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On quasi-stability of the vector Boolean problem of minimizing absolute deviations of linear functions from zero*

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Abstract

We consider a multi-criterion Boolean programming problem with partial criteria of the kind MIN MODUL of linear functions. We investigate such type of stability which can be understood as a discrete analogue of the Hausdorff lower semi-continuity. A formula of the quasi-stability radius is obtained.

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1 Introduction

Let us consider a vector (m-criterion) Boolean programming problem with a finite solution set X

$$Z^m(A, b) : \ \min\{f(x, A, b) : \ x \in X\},$$

where

$$f(x, A, b) = (|A_1x + b_1|, |A_2x + b_2|, \dots, |A_mx + b_m|),$$

$X \subseteq \mathbf{E}^n = \{0,1\}^n$, $|X| \geq 2$, $n \geq 2$, A_i denotes the i–th row of matrix $A = [a_{ij}]_{m\times n} \in \mathbf{R}^{m\times n}$, $i \in N_m = \{1, 2, \dots, m\}$, $m \geq 1$, $b = (b_1, b_2, \dots, b_m)^T \in \mathbf{R}^m$, $x = (x_1, x_2, \dots, x_n)^T$.

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By problem $Z^m(A,b)$ we understand the problem of finding the set of efficient solutions (Pareto set)

$$P^m(A,b) = \{x \in X : X_x(A,b) = \emptyset\},$$

where

$$X_x(A,b) = \{x' \in X : f(x,A,b) \geq f(x',A,b) \ \& \ f(x,A,b) \neq f(x',A,b)\}.$$

By virtue of inequalities $1 < |X| < \infty$, the set $P^m(A,b)$ is nonempty for any $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$.

Note, that the vector function $f(x,A,b)$ shows a measure of inconsistency of the following system of linear boolean equalities

$$Ax + b = \mathbf{0}^{(m)}, \quad x \in X. \tag{1}$$

where $\mathbf{0}^{(m)} = (0,0,\dots,0) \in \mathbf{R}^m$.

Thus, minimization of function $|A_i x + b_i|$ on the finite set X is equivalent to the minimization of absolute deviations of linear functions $A_i x + b_i$ from zero. Therefore, problem $Z^m(A,b)$ is reduced to finding the set of all solutions to system (1) under the condition that this system is consistent. Otherwise, Pareto set $P^m(A,b)$ can be considered as the set of quasi solutions to system (1). It is easy to see that the system of equalities is consistent if and only if the set of efficient vector estimates

$$f(P^m(A,b)) = \{y \in \mathbf{R}^m : \ y = f(x,A,b), \ x \in P^m(A,b)\}$$

contains only zero vector $\mathbf{0}^{(m)}$.

For each number $k \in \mathbf{N}$, we endow the space $\mathbf{R}^k$ with two metrics l_1 and l_∞, i.e. we define the norms of a vector $z = (z_1, z_2, \dots, z_k) \in \mathbf{R}^k$ as follows:

$$||z||_1 = \sum_{j \in N_k} |z_j|, \quad ||z||_\infty = \max_{j \in N_k} |z_j|.$$

Under a norm of matrix we understand the norm of vector, composed from all its elements.

For any positive number $\varepsilon > 0$, we define the set of perturbing pairs

$$\Omega(\varepsilon) = \{(A', b') \in \mathbf{R}^{m\times(n+1)} : \max\{||A'||_\infty, ||b'||_\infty\} < \varepsilon\}.$$

As usual [1 - 6], problem $Z^m(A, b)$ is called quasi-stable, if the set

$$\Xi = \{\varepsilon > 0 : \forall\ (A', b') \in \Omega(\varepsilon)\ (P^m(A, b) \subseteq P^m(A + A', b + b'))\} \neq \emptyset.$$

Therefore, problem $Z^m(A, b)$ is quasi-stable, if there exists a neighborhood of pair (A, b) in the space of perturbing parameters $\mathbf{R}^{m\times(n+1)}$ with metric l_∞, in which Pareto optimality of all solutions is preserved. Note, that the quasi-stability of the problem is a discrete analogue of the Hausdorff lower semi-continuity [7] of the optimal mapping at point (A, b)

$$P^m : \ \mathbf{R}^{m\times(n+1)} \to 2^X,$$

i.e. of the point-to-set (many-valued) mapping, that puts in correspondence the Pareto set $P^m(A, b)$ to the collection of parameters (pair (A, b)) of the problem.

