontier of P_g, but $\chi(P_g)$ – Euler formula for the considered polygon with n holes, $\chi(\delta P_g)$ represents the Euler formula of this frontier.

We'll illustrate those affirmed, in case of polygon P_g from Figure 5. It's easily determined that $d^{'} = \frac{1}{3}$. As sequel, the polygon P_g with holes can be placed in network $D\left(\frac{1}{3}\right)$.

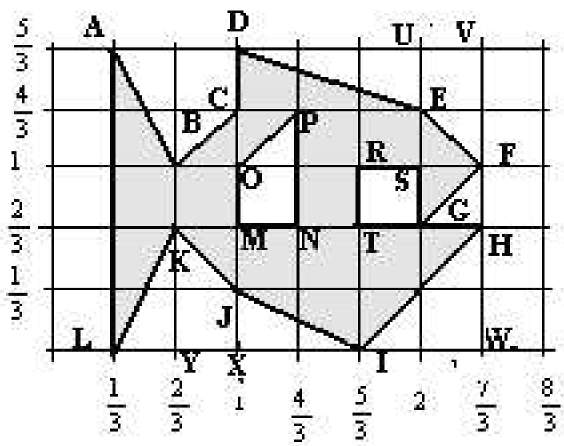


Figure 5.

In this case, we have $i^{'} = 3$, $b^{'} = 25$ and $\chi(P_g) = 1 - n = 1 - 2 = -1$, $\chi(\delta P_g) = b^{'} - M_{b^{'}} = 25 - 26 = -1$, as continuation, the polygon has the area

$$S(P_g) = \left(3 + \frac{25}{2} - (-1) + \frac{1}{2} \cdot (-1) \right) \cdot \left(\frac{1}{3} \right)^2 = \frac{32}{18} \ .$$

Dividing the given polygon into simpler polygons, it's easily verified that we obtain the same result just so:

$$S(P_g) = S(AVWL) - 2\cdot(S(LKY) + S(KJXY)) - S(DEU) - S(UEFV) - S(FGH) - S(HIW) - S(IJX) - S(OMPN) - S(RSGT) =$$

$$= AL \cdot LW - 2 \cdot \left(\frac{LY \cdot YK}{2} - \frac{KY + JX}{2} \cdot XY \right) - \frac{DU \cdot UE}{2} - \frac{UE + VF}{2} \cdot UV - \frac{FH \cdot GH}{2} - \frac{HW \cdot WI}{2} - \frac{IX \cdot XJ}{2} - \frac{OM + PN}{2} \cdot \frac{1}{3} - RS^2 =$$

$$= \frac{5}{3} \cdot \frac{6}{3} - 2\cdot \left(\frac{\frac{1}{3} \cdot \frac{2}{3}}{2} + \frac{\frac{2}{3} + \frac{1}{3}}{2} \cdot \frac{1}{3} \right) - \frac{\frac{3}{3} \cdot \frac{1}{3}}{2} - \frac{\frac{1}{3} + \frac{2}{2}}{2} \cdot \frac{1}{3} - \frac{\frac{1}{3} \cdot \frac{1}{3}}{2} - \frac{\frac{2}{3} \cdot \frac{2}{3}}{2} - \frac{\frac{2}{3} \cdot \frac{1}{3}}{2} - \frac{\frac{1}{3} + \frac{2}{3}}{2} \cdot \frac{1}{3} - \left(\frac{1}{3}\right)^2 = \frac{32}{18} \, .$$

4 Generalizations of Pico formula in case of 3 - dimensional polyhedron

Suppose P is a 3 - dimensional polyhedron without holes which contains $k+1$ vertexes with rational coordinates. As in the case of polygons, on the basis of non-descending ranges of coordinates

$$x_1, \quad x_2, \quad \ldots, \quad x_k, \quad x_{k+1}$$

$$y_1, \quad y_2, \quad \ldots, \quad y_k, \quad y_{k+1}$$

$$z_1, \quad z_2, \quad \ldots, \quad z_k, \quad z_{k+1}$$

we form the multitude $D = \{d_1^x, \quad d_2^x, \quad \ldots, \quad d_k^x, \quad d_1^y, \quad d_2^y, \quad \ldots, \quad d_k^y \quad d_1^z, \quad d_2^z, \quad \ldots, \quad d_k^z\}$ where

$$d_i^x = x_{i+1} - x_i, \quad \forall i = \overline{1, k} \ ,$$

$$d_i^y = y_{i+1} - y_i, \quad \forall i = \overline{1, k} \ ,$$

$$d_i^z = z_{i+1} - z_i, \quad \forall i = \overline{1, k} \ .$$

We'll denote by $d^{'}$ some of common divisors of the elements from multitude D. (We mention that for any multitude of rational numbers there exists, at least, one common divisor). In the space $\mathbb{R}^3$ we pass planes parallel to planes XOY, XOZ and YOZ so that the distance between any two parallel and neighbouring planes is equal to $d^{'}$. Thus in space $\mathbb{R}^3$ we obtain a cubic network which we'll denote by $Q(d^{'})$. In this case we can say that P is a $d^{'}$-rational polyhedron. In a sequel we'll study the problem of volume calculation of polyhedron P when this is a pyramid or prism.

