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The graphic representations for the one-dimensional solutions of problem from elastic mechanic deformations of two-component mixture

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Abstract

In this paper we find the analytical solution of simple one-dimensional unsteady elastic problem of two-component mixture using Laplace integral transformation. The integral transformations simplify the initial motion systems for finding analytical solutions. The analytical solutions are represented as the graphic on time dependence in the fixed point of medium, and the graphic on the horizontal coordinate at the fixed time.

Keywords: Two-component medium, dynamic problem, unsteady elasticity equations.

The continuum theory of mixtures has been a subject of study in recent years. The linearized theory of elasticity for the indicated medium was given by T. R. Steel [1]. The two-dimensional problems for the isotropic mixture are considered by T.R. Steel [2] and M.O. Basheleishvili [3]. Some three-dimensional basic problems for indicated medium are considered by D.G. Natroshvili, A.J. Jagmaidze and M.J. Svanadze [4].

Let us consider the two-component medium where there is the dependence only on the horizontal coordinate of material. In the one-dimensional case the basic equation of the theory on the elastic mixture of the two-component medium has the form [5,6,7]:

$$A_{11}\frac{\partial^2 U_1}{\partial x^2} + B_{11}\frac{\partial^2 U_2}{\partial x^2} = \rho_{11}\frac{\partial^2 U_1}{\partial t^2} + \rho_{12}\frac{\partial^2 U_2}{\partial t^2} + b\left(\frac{\partial U_1}{\partial t} - \frac{\partial U_2}{\partial t}\right),$$

$$A_{21}\frac{\partial^2 U_2}{\partial x^2} + B_{21}\frac{\partial^2 U_1}{\partial x^2} = \rho_{12}\frac{\partial^2 U_1}{\partial t^2} + \rho_{22}\frac{\partial^2 U_2}{\partial t^2} - b\left(\frac{\partial U_1}{\partial t} - \frac{\partial U_2}{\partial t}\right) (1)$$

where U_1, U_2 are partial displacements of material, and $A_{i1} = \lambda_i + 2\mu_i + (-1)^i \frac{\rho_{3-i}\alpha_2}{\rho}$; $A_{i2} = \mu_i - \lambda_5$; $B_{i1} = \lambda_{2+i} + 2\mu_3 + \frac{\rho_i\alpha_2}{\rho}(-1)^i$; $B_{i2} = \lambda_5 + \mu_3$; $(i = 1, 2)$ are the known constants characterizing the physical properties of material, and ρ_1, ρ_2 – are the partial densities; b – coefficient of diffusion, $\mu_1, \mu_2, \mu_3, \lambda_1, \lambda_2, ..., \lambda_5$ are elastic constants of the mixture [1,6,7].

In the sequel it will be assumed that the following conditions are fulfilled [1,6,7]:

$$\mu_1 > 0, \quad \mu_1\mu_2 > \mu_3^2, \quad \lambda_1 - \frac{\rho_2\alpha_2}{\rho_1+\rho_2} + \frac{2}{3}\mu_1 > 0, \quad \lambda_5 \le 0,$$
$$\rho_{11} > 0, \quad \rho_{11}\rho_{22} > \rho_{12}^2,$$
$$\left(\lambda_1 - \frac{\rho_2\alpha_2}{\rho_1+\rho_2} + \frac{2}{3}\mu_1\right)\left(\lambda_2 + \frac{\rho_1\alpha_2}{\rho_1+\rho_2} + \frac{2}{3}\mu_2\right) > \left(\lambda_3 - \frac{\rho_1\alpha_2}{\rho_1+\rho_2} + \frac{2}{3}\mu_3\right)^2$$