As a result of the said above, under the quasi-stability radius of $Z^m(A, b)$ we will understand the number

$$\rho^m(A, b) = \begin{cases} \sup\ \Xi, & \text{if } \Xi \neq \emptyset, \\ 0, & \text{if } \Xi = \emptyset. \end{cases}$$

For any $x, x' \in X$ and $i \in N_m$, put

$$\alpha_i(x, x') = \min\{\beta_i(x, x', s) : \ s \in \{-1; 1\}\},$$

$$\beta_i(x, x', s) = \frac{|A_i(sx + x') + b_i(s + 1)|}{||sx + x'||_1 + |s + 1|},$$

$$K(x, x') = \{i \in N_m : \ |A_i x + b_i| \leq |A_i x' + b_i|\}.$$

It is evident, that $K(x, x') \neq \emptyset$, if $x \in P^m(A, b)$. For any number $z \in \mathbf{R}$, denote

$$\text{sg } z = \begin{cases} 1, & \text{if } z \geq 0, \\ -1, & \text{if } z < 0. \end{cases}$$

We will also use the following implication:

$$\exists s \in \{-1; 1\} \ \ \forall s' \in \{-1; 1\} \ \ (sz > s'z') \Rightarrow |z| > |z'|, \qquad (2)$$

which holds for any numbers $z, z' \in \mathbf{R}$.

2 The Results

Theorem. *The quasi-stability radius of problem $Z^m(A, b)$ is expressed by the formula*

$$\rho^m(A, b) = \min_{x \in P^m(A,b)} \min_{x' \in X \setminus \{x\}} \max_{i \in K(x,x')} \alpha_i(x, x'). \qquad (3)$$

Proof. Denote by φ the right side of (3). It is easy to see, that $\varphi \geq 0$.

At first we will prove the inequality $\rho^m(A, b) \geq \varphi$. Suppose $\varphi > 0$ (this inequality is evident where $\varphi = 0$). Let $(A', b') \in \Omega(\varphi)$. Then by definition of φ, for any solutions $x \in P^m(A, b)$ and $x' \in X \setminus \{x\}$, there exists $k \in K(x, x')$ such that

$$\max\{||A'||_\infty, ||b'||_\infty\} < \varphi \leq \alpha_k(x, x'). \qquad (4)$$

Taking into account $\alpha_k(x, x') > 0$, we have

$$|A_k x + b_k| < |A_k x' + b_k|.$$

From this, assuming $\sigma = \text{sg } (A_k x' + b_k)$, we obtain that the following equalities hold:

$$A_k(sx + \sigma x') + b_k(\sigma + s) = |A_k(\sigma s x + x') + (1 + \sigma s) b_k|, \ \ s \in \{-1; 1\}.$$

Therefore, applying (4), we derive

$$s((A_k + A'_k)x + (b_k + b'_k)) + \sigma((A_k + A'_k)x' + (b_k + b'_k)) =$$

$$= |A_k(\sigma s x + x') + (1 + \sigma s) b_k| + \sigma(A'_k(\sigma s x + x') + b'_k(1 + \sigma s)) \geq$$

$$\geq |A_k(\sigma s x + x') + (1 + \sigma s) b_k| - (||A'||_\infty ||\sigma s x + x'||_1 +$$