Definition 3.1 *One polyhedron P, the vertexes of which are situated in the knots of network $Q(d^{'})$, is named $d^{'}$-rational elementary polyhedron, if with exception of knots in which the vertexes of P are situated, this contains no other knots from $Q(d^{'})$.*

Easily can be observed that if P is a 1-rational elementary tetrahedron with height $h = 1$, then the volume of this is $V = \frac{1}{6}$. Surely, on the basis of those exposed earlier we obtain

$$V = \frac{1}{3} \cdot S_b \cdot h = \frac{1}{3} \cdot \left(i + \frac{b}{2} - 1\right) \cdot h = \frac{1}{3} \cdot \left(0 + \frac{3}{2} - 1\right) \cdot 1 = \frac{1}{6} \ .$$

In the case of $d^{'}$-rational tetrahedron with the height H we have

$$V = \frac{(d^{'})^2 \cdot h}{6} \ .$$

Theorem 3.1 *Any straight prism, the vertexes of which are situated in the knots of rational network $Q(d^{'})$, can be divided in $d^{'}$-rational elementary prisms.*

Proof: Firstly we observe that any polygon from plane, the vertexes of which are situated in the knots of rational network $D(d^{'})$ can be divided into $d^{'}$-rational elementary triangles, that is triangles which with exception of knots in which the vertexes of these are placed, contain no other knots of the network $D(d^{'})$. In this case we'll say that a triangulation of the polygon, determined by network $D(d^{'})$ is given.

One of possible triangulations of any polygon can iteratively be obtained in the following way:

1. We denote by $N(P)$ the multitude of knots of unitary network which belong to polygon P, that are situated on the frontier $bd(P)$ or in the interior $int(P)$ of this. Evidently $N(P) \neq \oslash$.

2. We choose an element $t \in N(P) \cap bd(P)$.

3. We form the multitude $\Gamma(t)$ of all $\omega \in N(P)$ knots, $\omega \neq t$, for which the segment $[t, \quad \omega]$ belongs to the polygon P, which contains no other knots of the network excepting t and ω, which can be united by a curve that belongs integrally to the polygon P.

4. We denote by $q^{'}$ and $q^{''}$ the elements from $\Gamma(t)$ which belong to the frontier $bd(P)$ and for which the curve line $[q^{'}, \quad t, \quad q^{''}]$ is placed on this frontier. We execute an order of elements of multitude $\Gamma(t)$ correspondingly to the clockwise direction, beginning with one of the knots $q^{'}$ or $q^{''}$, finishing with the second, with the condition that the curve which unites consecutive elements from $\Gamma(t)$ belongs as a whole to the polygon. Suppose $\Gamma(t) = \{s_1 = q^{'}, \quad s_2, \quad ..., \quad s_{k-1}, \quad s_k = q^{''}\}$ (see Figure 6 a)). We draw up a curve, consequently uniting elements from $\Gamma(t)$. We denote by P_Γ the polygon with the frontier $bd(P_\Gamma) = [s_1 = q^{'}, \quad s_2, \quad ..., \quad s_{k-1}, \quad s_k = q^{''}, \quad t, \quad q]$. We mention that $int(P_\Gamma)$ contains no knots of the network. Surely, in the opposite case such knot z will belong to a triangle $[s_{i-1}, \quad t, \quad s_i]$, $i = \overline{2,k}$

which means that in the interior of the cone s_{i-1}, t, s_1 there exist nodes which weren't taken into consideration at forming the set $\Gamma(t)$. It is obvious that P_Γ is a triangulated polygon.

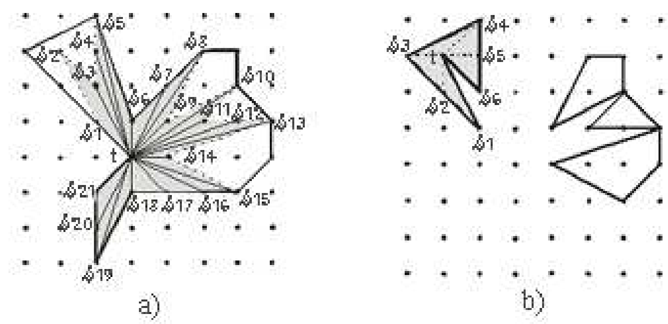


Figure 6.

5. Let us denote by $Bd(P)$ the common frontier between P and P_Γ and eliminate the set $Bd(P) \cap int(P_\Gamma)$ from P. If we obtain as a result a void set of the points from the plan then we obtain the triangulation of the initial polygon. Otherwise we denote by P the remained part of the polygon. We mention that in general case the obtained domain P can be a reunion of simple polygons. (see Figure 6 b). Let us return to the step 2 and continue the triangulation procedure.

If for the polygon P from Figure 6 a) we consecutively apply the described algorithm, then we obtain the situations described in the Figure 6 a), b) and Figure 7 a), b).