For system (1) the following conditions on the boundary $x = 0$ are given:

$$U_1|_{x=0} = 0, \quad U_2|_{x=0} = 0, \tag{2}$$

and the initial states are the following:

$$\begin{array}{ll} U_1|_{t=0} = 0, & \left.\frac{\partial U_1}{\partial t}\right|_{t=0} = -V_0 \\ U_2|_{t=0} = 0, & \left.\frac{\partial U_2}{\partial t}\right|_{t=0} = -V_0 \end{array} \tag{3}$$

For solving the problem (1-3) we use the integral Laplace transform of the time, the operator of this transform is defined by:

$$\mathrm{Z}\left\{u(x,t)\right\} = \int_0^\infty e^{-st}u(x,t)dt = U(x,s), \tag{4}$$

The following are the proprieties of unilateral Laplace transform:

$$\mathrm{Z}\left\{\frac{\partial u(x,t)}{\partial t}\right\} = sU(x,s) - U(x,+0),$$
$$\mathrm{Z}\left\{\frac{\partial^2 u(x,t)}{\partial t^2}\right\} = s^2U(x,s) - sU(x,+0) - U_t(x,+0), \tag{5}$$

$$\mathrm{Z}\left\{\frac{\partial u(x,t)}{\partial x}\right\} = \frac{\partial}{\partial x}\mathrm{Z}\left\{u(x,t)\right\} = \frac{\partial U(x,s)}{\partial x},$$

$$\mathrm{Z}\left\{\frac{\partial^2 u(x,t)}{\partial x \partial t}\right\} = \frac{\partial}{\partial x}\mathrm{Z}\left\{\frac{\partial u(x,t)}{\partial t}\right\} = \frac{\partial}{\partial x}\left[sU(x,s) - U(x,+0)\right],$$

In the case where diffusion is zero we will search the solutions of the problem (1) with boundary conditions (2) and initial conditions (3) as combination of:

$$\begin{aligned} U_1 &= u_1 + u_2, \\ U_2 &= \beta_1 u_1 + \beta_2 u_2, \end{aligned} \tag{6}$$

where the constants β_i (i=1,2) are from the following equation:

$$\frac{A_{11} + \beta_i B_{11}}{\rho_{11} + \beta_i \rho_{12}} = \frac{B_{21} + \beta_i A_{21}}{\rho_{12} + \beta_i \rho_{22}}, \quad (i = 1, 2) \tag{7}$$

i.e. β_i are the solutions of the quadratic equation:

$$A^* \beta_i^2 + B^* \beta_i + C^* = 0$$

where $A^* = B_{11}\rho_{22} - A_{21}\rho_{12}$ $B^* = B_{11}\rho_{12} + A_{11}\rho_{22} - A_{21}\rho_{11} - B_{21}\rho_{12}$ $C^* = A_{11}\rho_{12} - B_{21}\rho_{11}$.

Then the system (1) results from (6) in the following form:

$$\begin{aligned} & a_i^2 \frac{\partial^2 u_i}{\partial x^2} = \frac{\partial^2 u_i}{\partial t^2} \\ & a_i^2 = \frac{A_{11} + \beta_i B_{11}}{\rho_{11} + \beta_i \rho_{12}} \qquad (i = 1, 2), \end{aligned} \tag{8}$$

and the initial conditions (3) are the following:

$$\begin{aligned} u_1|_{t=0} &= 0, \quad \left.\frac{\partial u_1}{\partial t}\right|_{t=0} = -V_0 \frac{\beta_2 - 1}{\beta_2 - \beta_1} \\ u_2|_{t=0} &= 0, \quad \left.\frac{\partial u_2}{\partial t}\right|_{t=0} = -V_0 \frac{1 - \beta_1}{\beta_2 - \beta_1}. \end{aligned} \tag{9}$$

Let us apply the Laplace transform (4) to the system (8) and according to the proprieties (5) we get:

$$\frac{\partial^2 u_i}{\partial x^2} = \frac{s^2}{a_i^2} u_i(x,s) + \frac{V_0}{a_i^2} \frac{(-1)^{i+1}(\beta_{3-i} - 1)}{\beta_2 - \beta_1} \qquad (i = 1, 2) \tag{10}$$