$$+||b'||_\infty|1+\sigma s|) \geq |A_k(\sigma sx+x')+(1+\sigma s)b_k|-$$

$$-\max(||A'||_\infty,||b'||_\infty)(||\sigma sx+x'||_1+|1+\sigma s|)>$$

$$>|A_k(\sigma sx+x')+(1+\sigma s)b_k|-$$

$$-\alpha_k(x,x')(||\sigma sx+x'||_1+|1+\sigma s|)\geq$$

$$\geq |A_k(\sigma sx+x')+(1+\sigma s)b_k|-$$

$$-\beta_k(x,x',\sigma s)(||\sigma sx+x'||_1+|1+\sigma s|)=0.$$

Thus we have

$$\sigma((A_k+A'_k)x'+(b_k+b'_k))>s((A_k+A'_k)x+(b_k+b'_k)),\ \ s\in\{-1;1\}.$$

Therefore, taking into account (2), we obtain

$$|(A_k+A'_k)x'+(b_k+b'_k)|>|(A_k+A'_k)x+(b_k+b'_k)|,$$

which implies $x\in P^m(A+A',b+b')$.

Summarizing the said above, we conclude that for any $x\in P^m(A,b)$ and for any perturbing pair $(A',b')\in\Omega(\varphi)$ the solution $x\in P^m(A+A',b+b')$. Hence, $\rho^m(A,b)\geq\varphi$.

It remains to prove inequality $\rho^m(A,b)\leq\varphi$. Using the definition of number φ we conclude that there exist such solutions $x^0\in P^m(A,b),\ x^*\in X\backslash\{x^0\}$, that for any index $i\in K(x^0,x^*)$ the following inequality holds:

$$\alpha_i(x^0,x^*)\leq\varphi. \tag{5}$$

Let $\varepsilon>\varphi$. Let us show, that there exists perturbing pair $(A',b')\in\Omega(\varepsilon)$ satisfying the condition $x^0\notin P^m(A+A',b+b')$.

Suppose

$$N(x^0,x^*)=|\{j\in N_n:\ x^0_j=1\ \&\ x^*_j=0\}|,$$

$$\sigma^0=\mathrm{sign}(A_ix^0+b_i),$$

$$\sigma^*=\mathrm{sign}(A_ix^*+b_i).$$

It is easy to see that at least one of the numbers $N(x^0, x^*)$ or $N(x^*, x^0)$ is positive and

$$\max\{N(x^*, x^0),\ N(x^0, x^*)\} \leq ||x^0 + x^*||_1. \tag{6}$$

To build all the rows

$$(A'_i, b'_i), \quad i \in N_m$$

of the needed perturbing pair (A', b'), consider four possible cases.

Case 1: $i \in K(x^0, x^*)$, $\beta_i(x^0, x^*, -1) < \beta_i(x^0, x^*, 1)$. Then

$$0 < \beta_i(x^0, x^*, 1) \leq \frac{|A_i x^0 + b_i| + |A_i x^* + b_i|}{||x^0 + x^*||_1 + 2} \tag{7}$$

and under inequality (5) there exists a number $\delta_i < \varepsilon$ such that

$$0 \leq \beta_i(x^0, x^*, -1) < \delta_i < \beta_i(x^0, x^*, 1). \tag{8}$$

From this, assuming

$$A'_i = (a'_{i1}, a'_{i2}, \ldots, a'_{in}),\ b'_i = 0,$$

where

$$a'_{ij} = \begin{cases} \sigma^0 \delta_i, & \text{if } x^0_j = 1,\ x^*_j = 0, \\ -\sigma^* \delta_i, & \text{if } x^0_j = 0,\ x^*_j = 1, \\ 0 & \text{in the other cases,} \end{cases} \tag{9}$$

we have $\max\{||A'_i||_\infty, |b'_i|\} = \delta_i$ and

$$\begin{gathered} \sigma^0((A_i + A'_i)x^0 + (b_i + b'_i)) - \sigma^*((A_i + A'_i)x^* + (b_i + b'_i)) = \\ = |A_i x^0 + b_i| - |A_i x^* + b_i| + \delta_i(N(x^0, x^*) + N(x^*, x^0)) \geq \\ \geq -|A_i(x^* - x^0)| + \delta_i ||x^* - x^0||_1 > \\ > -|A_i(x^* - x^0)| + \beta_i(x^0, x^*, -1)||x^* - x^0||_1 = 0, \end{gathered} \tag{10}$$

