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Nash equilibria set computing in finite extended games

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Abstract

The Nash equilibria set (NES) is described as an intersection of graphs of best response mappings. The problem of NES computing for multi-matrix extended games is considered. A method for NES computing is studied.

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1 Introduction

The Nash equilibria set (NES) is determined as an intersection of graphs of best response mappings [5, 8, 9]. This idea yields a natural method for NES computing in mixed extended two-player $m \times n$ games and n-player $m_1 \times m_2 \times \cdots \times m_n$ games.

Consider a noncooperative finite strategic game:

$$\Gamma = \langle N, \{S_p\}_{p \in N}, \{a^p_{\mathbf{s}} = a^p_{s_1 s_2 \ldots s_n}\}_{p \in N} \rangle,$$

where $N = \{1, 2, ..., n\}$ is a set of players, $S_p = \{1, 2, \ldots, m_p\}$ is a set of strategies of player $p \in N$, $\#S_p = m_p < +\infty$, $p \in N$ and $a^p_{\mathbf{s}} = a^p_{s_1 s_2 \ldots s_n}$ is a player's $p \in N$ payoff function defined on the Cartesian product $S = \underset{p \in N}{\times} S_p$ (payoff of the player $p \in N$ is done by n dimensional matrix

$A^p[m_1 \times m_2 \times \cdots \times m_n]$). Elements $\mathbf{s} = (s_1, s_2, ..., s_n) \in S$ are named outcomes of the game (situations or strategy profiles).

The mixed extension of Γ is

$$\widetilde{\Gamma} = \langle X_p, f_p(\mathbf{x}), p \in N \rangle,$$

where

$$f_p(\mathbf{x}) = \sum_{s_1=1}^{m_1} \sum_{s_2=1}^{m_2} \cdots \sum_{s_n=1}^{m_n} a^p_{s_1 s_2 \ldots s_n} x^1_{s_1} x^2_{s_2} \ldots x^n_{s_n} = \\ = \sum_{s_1=1}^{m_1} \sum_{s_2=1}^{m_2} \cdots \sum_{s_n=1}^{m_n} a^p_{\mathbf{s}} \prod_{p=1}^{n} x^p_{s_p},$$

$$\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n) \in X = \underset{p \in N}{\times} X_p \subset R^m,$$

$$m = m_1 + m_2 + \cdots + m_n,$$

$$X_p = \left\{ \mathbf{x}^p = (x^p_1, \ldots, x^p_{m_p}) : \begin{array}{c} x^p_1 + \cdots + x^p_{m_p} = 1, \\ x^p_1 \geq 0, \ldots, x^p_{m_p} \geq 0 \end{array} \right\}$$

is a set of mixed strategies of the player $p \in N$.

Definition. *The outcome $\hat{\mathbf{x}} \in X$ of the game is a Nash equilibrium [4] if*

$$f_p(\mathbf{x}^p, \hat{\mathbf{x}}^{-p}) \leq f_p(\hat{\mathbf{x}}^p, \hat{\mathbf{x}}^{-p}), \forall \mathbf{x}^p \in X_p, \ \forall p \in N,$$

where

$$\hat{\mathbf{x}}^{-p} = (\hat{\mathbf{x}}^1, \hat{\mathbf{x}}^2, \ldots, \hat{\mathbf{x}}^{p-1}, \hat{\mathbf{x}}^{p+1}, \ldots, \hat{\mathbf{x}}^n),$$

$$\hat{\mathbf{x}}^{-p} \in X_{-p} = X_1 \times X_2 \times \cdots \times X_{p-1} \times X_{p+1} \times \cdots \times X_n,$$

$$\hat{\mathbf{x}}^p || \hat{\mathbf{x}}^{-p} = (\hat{\mathbf{x}}^p, \hat{\mathbf{x}}^{-p}) = (\hat{\mathbf{x}}^1, \hat{\mathbf{x}}^2, \ldots, \hat{\mathbf{x}}^{p-1}, \hat{\mathbf{x}}^p, \hat{\mathbf{x}}^{p+1}, \ldots, \hat{\mathbf{x}}^n) = \hat{\mathbf{x}} \in X.$$

It is well known that not all the games in pure strategies have NE, but all the extended games have NE [4], i.e. $NE(\widetilde{\Gamma}) \neq \emptyset$.

There are diverse alternative formulations of a NE [2]: as a fixed point of the best response correspondence, as a fixed point of a function, as a solution of a nonlinear complementarity problem, as a solution of a stationary point problem, as a minimum of a function on a polytope, as a semi-algebraic set. The NES may be considered as an intersection of graphs of best response multivalued mappings (correspondences) [5, 8, 9] $\operatorname{Arg}\max\limits_{\mathbf{x}^p \in X_p} f_p\left(\mathbf{x}^p, \mathbf{x}^{-p}\right) : X_{-p} \multimap X_p,\ p = \overline{1,n}$:

$$NE(\widetilde{\Gamma}) = \bigcap_{p \in N} Gr_p,$$

$$Gr_p = \left\{ (\mathbf{x}^p, \mathbf{x}^{-p}) \in X : \begin{array}{l} \mathbf{x}^{-p} \in X_{-p}, \\ \mathbf{x}^p \in \operatorname{Arg}\max\limits_{\mathbf{x}^p \in X_p} f_p\left(\mathbf{x}^p, \mathbf{x}^{-p}\right) \end{array} \right\},\ p \in N.$$

The simplest solvable problems of NES determination are problems in the mixed extension of two-person 2×2 game [2, 3, 5], 2×3 game [9], and three-person $2 \times 2 \times 2$ game [8]. In this paper a method for NES computing in mixed extended $m \times n$ games and multi-matrix mixed extended games is analyzed.

According to [6]: *"The computational complexity of finding one equilibrium is unclear... Gilboa and Zemel [1] show that finding an equilibrium of a bi-matrix game with maximum payoff sum is NP-hard, so for this problem no efficient algorithm is likely to exist. The same holds for other problems that amount essentially to examining all equilibria, like finding an equilibrium with maximum support".* Consequently, the problem of Nash equilibria set computing is NP-hard. And so, from the complexity point of view proposed algorithms are admissible.