The solutions of these equations are:

$$u_i = \frac{V_0}{s^2}\frac{(-1)^{i+1}(\beta_{3-i}-1)}{\beta_2-\beta_1}\left(e^{-\frac{s}{a_i}x}-1\right) \qquad (i=1,2) \qquad (11)$$

We apply the inverse of Laplace transform to the solution (11), and according to combinations (6) we obtain the analytical solutions of the problem (1-3) when b=0:

$$\begin{aligned} U_1 &= V_0\frac{\beta_2-1}{\beta_2-\beta_1}\left(t-\frac{x}{a_1}\right)\mathrm{H}\left(t-\frac{x}{a_1}\right)+ \\ &\quad +V_0\frac{1-\beta_1}{\beta_2-\beta_1}\left(t-\frac{x}{a_2}\right)\mathrm{H}\left(t-\frac{x}{a_2}\right)-tV_0 \\ U_2 &= V_0\beta_1\frac{\beta_2-1}{\beta_2-\beta_1}\left(t-\frac{x}{a_1}\right)\mathrm{H}\left(t-\frac{x}{a_1}\right)+ \\ &\quad +V_0\beta_2\frac{1-\beta_1}{\beta_2-\beta_1}\left(t-\frac{x}{a_2}\right)\mathrm{H}\left(t-\frac{x}{a_2}\right)-tV_0. \end{aligned} \qquad (12)$$

From the analytical solutions (12) and by the proprieties of the Hevisaid function we obtain the continuum solutions on the following coordinate subareas.

For $t<\frac{x}{\alpha_1}$:

$$U_1=-V_0t, \quad U_2=-V_0t.$$

For $t\in\left[\frac{x}{\alpha_1},\frac{x}{\alpha_2}\right]$:

$$U_1 = V_0\left[\frac{(1-\beta_2)}{\alpha_1(\beta_2-\beta_1)}x+\frac{(\beta_1-1)}{(\beta_2-\beta_1)}t\right],$$

$$U_2 = V_0\left[\frac{\beta_1(1-\beta_2)}{\alpha_1(\beta_2-\beta_1)}x+\frac{\beta_2(\beta_1-1)}{(\beta_2-\beta_1)}t\right],$$

$$\varepsilon_x=\frac{V_0(1-\beta_2)}{\alpha_1(\beta_2-\beta_1)}, \quad q_x=\frac{V_0\beta_1(1-\beta_2)}{\alpha_1(\beta_2-\beta_1)},$$

$$\sigma_{xx}=a_2+(\lambda_1+2\mu_1)\frac{V_0(1-\beta_2)}{\alpha_1(\beta_2-\beta_1)}, \qquad (13)$$

$$\sigma_{yy} = -a_2 + \lambda_1 \frac{V_0 (1-\beta_2)}{\alpha_1 (\beta_2 - \beta_1)} + \lambda_3 \frac{V_0 \beta_1 (1-\beta_2)}{\alpha_1 (\beta_2 - \beta_1)},$$

$$\pi_{xx} = a_2 + (\lambda_2 + 2\mu_2) \frac{V_0 \beta_1 (1-\beta_2)}{\alpha_1 (\beta_2 - \beta_1)} + (\lambda_4 + 2\mu_3) \frac{V_0 (1-\beta_2)}{\alpha_1 (\beta_2 - \beta_1)},$$

$$\pi_{yy} = a_2 + \lambda_2 \frac{V_0 \beta_1 (1-\beta_2)}{\alpha_1 (\beta_2 - \beta_1)}.$$

For graphical representations of the analytical solutions we use such mathematical processors as Mathematica 5. The Mathematica 5 has the standard procedure for algebraic analysis and for graphic representations.