$$\begin{gathered} \sigma^0((A_i + A'_i)x^0 + (b_i + b'_i)) + \sigma^*((A_i + A'_i)x^* + (b_i + b'_i)) = \\ = |A_i x^0 + b_i| + |A_i x^* + b_i| + \delta_i(N(x^0, x^*) - N(x^*, x^0)). \end{gathered} \tag{11}$$

The right side of the last equality is denoted by ψ. If $N(x^*, x^0) = 0$, then in view of (7) we have $\psi > 0$. If $N(x^*, x^0) > 0$, then taking into account (6), (7) and (8), we have

$$\delta_i N(x^*, x^0) < |A_i x^0 + b_i| + |A_i x^* + b_i|.$$

Therefore we obtain

$$\psi > \delta_i N(x^0, x^*) \geq 0.$$

Thus $\psi > 0$ and we have

$$\sigma^0((A_i + A_i')x^0 + (b_i + b_i')) > s\sigma^*((A_i + A_i')x^* + (b_i + b_i')),\ s \in \{-1; 1\}.$$

From this, applying (2) we find, that

$$|(A_i + A_i')x^0 + (b_i + b_i')| > |(A_i + A_i')x^* + (b_i + b_i')|. \qquad (12)$$

Case 2: $i \in K(x^0, x^*)$, $\beta_i(x^0, x^*, -1) > \beta_i(x^0, x^*, 1)$. Then under (5) there exists a number $\delta_i < \varepsilon$ with conditions

$$0 \leq \beta_i(x^0, x^*, 1) < \delta_i < \beta_i(x^0, x^*, -1).$$

From this, denoting

$$A_i' = (-\sigma^*\delta, -\sigma^*\delta, \ldots, -\sigma^*\delta) \in \mathbf{R}^n,\ b_i' = -\sigma^*\delta,$$

we have $\max\{||A_i'||_\infty, |b_i'|\} = \delta_i$ and

$$\begin{gathered}
-\sigma^*((A_i + A_i')x^0 + (b_i + b_i')) - \sigma^*((A_i + A_i')x^* + (b_i + b_i')) = \\
= -\sigma^*(A_i(x^0 + x^*) + 2b_i) + \delta_i(||x^0||_1 + ||x^*||_1 + 2) > \\
> -|A_i(x^0 + x^*) + 2b_i| + \beta_i(x^0, x^*, 1)(||x^0 + x^*||_1 + 2) = 0, \\
-\sigma^*((A_i + A_i')x^0 + (b_i + b_i')) + \sigma^*((A_i + A_i')x^* + (b_i + b_i')) = \\
= \sigma^* A_i(x^* - x^0) - \delta_i(||x^*||_1 - ||x^0||_1) = \\
= |A_i(x^* - x^0)| - \delta_i(||x^*||_1 - ||x^0||_1) > \\
> |A_i(x^* - x^0)| - \beta_i(x^0, x^*, -1)||x^* - x^0||_1 = 0.
\end{gathered}$$

Thus the following equalities hold

$$-\sigma^*((A_i + A_i')x^0 + (b_i + b_i')) > s\sigma^*((A_i + A_i')x^* + (b_i + b_i')),\ s \in \{-1; 1\}$$

Then, applying (2), we obtain (12).

Case 3: $i \in K(x^0, x^*)$, $\beta_i := \beta_i(x^0, x^*, -1) = \beta_i(x^0, x^*, 1) = \alpha_i(x^0, x^*)$. Then

$$\beta_i \leq \frac{|A_i x^0 + b_i| + |A_i x^* + b_i|}{||x^0 + x^*||_1 + 2}. \tag{13}$$

Consider two variants.