As it is easy to see, the algorithms for NES computing in multi-matrix extended games contain particularly algorithm that computes NES in extended $m \times n$ two-matrix games. But, two-matrix game has peculiar features that permit to give a more expedient algorithm. Examples have to give the reader the opportunity to easy and prompt grasp of the paper.

2 NES in two-player mixed extended $m \times n$ games

Consider a two-matrix $m \times n$ game Γ with matrices:

$$A = (a_{ij}),\ B = (b_{ij}),\ i = \overline{1,m},\ j = \overline{1,n}.$$

Let $A^i, i = \overline{1,m}$ denote the lines of matrix A,

$\mathbf{b}^j, j = \overline{1,n}$ denote the columns of matrix B,

$X = \{\mathbf{x} \in R^m : x_1 + x_2 + ... + x_m = 1, \mathbf{x} \geq 0\}$,

$Y = \{\mathbf{y} \in R^n : y_1 + y_2 + ... + y_n = 1, \mathbf{y} \geq 0\}$,

$$f_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_iy_j = (A^1\mathbf{y})x_1 + (A^2\mathbf{y})x_2 + ... + (A^m\mathbf{y})x_n,$$

$$f_2(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}x_iy_j = (\mathbf{x}^T\mathbf{b}^1)y_1 + (\mathbf{x}^T\mathbf{b}^2)y_2 + ... + (\mathbf{x}^T\mathbf{b}^n)y_n.$$

The game $\widetilde{\Gamma} = \langle X, Y;\, f_1, f_2\rangle$ is the mixed extension of Γ.

If the strategy of the second player is fixed, then the first player has to solve a linear programming problem:

$$f_1(\mathbf{x}, \mathbf{y}) \to \max, \mathbf{x} \in X. \tag{1}$$

Evidently, this problem is a linear programming parametric problem with the parameter-vector $\mathbf{y} \in Y$.

Analogically, the second player has to solve the linear programming parametric problem:

$$f_2(\mathbf{x}, \mathbf{y}) \to \max, \mathbf{y} \in Y, \tag{2}$$

with the parameter-vector $\mathbf{x} \in X$.

Denote $\mathbf{ex}^T = (1, \ldots, 1) \in R^m, \mathbf{ey}^T = (1, \ldots, 1) \in R^n$. The solution of linear programming problem is realized on the vertices of polytopes of feasible solutions. In the problems (1) and (2) the sets X and Y have m and, respectively, n vertices — the axis unit vectors $\mathbf{e}^{x_i} \in R^m, i = \overline{1,m}$, and $\mathbf{e}^{y_j} \in R^n, j = \overline{1,n}$. Thus, in accordance with the simplex method and its optimality criterion [7], in the parametric problem (1) the parameter set Y is partitioned into such m subsets

$$Y^i = \left\{ \mathbf{y} \in R^n : \begin{array}{l} (A^k - A^i)\mathbf{y} \leq 0, k = \overline{1,m}, \\ \mathbf{ey}^T\mathbf{y} = 1, \\ \mathbf{y} \geq 0 \end{array} \right\}, \quad i = \overline{1,m},$$

for which one of the optimal solution of the linear programming problem (1) is $\mathbf{e}^{x_i}$ — the corresponding x_i axis unit vector.

Let $U = \{i \in \{1,2,...,m\} : Y^i \neq \emptyset\}$. In conformity with the optimality criterion of the simplex method $\forall i \in U$ and $\forall I \in 2^{U\setminus\{i\}}$ all the points of

$$\mathbf{Conv}\{\mathbf{e}^{x_k}, k \in I \cup \{i\}\} = \left\{ \mathbf{x} \in R^m : \begin{array}{l} \mathbf{ex}^T\mathbf{x} = 1, \\ \mathbf{x} \geq \mathbf{0}, \\ x_k = 0, k \notin I \cup \{i\} \end{array} \right\}$$

are optimal for parameters

$$\mathbf{y} \in Y^{iI} = \left\{ \mathbf{y} \in R^n : \begin{array}{l} (A^k - A^i)\mathbf{y} = 0, k \in I, \\ (A^k - A^i)\mathbf{y} \leq 0, k \notin I \cup \{i\}, \\ \mathbf{ey}^T\mathbf{y} = 1, \\ \mathbf{y} \geq 0. \end{array} \right\}$$

Evidently $Y^{i\emptyset} = Y^i$. Hence,

$$Gr_1 = \bigcup_{i \in U,\ I \in 2^{U\setminus\{i\}}} \mathbf{Conv}\{\mathbf{e}^{x_k}, k \in I \cup \{i\}\} \times Y^{iI}.$$

In the parametric problem (2) the parameter set X is partitioned into such n subsets

$$X^j = \left\{ \mathbf{x} \in R^m : \begin{array}{l} (\mathbf{b}^k - \mathbf{b}^j)\mathbf{x} \leq 0, k = \overline{1,n}, \\ \mathbf{ex}^T\mathbf{x} = 1, \\ \mathbf{x} \geq 0, \end{array} \right\}, \quad j = \overline{1,n},$$

for which one of the optimal solution of the linear programming problem (2) is $\mathbf{e}^{y_j}$ — the corresponding y_j axis unit vector.

Let $V = \{j \in \{1,2,...,n\} : X^j \neq \emptyset\}$. In conformity with the optimality criterion of the simplex method $\forall j \in V$ and $\forall J \in 2^{V\setminus\{j\}}$ all the points of

$$\mathbf{Conv}\{\mathbf{e}^{y_k}, k \in J \cup \{j\}\} = \left\{ \mathbf{y} \in R^n : \begin{array}{l} \mathbf{ey}^T\mathbf{y} = 1, \\ \mathbf{y} \geq \mathbf{0}, \\ y_k = 0, k \notin J \cup \{j\} \end{array} \right\}$$

are optimal for parameters

$$\mathbf{x} \in X^{jJ} = \left\{ \mathbf{x} \in R^m : \begin{array}{l} (\mathbf{b}^k - \mathbf{b}^j)\mathbf{x} = 0, k \in J, \\ (\mathbf{b}^k - \mathbf{b}^j)\mathbf{x} \leq 0, k \notin J \cup \{j\}, \\ \mathbf{e}\mathbf{x}^T\mathbf{x} = 1, \\ \mathbf{x} \geq 0. \end{array} \right\}$$