Finally we'll obtain a triangulation of the initial polygon. Such a triangulation is presented in Figure 8. Of course, the triangulation of the polygon on the basis of the described algorithm is not obtained univocally. Surely, the number of the triangles into which the polygon will be divided is always the same.

Now let P be an arbitrary prism, the vertexes of which are situated in the nodes of the network $Q(d^{'})$. Let us triangulate the polygon

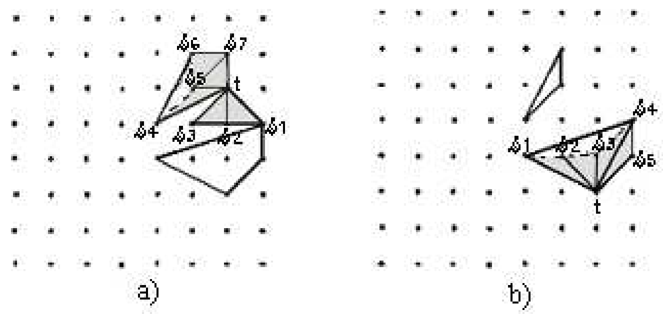


Figure 7.

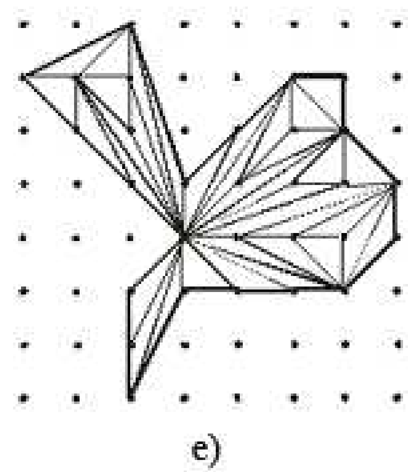


Figure 8.

from the base of the prism according to the described procedure. In accordance with the obtained base configuration we trace a vertical section of the prism P and the planes parallel to the base through different nodes of the network. As a result we obtain a division of prism P into $d^{'}$-rational elementary prisms.

Consequence. *Any prism the vertexes of which are situated in the nodes of the network* $Q(d^{'})$ *can be divided into* $(2i + b - 2)(n - 1)$ $d^{'}$*-rational elementary prisms, where* i *is the number of nodes of the network situated in the interior of the polygon on the base of the prism,* b *– the number of nodes on the polygon frontier, and* n *– the number of nodes on a lateral edge.*

Proof: According to the Euler formula, the number of triangular domains into which a plane can be divided using k points is $2k - 4$. In the case of the studied polygon $k = i + b$. Because the domains from the polygon exterior do not interest us we obtain that the polygon can be divided into $2k - 4 - 1 - (b - 3) = 2(i + b) - 4 - 1 - b + 3 = 2i + b - 2$ triangles. Because there are n nodes of the network on the lateral edge of the prism we trace $(n - 1)$ sections parallel to the base and thus we obtain $(2i + b - 2)(n - 1)$ $d^{'}$-rational elementary prisms.

Theorem 3.2. *The volume of a prism with the vertexes in the nodes of a network* $Q(d^{'})$ *can be calculated by the formula:*

$$V = \frac{1}{2} \cdot (2i + b - 2) \cdot (n - 1) \cdot (d^{'})^3 \ ,$$

where i *is the number of nodes of the network* $Q(d^{'})$ *which is situated in the interior of the polygon on the prism base,* b *– the number of nodes on the polygon frontier, and* n *– the number of nodes on a lateral edge.*

Proof: It is known that a $d^{'}$-rational elementary prism can be divided into three $d^{'}$-rational elementary pyramids of equal volume. Thus, according to those mentioned above, the volume of such a prism is $\frac{1}{2} \cdot (d^{'})^2 \cdot h = \frac{1}{2} \cdot (d^{'})^3$. Taking into consideration the consequence 3.1 we obtain the affirmation of the Theorem 3.2.

Theorem 3.3. *The volume of a pyramid, the vertexes of which are in the nodes of the network $Q(d^{'})$ is calculated by the formula*

$$V = \frac{1}{6} \cdot (2i + b - 2) \cdot k \cdot (d^{'})^3 \ ,$$

where i and b represent the number of network nodes, that belong to the interior of the base of the pyramid and to the frontier of this base respectively, and k – the number of nodes on the height traced from the pyramid vertex.

Proof: According to those described above let us make a triangulation of pyramid base. Joining every node of the network that belong to the base with the vertex of pyramid, we obtain $(2i + b - 2)$ $d^{'}$-rational elementary pyramids. Each of these pyramids has the height $h = d^{'} \cdot (k - 1)$, where k is the number of nodes that belong to the height traced from pyramid vertex. Thus for initial pyramid we have

$$V = (2i + b - 2) \cdot \frac{(d^{'})^2 \cdot h}{6} = \frac{1}{6} \cdot (2i + b - 2) \cdot (k - 1) \cdot (d^{'})^2 \ .$$

Theorem is proved.

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