For example we have the next value of constants characterizing the physical properties of material:

$$\begin{array}{l} \lambda_1 = 0,4026 \cdot 10^9 \ kg/m^2, \quad \mu_1 = 0,2493 \cdot 10^9 \ kg/m^2, \\ \lambda_3 = \lambda_4 = 0,0672 \cdot 10^9 \ kg/m^2, \quad \lambda_2 = 0,0295 \cdot 10^9 \ kg/m^2 \\ \mu_2 = \mu_3 = 0 \\ \rho_1 = 0,82 \cdot 10^{-5} \ s^2 \cdot kg/m^2, \quad \rho_2 = 0,26 \cdot 10^{-5} \ s^2 \cdot kg/m^2 \\ \rho_{12} = 0,19 \cdot 10^{-7} \ s^2 \cdot kg/m^2, \\ V = 3000 \ m/s \end{array} \tag{14}$$

The program script for graphical representation is:

```
V = 3000;
 l1 = 0.4026 10^9; m1 = 0.2493 10^9;
  l3 = (l4 = 0.0672 10^9 );
  l2 = 0.0295 10^9; m2 = (m3 = 0 );
  r1 = 0.82 10^(-5 ); r2 = 0.26 10^(-5 );
  r12 = 0.19 10^(-7 );
   (r = r1 r2 + r12^2; )
   (A11 = l1 + 2 m1 - 0.001 r2/r; )
   (B11 = l3 + 2 m3 - 0.001 r2/r; )
   (A21 = l2 + 2 m2 + 0.001 r2/r; )
   (B21 = l4 + 2 m3 + 0.001 r2/r; )
  a = B11 r2 - A21 r12;
  b = B11 r12 + A11 r2 - A21 r1 - B21 r12;
```

```
 (c = A11 r12 - B21 r1; )
 (beta = s /. Solve[a s^2 + b s + c [Equal] 0, s]; )
b1 = beta[ [1]]; b2 = beta[[2]];
a1 = Sqrt[(A11 + b1 B11)/ (r1 + b1 r12) ]
a2 = Sqrt[ (A11 + b2 B11)/ (r1 + b2 r12)]
 (U1[x_, t_] =
   V   (( ((b1 - 1) )/ ((b2 - b1) ) ((t - x/a1) )
          UnitStep[
           t - x/a1] + ((1 - b1) )/((b2 - b1) ) ((t -
           - x/a2) )
          UnitStep[t - x/a2]) ) - t V; )
 (U2[x_, t_] =
   V   ((b1 ((b1 - 1) )/ ((b2 - b1) ) ((t - x/a1) )
          UnitStep[t - x/a1] +
         b2 ((1 - b1) )/ ((b2 - b1) ) ((t - x/a2) )
          UnitStep[t - x/a2]) ) - t V; )
Plot[{U1[1, t], U2[1, t]}, {t, 0, Max[1/a1, 1/a2] +
   +((1/a1 + 1/a2) )/2},
  AxesLabel [Rule] {x, {U1, U2}}]
Plot3D[
  U1[x, t], {x, 0, 100}, {t, 0,
    Max[100/a1, 100/a2] + ((100/a1 + 100/a2) )/2}]
Plot3D[
  U2[x, t], {x, 0, 100}, {t, 0,
    Max[100/a1, 100/a2] + ((100/a1 + 100/a2) )/2}] )] )
```

As the result of execution of this script we obtain the graphics of functions U_1 and U_2 (Figure 1).

For the area $(x,t) \in [0,100] \times [0, Max(100/\alpha_1, 100/\alpha_2) + ((100/\alpha_1 + 100/\alpha_2))/2]$ we obtain the 3D graphics for U_1 (Figure 2) and for U_2 (Figure 3).