Evidently $X^{j\emptyset} = X^j$. Hence

$$Gr_2 = \bigcup_{j \in V,\ J \in 2^{V \setminus \{j\}}} X^{jJ} \times \mathbf{Conv} \left\{ \mathbf{e}^{y_k}, k \in J \cup \{j\} \right\}.$$

Finally,

$$NE(\widetilde{\Gamma}) = Gr_1 \bigcap Gr_2 = \bigcup_{\substack{i \in U,\ I \in 2^{U \setminus \{i\}} \\ j \in V,\ J \in 2^{V \setminus \{j\}}}} X_{iI}^{jJ} \times Y_{jJ}^{iI},$$

where $X_{iI}^{jJ} \times Y_{jJ}^{iI}$ is a convex component of NES,

$$\begin{array}{l} X_{iI}^{jJ} = \mathbf{Conv} \left\{ \mathbf{e}^{x_k}, k \in I \cup \{i\} \right\} \bigcap X^{jJ}, \\ Y_{jJ}^{iI} = \mathbf{Conv} \left\{ \mathbf{e}^{y_k}, k \in J \cup \{j\} \right\} \bigcap Y^{iI}, \end{array}$$

$$X_{iI}^{jJ} = \left\{ \mathbf{x} \in R^m : \begin{array}{l} (\mathbf{b}^k - \mathbf{b}^j)\mathbf{x} = 0, k \in J, \\ (\mathbf{b}^k - \mathbf{b}^j)\mathbf{x} \leq 0, k \notin J \cup \{j\}, \\ \mathbf{e}\mathbf{x}^T\mathbf{x} = 1, \mathbf{x} \geq 0, \\ x_k = 0, k \notin I \cup \{i\} \end{array} \right\}$$

is a set of strategies $\mathbf{x} \in X$ with support from $\{i\} \cup I$ and for which points of $\mathbf{Conv} \left\{ \mathbf{e}^{y_k}, k \in J \cup \{j\} \right\}$ are optimal,

$$Y_{jJ}^{iI} = \left\{ \mathbf{y} \in R^n : \begin{array}{l} (A^k - A^i)\mathbf{y} = 0, k \in I, \\ (A^k - A^i)\mathbf{y} \leq 0, k \notin I \cup \{i\}, \\ \mathbf{e}\mathbf{y}^T\mathbf{y} = 1, \mathbf{y} \geq 0, \\ y_k = 0, k \notin J \cup \{j\} \end{array} \right\}$$

is a set of strategies $\mathbf{y} \in Y$ with support from $\{j\} \cup J$ and for which points of $\mathbf{Conv}\{\mathbf{e}^{x_k}, k \in I \cup \{i\}\}$ are optimal.

Theorem 1. $NE(\widetilde{\Gamma}) = \bigcup\limits_{\substack{i \in U,\ I \in 2^{U\setminus\{i\}} \\ j \in V,\ J \in 2^{V\setminus\{j\}}}} X_{iI}^{jJ} \times Y_{jJ}^{iI}.$

The proof of the theorem is performed above.

Theorem 2. *If* $X_{iI}^{j\emptyset} = \emptyset$*, then* $X_{iI}^{jJ} = \emptyset$ *for all* $J \in 2^V$*.*

For the proof it is sufficient to maintain that $X_{iI}^{jJ} \subseteq X_{iI}^{j\emptyset}$ for $J \neq \emptyset$.

Theorem 3. *If* $Y_{jJ}^{i\emptyset} = \emptyset$*, then* $Y_{jJ}^{iI} = \emptyset$ *for all* $I \in 2^U$*.*

Theorem 3 is equivalent to theorem 2.

From the above the algorithm for NES computing follows:

$$\begin{aligned} NE = \emptyset; \quad & U = \{i \in \{1,2,...,m\} : Y^i \neq \emptyset\}; \quad UX = U; \\ & V = \{j \in \{1,2,...,n\} : X^j \neq \emptyset\}; \end{aligned}$$

```
for  i ∈ U  do
 {
     UX = UX \ {i};
     for  I ∈ 2^UX  do
       {
           VY = V;
           for  j ∈ V  do
             {
                  if  (X_iI^{j∅} = ∅)  break;
                  VY = VY \ {j};
                  for  J ∈ 2^VY  do
                       if  (Y_jJ^{iI} ≠ ∅)  NE = NE ∪ (X_iI^{jJ} × Y_jJ^{iI});
             }
       }
 }
```

Algorithm executes the interior operator **if** no more then

$$\begin{array}{l} 2^{m-1}(2^{n-1}+2^{n-2}+\cdots+2^{1}+2^{0})+ \\ +2^{m-2}(2^{n-1}+2^{n-2}+\cdots+2^{1}+2^{0})+ \\ \ldots \\ +2^{1}(2^{n-1}+2^{n-2}+\cdots+2^{1}+2^{0})+ \\ +2^{0}(2^{n-1}+2^{n-2}+\cdots+2^{1}+2^{0})= \\ =(2^{m}-1)(2^{n}-1) \end{array}$$

times. So, the following theorem is true.

Theorem 4. *The algorithm examines no more then* $(2^m-1)(2^n-1)$ *polytopes of the* $X_{iI}^{jJ} \times Y_{jJ}^{iI}$ *type.*

If all the players' strategies are equivalent, then NES consists of $(2^m-1)(2^n-1)$ polytopes.

Evidently, for practical reasons algorithm may be improved by identifying equivalent, dominant and dominated strategies in pure game [5, 8, 9] with the following pure and extended game simplification. *"In a nondegenerate game, both players use the same number of pure strategies in equilibrium, so only supports of equal cardinality need to be examined"* [6]. This property may be used to minimize essentially the number of components $X_{iI}^{jJ} \times Y_{jJ}^{iI}$ examined in nondegenerate game.

Example 1. Matrices of the two person game [6] are

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 3 \\ 6 & 5 & 3 \end{bmatrix}.$$

The exterior cycle is executed for the value $i = 1$.

As

$$X_{1\emptyset}^{1\emptyset} = \left\{ \mathbf{x} : \begin{array}{c} 2x_1 - x_2 \leq 0, \\ 3x_1 - 3x_2 \leq 0, \\ x_1 + x_2 = 1, \\ x_1 \geq 0, x_2 = 0 \end{array} \right\} = \emptyset$$

then the cycle for $j = 1$ is omitted.

Since

$$X_{1\emptyset}^{2\emptyset} = \left\{ \mathbf{x} : \begin{array}{l} -2x_1 + x_2 \le 0, \\ x_1 - 2x_2 \le 0, \\ x_1 + x_2 = 1, \\ x_1 \ge 0, x_2 = 0 \end{array} \right\} = \emptyset$$

then the cycle for j=2 is omitted.

As

$$X_{1\emptyset}^{3\emptyset} = \left\{ \mathbf{x} : \begin{array}{l} -3x_1 + 3x_2 \le 0, \\ -x_1 + 2x_2 \le 0, \\ x_1 + x_2 = 1, \\ x_1 \ge 0, x_2 = 0 \end{array} \right\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \ne \emptyset,$$

$$Y_{3\emptyset}^{1\emptyset} = \left\{ \mathbf{y} : \begin{array}{l} -y_1 + 2y_2 - y_3 \le 0, \\ y_1 + y_2 + y_3 = 1, \\ y_1 = 0, y_2 = 0, y_3 \ge 0 \end{array} \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \ne \emptyset,$$

the point $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is a Nash equilibrium for which $f_1 = 4$, $f_2 = 3$.

$$X_{1\{2\}}^{1\emptyset} = \left\{ \mathbf{x} : \begin{array}{l} 2x_1 - x_2 \le 0, \\ 3x_1 - 3x_2 \le 0, \\ x_1 + x_2 = 1, \\ x_1 \ge 0, x_2 \ge 0 \end{array} \right\} \ne \emptyset,$$

$$Y_{1\emptyset}^{1\{2\}} = \left\{ \mathbf{y} : \begin{array}{l} -y_1 + 2y_2 - y_3 = 0, \\ y_1 + y_2 + y_3 = 1, \\ y_1 \ge 0, y_2 = 0, y_3 = 0 \end{array} \right\} = \emptyset,$$

Since

$$X_{1\{2\}}^{1\{2\}} = \left\{ \mathbf{x} : \begin{array}{l} 2x_1 - x_2 = 0, \\ 3x_1 - 3x_2 \le 0, \\ x_1 + x_2 = 1, \\ x_1 \ge 0, x_2 \ge 0 \end{array} \right\} = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} \ne \emptyset,$$

$$Y_{1\{2\}}^{1\{2\}} = \left\{ \mathbf{y} : \begin{array}{l} -y_1 + 2y_2 - y_3 = 0, \\ y_1 + y_2 + y_3 = 1, \\ y_1 \ge 0, y_2 \ge 0, y_3 = 0 \end{array} \right\} = \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix} \ne \emptyset,$$

the point $\begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} \times \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix}$ is a Nash equilibrium for which $f_1 = 2/3$, $f_2 = 4$.

$$X_{1\{2\}}^{1\{3\}} = \left\{ \mathbf{x} : \begin{array}{l} 2x_1 - x_2 \le 0, \\ 3x_1 - 3x_2 = 0, \\ x_1 + x_2 = 1, \\ x_1 \ge 0, x_2 \ge 0 \end{array} \right\} = \emptyset,$$

$$X_{1\{2\}}^{1\{2,3\}} = \left\{ \mathbf{x} : \begin{array}{l} 2x_1 - x_2 = 0, \\ 3x_1 - 3x_2 = 0, \\ x_1 + x_2 = 1, \\ x_1 \ge 0, x_2 \ge 0 \end{array} \right\} = \emptyset,$$

$$X_{1\{2\}}^{2\emptyset} = \left\{ \mathbf{x} : \begin{array}{l} -2x_1 + x_2 \le 0, \\ x_1 - 2x_2 \le 0, \\ x_1 + x_2 = 1, \\ x_1 \ge 0, x_2 \ge 0 \end{array} \right\} \ne \emptyset,$$

$$Y_{2\emptyset}^{1\{2\}} = \left\{ \mathbf{y} : \begin{array}{l} -y_1 + 2y_2 - y_3 = 0, \\ y_1 + y_2 + y_3 = 1, \\ y_1 = 0, y_2 \ge 0, y_3 = 0 \end{array} \right\} = \emptyset,$$

As

$$X_{1\{2\}}^{2\{3\}} = \left\{ \mathbf{x} : \begin{array}{l} -2x_1 + x_2 \le 0, \\ x_1 - 2x_2 = 0, \\ x_1 + x_2 = 1, \\ x_1 \ge 0, x_2 \ge 0 \end{array} \right\} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} \ne \emptyset,$$
$$Y_{2\{3\}}^{1\{2\}} = \left\{ \mathbf{y} : \begin{array}{l} -y_1 + 2y_2 - y_3 = 0, \\ y_1 + y_2 + y_3 = 1, \\ y_1 = 0, y_2 \ge 0, y_3 = 0 \end{array} \right\} = \begin{pmatrix} 0 \\ 1/3 \\ 2/3 \end{pmatrix} \ne \emptyset,$$

the point $\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1/3 \\ 2/3 \end{pmatrix}$ is a Nash equilibrium for which $f_1 = 8$,

$f_2 = 3$.

$$X_{1\{2\}}^{3\emptyset} = \left\{ \mathbf{x} : \begin{array}{l} -2x_1 + x_2 \leq 0, \\ x_1 - 2x_2 = 0, \\ x_1 + x_2 = 1, \\ x_1 \geq 0, x_2 \geq 0 \end{array} \right\} \neq \emptyset,$$

$$Y_{3\emptyset}^{1\{2\}} = \left\{ \mathbf{y} : \begin{array}{l} -y_1 + 2y_2 - y_3 = 0, \\ y_1 + y_2 + y_3 = 1, \\ y_1 = 0, y_2 = 0, y_3 \geq 0 \end{array} \right\} = \emptyset.$$