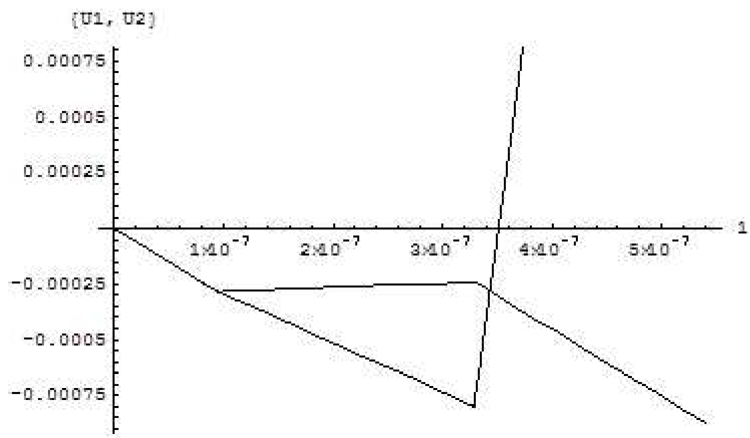

Figure 1. Graphics of functions U_1 and U_2 in dependence on time.

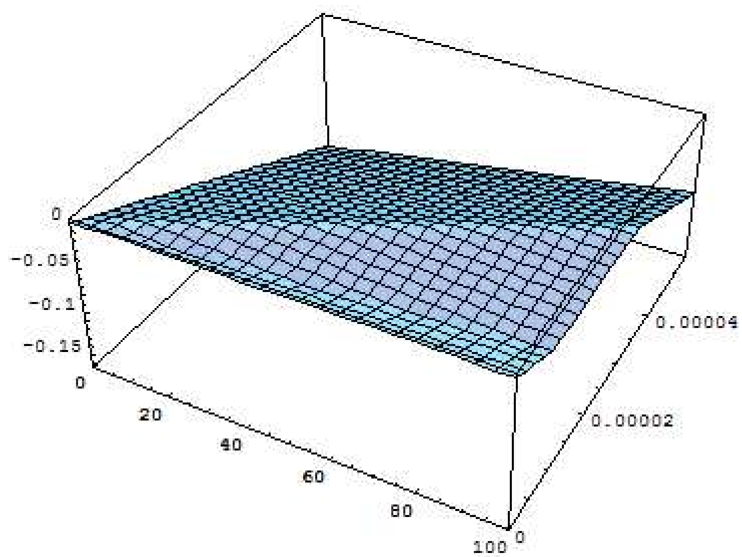

Figure 2. 3D graphics for function U_1.

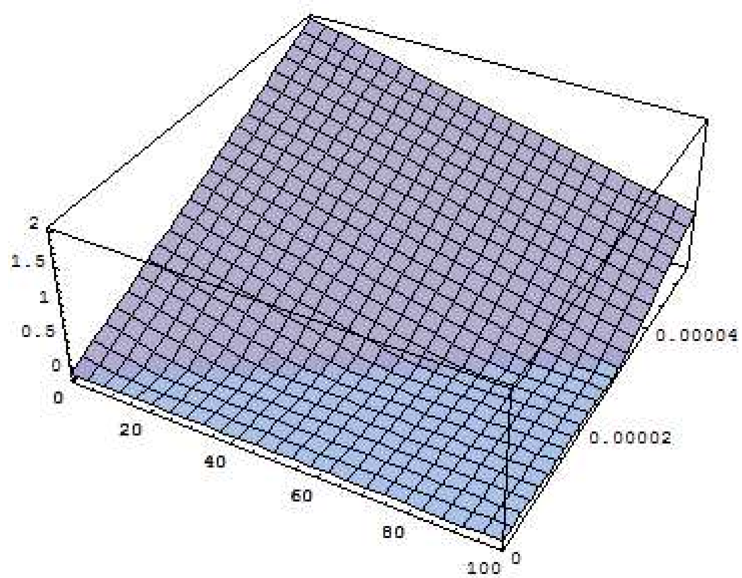

Figure 3. 3D graphics for function U_2.

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