The exterior cycle is executed for the value $i = 2$.

$$X_{2\emptyset}^{1\emptyset} = \left\{ \mathbf{x} : \begin{array}{l} 2x_1 - x_2 \leq 0, \\ 3x_1 - 3x_2 \leq 0, \\ x_1 + x_2 = 1, \\ x_1 = 0, x_2 \geq 0 \end{array} \right\} \neq \emptyset,$$

$$Y_{1\emptyset}^{2\emptyset} = \left\{ \mathbf{y} : \begin{array}{l} y_1 - 2y_2 + y_3 \leq 0, \\ y_1 + y_2 + y_3 = 1, \\ y_1 \geq 0, y_2 = 0, y_3 = 0 \end{array} \right\} = \emptyset,$$

$$X_{2\emptyset}^{1\{2\}} = \left\{ \mathbf{x} : \begin{array}{l} 2x_1 - x_2 = 0, \\ 3x_1 - 3x_2 \leq 0, \\ x_1 + x_2 = 1, \\ x_1 = 0, x_2 \geq 0 \end{array} \right\} = \emptyset,$$

$$X_{2\emptyset}^{1\{3\}} = \left\{ \mathbf{x} : \begin{array}{l} 2x_1 - x_2 \leq 0, \\ 3x_1 - 3x_2 = 0, \\ x_1 + x_2 = 1, \\ x_1 = 0, x_2 \geq 0 \end{array} \right\} = \emptyset,$$

$$X_{2\emptyset}^{1\{2,3\}} = \left\{ \mathbf{x} : \begin{array}{l} 2x_1 - x_2 = 0, \\ 3x_1 - 3x_2 = 0, \\ x_1 + x_2 = 1, \\ x_1 = 0, x_2 \geq 0 \end{array} \right\} = \emptyset.$$

Since

$$X_{2\emptyset}^{2\emptyset} = \left\{ \mathbf{x} : \begin{array}{l} -2x_1 + x_2 \leq 0, \\ x_1 - 2x_2 \leq 0, \\ x_1 + x_2 = 1, \\ x_1 = 0, x_2 \geq 0 \end{array} \right\} = \emptyset.$$

the cycle for $j = 2$ is omitted.

$$X_{2\emptyset}^{3\emptyset} = \left\{ \mathbf{x} : \begin{array}{l} -3x_1 + 3x_2 \le 0, \\ -x_1 + 2x_2 \le 0, \\ x_1 + x_2 = 1, \\ x_1 = 0, x_2 \ge 0 \end{array} \right\} = \emptyset.$$

Thus, the NES consists of three elements — one pure and two mixed Nash equilibria.

The following example 2 illustrates that a simple modification in the first example of one element of the cost matrix of the first player changes the NES to a continue power set that consists of one isolated point and one segment.

Example 2. If in the first example the element a_{23} of the matrix A is modified

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & \boxed{4} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 3 \\ 6 & 5 & 3 \end{bmatrix},$$

then the NES of the obtained game consists of one distinct point $\begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} \times \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix}$ for which $f_1 = 10/9, f_2 = 4$ and of one distinct segment $\left[\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ for which $f_1 = 4, f_2 = 3$.

A subsequent modification of the cost matrix of the second player in precedent game transforms the NES into non-convex continuum.

Example 3. If in the second example the first column of the matrix B is equal to the second column

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & \boxed{4} \end{bmatrix}, \quad B = \begin{bmatrix} \boxed{2} & 2 & 3 \\ \boxed{5} & 5 & 3 \end{bmatrix},$$

then the NES of the such game consists of four connected segments:

- $\left[\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 1-1/3\lambda \\ 1/3\lambda \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$
  $\lambda \in [0;1]$, for which $f_1 = 4,\ f_2 = 3$,
- $\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} \times \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix}\right] \equiv \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} \times \begin{pmatrix} 2/3-2/3\mu \\ 1/3-1/3\mu \\ \mu \end{pmatrix},$
  $\mu \in [0;1]$, for which $f_1 = 2/3 + 10/3\mu,\ f_2 = 3$,
- $\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}\right] \times \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 2/3-2/3\lambda \\ 1/3+1/3\lambda \end{pmatrix} \times \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix},$
  $\lambda \in [0;1]$, for which $f_1 = 2/3,\ f_2 = 3 + 2\lambda$,
- $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \times \left[\begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right] \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2/3\mu \\ 1-2/3\mu \\ 0 \end{pmatrix},$
  $\mu \in [0;1]$, for which $f_1 = 2 - 4/3\mu,\ f_2 = 5$.

The following example 4 supplements preceding examples and illustrates that one of the NE may be strong dominant (optimal by Pareto) among all the others NE.

Example 4. The NES of the extended game with matrices:

$$A = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 2 & -1 \\ -1 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & 2 \end{bmatrix},$$

consists of two isolated points and one segment:

- $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, for which $f_1 = 6,\ f_2 = 3$.
- $\begin{pmatrix} 1/14 \\ 12/14 \\ 2/14 \end{pmatrix} \times \begin{pmatrix} 1/4 \\ 1/2 \\ 1/4 \end{pmatrix}$, for which $f_1 = 45/28,\ f_2 = -1/8$.

- $\left[\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\3/5\\2/5\end{pmatrix}\right] \times \begin{pmatrix}0\\1\\0\end{pmatrix} \equiv \begin{pmatrix}0\\3/5+2/5\lambda\\2/5-2/5\lambda\end{pmatrix} \times \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\lambda \in [0;1]$, for which $f_1 = 2$, $f_2 = 1/5 + 4/5\lambda$.

Evidently, the NE $\begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}0\\0\\1\end{pmatrix}$, for which $f_1 = 6$, $f_2 = 3$, strongly dominates all the other NE and it is more preferable than the other NE for both of the players.

Example 1 confirms the well known property that the number of Nash equilibria in the nondegenerate game is odd [2, 6]. Examples 2 – 4 illustrate that in the degenerate game the number of convex components may be both the even and the odd. Examples 3 – 4 illustrate that in the degenerate game the player cannot increase own gain by modifying his Nash strategy, but, by modifying his Nash strategy the player may essentially modify (increase or decrease) the gain of the opponent.

3 NES in n-player mixed extended $m_1 \times m_2 \ldots m_n$ games

Consider the n-player extended game $\widetilde{\Gamma} = \langle X_p, f_p(\mathbf{x}), p \in N\rangle$ formulated in the Introduction. The cost function of the player p is linear if the strategies of the remaining players are fixed:

$$f_p(\mathbf{x}) = \left(\sum_{\mathbf{s}_{-p} \in S_{-p}} a^p_{1||\mathbf{s}_{-p}} \prod_{\substack{q=\overline{1,n}\\ q \neq p}} x^q_{s_q} \right) x^p_1 +$$

$$+ \left(\sum_{\mathbf{s}_{-p} \in S_{-p}} a^p_{2||\mathbf{s}_{-p}} \prod_{\substack{q=\overline{1,n}\\ q \neq p}} x^q_{s_q} \right) x^p_2 +$$

$$+\cdots+ \left(\sum_{\mathbf{s}_{-p} \in S_{-p}} a^p_{m_p||\mathbf{s}_{-p}} \prod_{\substack{q=\overline{1,n} \\ q \neq p}} x^q_{s_q} \right) x^p_{m_p}.$$

Thus, the player p has to solve a linear parametric problem with parameter vector $x^{-p} \in X^{-p}$:

$$f_p\left(\mathbf{x}^p, \mathbf{x}^{-p}\right) \to \max, \quad \mathbf{x}^p \in X_p, \quad p = \overline{1,n}.$$

The solution of this problem is realized on the vertices of polytope X_p that has m_p vertices — the x^p_i axis unit vectors $\mathbf{e}^{x^p_i} \in R^{m_i}, i = \overline{1, m_p}$. Thus, in accordance with the simplex method and its optimality criterion [7], the parameter set X_{-p} is partitioned into such m_p subsets

$$X_{-p}\left(i_p\right) =$$

$$= \left\{ \mathbf{x}^{-p} \in R^{m-m_p} : \begin{array}{c} \sum\limits_{\mathbf{s}_{-p} \in S_{-p}} \left(a^p_{k||\mathbf{s}_{-p}} - a^p_{i_p||\mathbf{s}_{-p}} \right) \prod\limits_{\substack{q=\overline{1,n} \\ q \neq p}} x^q_{s_q} \leq 0, \\ k = \overline{1, m_p}, \\ \sum\limits_{i=1}^{m_q} x^q_i = 1, \quad q = \overline{1,n},\ q \neq p, \\ \mathbf{x}^{-p} \geq 0 \end{array} \right\},$$

$i_p = \overline{1, m_p}$, for which one of the optimal solution of the linear programming problem (1) is $\mathbf{e}^{x^p_i}$.

Let $U_p = \{i_p \in \{1, ..., m_p\} : X_{-p}(i_p) \neq \emptyset\}, \mathbf{ep}^T = (1, \ldots, 1) \in R^{m_p}$. In conformity with the optimality criterion of the simplex method $\forall i_p \in U_p$ and $\forall I_p \in 2^{U_p \setminus \{i_p\}}$ all the points of

$$\mathbf{Conv}\left\{\mathbf{e}^{x^p_k}, k \in I_p \cup \{i_p\}\right\} = \left\{ \mathbf{x} \in R^m : \begin{array}{l} \mathbf{ep}^T \mathbf{x}^p = 1, \\ \mathbf{x}^p \geq 0, \\ x^p_k = 0, k \notin I_p \cup \{i_p\} \end{array} \right\}$$

are optimal for parameters $\mathbf{x}^{-p} \in X_{-p}\left(i_p I_p\right) \subset R^{m-m_p}$, where

$X_{-p}(i_pI_p)$ is a set of solutions of the system:

$$\begin{cases} \sum\limits_{\mathbf{s}_{-p}\in S_{-p}} \left(a^p_{k||\mathbf{s}_{-p}} - a^p_{i_p||\mathbf{s}_{-p}}\right) \prod\limits_{\substack{q=\overline{1,n}\\ q\neq p}} x^q_{s_q} = 0, \quad k\in I_p, \\ \sum\limits_{\mathbf{s}_{-p}\in S_{-p}} \left(a^p_{k||\mathbf{s}_{-p}} - a^p_{i_p||\mathbf{s}_{-p}}\right) \prod\limits_{\substack{q=\overline{1,n}\\ q\neq p}} x^q_{s_q} \leq 0, \quad k\notin I_p\cup i_p, \\ \mathbf{er}^T\mathbf{x}^r = 1, \quad r=\overline{1,n},\ r\neq p, \\ \mathbf{x}^r \geq 0, \quad r=\overline{1,n},\ r\neq p. \end{cases}$$

Evidently $X_{-p}(i_p\emptyset) = X_{-p}(i_p\emptyset)$. Hence,

$$Gr_p = \bigcup_{i_p\in U_p,\ I_p\in 2^{U_p\setminus\{i_p\}}} \mathbf{Conv}\left\{\mathbf{e}^{x^p_k}, k\in I_p\cup\{i_p\}\right\}\times X_{-p}(i_pI_p).$$

Finally,

$$NE(\widetilde{\Gamma}) = \bigcap_{p=1}^{n} Gr_p = \bigcup_{\substack{i_1\in U_1,\ I_1\in 2^{U_1\setminus\{i_1\}},\\ \dots\\ i_n\in U_n,\ I_n\in 2^{U_n\setminus\{i_n\}}}} X(i_1I_1\dots i_nI_n)$$

where $X(i_1I_1\dots i_nI_n) = NE(i_1I_1\dots i_nI_n)$ is a set of solutions of the system:

$$\begin{cases} \sum\limits_{\mathbf{s}_{-r}\in S_{-r}} \left(a^r_{k||\mathbf{s}_{-r}} - a^r_{i_r||\mathbf{s}_{-r}}\right) \prod\limits_{\substack{q=\overline{1,n}\\ q\neq r}} x^q_{s_q} = 0, \quad k\in I_r, \\ \sum\limits_{\mathbf{s}_{-r}\in S_{-r}} \left(a^r_{k||\mathbf{s}_{-r}} - a^r_{i_r||\mathbf{s}_{-r}}\right) \prod\limits_{\substack{q=\overline{1,n}\\ q\neq r}} x^q_{s_q} \leq 0, \quad k\notin I_r\cup i_r, \\ r=\overline{1,n}, \\ \mathbf{er}^T\mathbf{x}^r = 1, \quad \mathbf{x}^r\geq 0,\ r=\overline{1,n}, \\ x^r_k = 0, \quad k\notin I_r\cup\{i_r\},\ r=\overline{1,n}. \end{cases}$$

Theorem 5. $NE(\widetilde{\Gamma}) = \bigcup\limits_{\substack{i_1 \in U_1,\ I_1 \in 2^{U_1 \setminus \{i_1\}} \\ \dots \\ i_n \in U_n,\ I_n \in 2^{U_n \setminus \{i_n\}}}} X\left(i_1 I_1 \dots i_n I_n\right).$

The theorem 5 is an extension of theorem 1 to an n-player game. The proof is performed above.

The following theorem is a corollary of theorem 5.

Theorem 6. *$NE(\widetilde{\Gamma})$ consists of no more then*

$$(2^{m_1} - 1)(2^{m_2} - 1) \dots (2^{m_n} - 1)$$

components of the type $X(i_1 I_1 \dots i_n I_n)$.

In game for which all the players have equivalent strategies NES is partitioned in maximal number $(2^{m_1} - 1)(2^{m_2} - 1) \dots (2^{m_n} - 1)$ of components.

In general, in n-player game ($n \geq 3$) components $X(i_1 I_1 \dots i_n I_n)$ are non-convex.

An exponential algorithm for NES computing in n-player game simply follows from the expression in theorem 5. The algorithm requires to solve $(2^{m_1} - 1)(2^{m_2} - 1) \dots (2^{m_n} - 1)$ finite systems of multilinear ($n-1$-linear) and linear equations and inequalities in m variables. The last problem is itself a difficult one.

Example 5. It is considered a three-player extended $2 \times 2 \times 2$ (diadic) game [2] with matrices:

$$a_{1**} = \begin{bmatrix} 9 & 0 \\ 0 & 3 \end{bmatrix}, \quad a_{2**} = \begin{bmatrix} 0 & 3 \\ 9 & 0 \end{bmatrix},$$

$$b_{*1*} = \begin{bmatrix} 8 & 0 \\ 0 & 4 \end{bmatrix}, \quad b_{*2*} = \begin{bmatrix} 0 & 4 \\ 8 & 0 \end{bmatrix},$$

$$c_{**1} = \begin{bmatrix} 12 & 0 \\ 0 & 2 \end{bmatrix}, \quad c_{**2} = \begin{bmatrix} 0 & 6 \\ 4 & 0 \end{bmatrix}.$$

$$f_1(\mathbf{x},\mathbf{y},\mathbf{z}) = (9y_1z_1 + 3y_2z_2)x_1 + (9y_2z_1 + 3y_1z_2)x_2,$$

$$f_2(\mathbf{x},\mathbf{y},\mathbf{z}) = (8x_1z_1 + 4x_2z_2)y_1 + (8x_2z_1 + 4x_1z_2)y_2,$$

$$f_3(\mathbf{x},\mathbf{y},\mathbf{z}) = (12x_1y_1 + 2x_2y_2)z_1 + (4x_2y_1 + 6x_1y_2)z_2.$$

Totally, we have to consider $(2^2-1)(2^2-1)(2^2-1) = 27$ components. Further, we will enumerate only nonempty components. Thus, $NE(1\emptyset1\emptyset1\emptyset) = (1;0)\times(1;0)\times(1;0)$ (for which $f_1 = 9, f_2 = 8, f_3 = 12$) is the solution of the system:

$$\begin{cases} 9y_2z_1 + 3y_1z_1 - 9y_1z_1 - 3y_2z_2 = 3(y_2-y_1)(3z_1-z_2) \le 0, \\ 8x_2z_1 + 4x_1z_2 - 8x_1z_1 - 4x_2z_2 = 4(x_2-x_1)(4z_1-z_2) \le 0, \\ 4x_2y_1 + 6x_1y_2 - 12x_1y_1 + 2x_2y_2 = 2(3x_1-x_2)(y_2-2y_1) \le 0, \\ x_1+x_2 = 1, x_1 \ge 0, x_2 = 0, \\ y_1+y_2 = 1, y_1 \ge 0, y_2 = 0, \\ z_1+z_2 = 1, z_1 \ge 0, z_2 = 0. \end{cases}$$

$NE(1\emptyset2\emptyset2\emptyset) = (1;0)\times(0;1)\times(0;1)$ and $f_1 = 3, f_2 = 4, f_3 = 6$:

$$\begin{cases} 3(y_2-y_1)(3z_1-z_2) \le 0, \\ -4(x_2-x_1)(4z_1-z_2) \le 0, \\ -2(3x_1-x_2)(y_2-2y_1) \le 0, \\ x_1+x_2 = 1, x_1 \ge 0, x_2 = 0, \\ y_1+y_2 = 1, y_1 = 0, y_2 \ge 0, \\ z_1+z_2 = 1, z_1 = 0, z_2 \ge 0. \end{cases}$$

$NE(2\emptyset1\emptyset2\emptyset) = (0;1)\times(1;0)\times(0;1)$ and $f_1 = 3, f_2 = 4, f_3 = 4$:

$$\begin{cases} -3(y_2-y_1)(3z_1-z_2) \le 0, \\ 4(x_2-x_1)(4z_1-z_2) \le 0, \\ -2(3x_1-x_2)(y_2-2y_1) \le 0, \\ x_1+x_2 = 1, x_1 = 0, x_2 \ge 0, \\ y_1+y_2 = 1, y_1 \ge 0, y_2 = 0, \\ z_1+z_2 = 1, z_1 = 0, z_2 \ge 0. \end{cases}$$

$NE(2\emptyset 2\emptyset 1\emptyset) = (0;1) \times (0;1) \times (1;0)$ and $f_1 = 9, f_2 = 8, f_3 = 2$:

$$\begin{cases} -3(y_2 - y_1)(3z_1 - z_2) \leq 0, \\ -4(x_2 - x_1)(4z_1 - z_2) \leq 0, \\ 2(3x_1 - x_2)(y_2 - 2y_1) \leq 0, \\ x_1 + x_2 = 1, x_1 = 0, x_2 \geq 0, \\ y_1 + y_2 = 1, y_1 = 0, y_2 \geq 0, \\ z_1 + z_2 = 1, z_1 \geq 0, z_2 = 0. \end{cases}$$

$NE(1\emptyset 1\{2\}1\{2\}) = (1;0) \times (1/3;2/3) \times (1/5;4/5)$ and $f_1 = 11/5$, $f_2 = 8/3$, $f_3 = 4$:

$$\begin{cases} 3(y_2 - y_1)(3z_1 - z_2) \leq 0, \\ 4(x_2 - x_1)(4z_1 - z_2) = 0, \\ 2(3x_1 - x_2)(y_2 - 2y_1) = 0, \\ x_1 + x_2 = 1, x_1 \geq 0, x_2 = 0, \\ y_1 + y_2 = 1, y_1 \geq 0, y_2 \geq 0, \\ z_1 + z_2 = 1, z_1 \geq 0, z_2 \geq 0. \end{cases}$$

$NE(1\{2\}2\emptyset 1\{2\}) = (2/5;3/5) \times (0;1) \times (1/4;3/4)$ and $f_1 = 9/4$, $f_2 = 12/5, f_3 = 21/10$:

$$\begin{cases} 3(y_2 - y_1)(3z_1 - z_2) = 0, \\ -4(x_2 - x_1)(4z_1 - z_2) \leq 0, \\ 2(3x_1 - x_2)(y_2 - 2y_1) = 0, \\ x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0, \\ y_1 + y_2 = 1, y_1 = 0, y_2 \geq 0, \\ z_1 + z_2 = 1, z_1 \geq 0, z_2 \geq 0. \end{cases}$$

$NE(1\{2\}1\{2\}1\emptyset) = (1/2;1/2) \times (1/2;1/2) \times (1;0)$ and $f_1 = 9/2$, $f_2 = 4, f_3 = 7/2$:

$$\begin{cases} 3(y_2 - y_1)(3z_1 - z_2) = 0, \\ -4(x_2 - x_1)(4z_1 - z_2) = 0, \\ 2(3x_1 - x_2)(y_2 - 2y_1) \leq 0, \\ x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0, \\ y_1 + y_2 = 1, y_1 \geq 0, y_2 \geq 0, \\ z_1 + z_2 = 1, z_1 \geq 0, z_2 = 0. \end{cases}$$

$$NE(1\{2\}1\{2\}1\{2\}) =$$

$$\left\{ \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \times \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} \times \begin{pmatrix} 1/4 \\ 3/4 \end{pmatrix}, \begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix} \times \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \times \begin{pmatrix} 1/5 \\ 4/5 \end{pmatrix} \right\}$$

for which $f_1 = 9/4, f_2 = 5/2, f_3 = 8/3$ and respective $f_1 = 21/10$, $f_2 = 12/5, f_3 = 63/25$:

$$\begin{cases} 3(y_2 - y_1)(3z_1 - z_2) = 0, \\ -4(x_2 - x_1)(4z_1 - z_2) = 0, \\ 2(3x_1 - x_2)(y_2 - 2y_1) = 0, \\ x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0, \\ y_1 + y_2 = 1, y_1 \geq 0, y_2 \geq 0, \\ z_1 + z_2 = 1, z_1 \geq 0, z_2 \geq 0. \end{cases}$$

Thus the game has 9 Nash equilibria. Remark that the last component of the NES consists of two distinct points. Hence, it is non-convex and non-connected.

4 Conclusions

The idea to consider NES as an intersection of the graphs of best response mappings yields to a simply NES representation and to a method of NES computing. Taking into account the computational complexity of the problem, the proposed exponential algorithms are pertinent.

The NES in two-matrix extended games may be partitioned into finite number of polytopes, no more then $(2^m - 1)(2^n - 1)$. The proposed algorithm examines, in general, a much more smaller number of sets of the type $X_{iI}^{jJ} \times Y_{jJ}^{iI}$.

The NES in multi-matrix extended games may be partitioned into finite number of components, no more then $(2^{m_1} - 1) \dots (2^{m_n} - 1)$, but they, in general, are non-convex and moreover non-polytopes. The algorithmic realization of the method is closely related with the problem of solving the systems of multilinear ($n - 1$-linear and simply linear) equations and inequalities, that represents in itself a serious obstacle to efficient NES computing.

References

[1] I. Gilboa, E. Zemel. *Nash and correlated equilibria: some complexity considerations,* Games and Economic Behavior, Volume 1, 1989, pp. 80–93.

[2] R.D. McKelvey, A. McLenan. *Computation of equilibria in finite games,* Handbook of Computational Economics, Volume 1, Elsevier, 1996, pp. 87–142.

[3] H. Moulen. *Théorie des jeux pour l'éqonomie et la politique,* Paris, 1981 (in Russian: Games Theory, Moscow, Mir, 1985, 200 p.).

[4] J.F. Nash. *Noncooperative game,* Annals of Mathematics, Volume 54, 1951, pp. 280–295.

[5] M. Sagaidac, V. Ungureanu. *Operational research,* Chişinău, CEP USM, 2004, 296 p. (in Romanian).

[6] von Stengel, Bernhard. *Computing equilibria for two-person games,* Handbook of Game Theory with Economic Applications, Volume 3, Elsevier, 2002, pp. 1723–1759.

[7] V. Ungureanu. *Mathematical programming,* Chişinău, USM, 2001, 344 p. (in Romanian).

[8] V. Ungureanu, A. Botnari. *Nash equilibria sets in mixed extension of* $2 \times 2 \times 2$ *games,* Computer Science Journal of Moldova, Volume 13, no. 1 (37), 2005, pp. 13–28.

[9] V. Ungureanu, A. Botnari. *Nash equilibria sets in mixed extended* 2×3 *games,* Computer Science Journal of Moldova, Volume 13, no. 2 (38), 2005, pp. 136–150.

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