At first let $\beta_i = 0$. Then it is easy to see, that

$$A_i x^0 + b_i = A_i x^* + b_i = 0. \tag{14}$$

If $N(x^0, x^*) > 0$, then, specifying components of the vector

$$A_i' = (a_{i1}', a_{i2}', \ldots, a_{in}')$$

by the rule

$$a_{ij}' = \begin{cases} \delta_i, & \text{if } x_j^0 = 1,\ x_j^* = 0, \\ 0 & \text{in the other cases,} \end{cases}$$

$$b_i' = 0,$$

where

$$0 \leq \varphi < \delta_i < \varepsilon,$$

we have $\max\{||A_i'||_\infty, |b_i'|\} = \delta_i$ and from (14), inequality (12) holds.

If $N(x^0, x^*) = 0$, then $N(x^*, x^0) > 0$, i.e. we may choose index $k \in N_n$ such that

$$x_k^* = 1, \quad x_k^0 = 0.$$

Then, assuming

$$A_i' = (a_{i1}', a_{i2}', \ldots, a_{in}'),\ b_i' = \delta_i$$

where

$$a_{ij}' = \begin{cases} -\delta_i/2, & \text{if } j = k, \\ 0 & \text{in the other cases ,} \end{cases}$$

$$0 \le \varphi < \delta_i < \varepsilon,$$

we have, that from (14) the inequality (12) holds, and also $\max\{||A'_i||_\infty, |b'_i|\} = \delta_i$.

Now let $\beta_i > 0$. Assuming,

$$\beta_i < \delta_i < \varepsilon, \tag{15}$$

we build the row A'_i by formula (9), and assume b'_i to equal zero. Then $\max\{||A'_i||_\infty, |b'_i|\} = \delta_i$ and the relations (10) and (11) hold. As in the case 1, we can show, that $\psi > 0$. If $N(x^*, x^0) = 0$, then from (13) we have $\psi > 0$. If $N(x^*, x^0) > 0$, then taking into account (6) and (13) we may additionally to condition (15) impose the following condition on the number δ_i:

$$\delta_i N(x^*, x^0) < |A_i x^0 + b_i| + |A_i x^* + b_i|.$$

Therefore (see case 1) $\psi > \delta_i N(x^0, x^*) \ge 0$. Hence, we have (12).

Case 4: $i \in N_m \backslash K(x^0, x^*)$. Then, assuming $A'_i = \mathbf{0}^{(n)}$, $b'_i = 0$, we have (12).

As a result of the above reasoning, we obtain a pair $(A', b') \in \Omega(\varepsilon)$ such that $x^* \in X_{x^0}(A + A', b + b')$. Thus for solution $x^0 \in P^m(A, b)$ and for any $\varepsilon > \varphi$ there exists a perturbing pair $(A', b') \in \Omega(\varepsilon)$ such that $x^0 \notin P^m(A + A', b + b')$. Consequently $\rho^m(A, b) \le \varphi$.

That completes the proof.

Let $S^m(A, b)$ (otherwise – the set of strictly efficient solutions) be Smale set [8] defined by the rule

$$x \in S^m(A, b) \;\Leftrightarrow\; \{x' \in X\backslash\{x\} :\; f(x, A, b) \ge f(x', A, b)\} = \emptyset.$$

It is evident that the inclusion $S^m(A, b) \subseteq P^m(A, b)$ holds for any A, b.

Corollary. *The problem $Z^m(A, b)$ is quasi-stable if and only if*

$$P^m(A, b) = S^m(A, b). \tag{16}$$

Proof. Sufficiency. Let equality (16) holds. Then, for any solutions $x \in P^m(A,b)$ and $x' \in X\backslash\{x\}$ there exists an index $k \in N_m$, such that

$$|A_k x + b_k| < |A_k x' + b_k|,$$

which implies $k \in K(x,x')$.
Therefore, taking into account evident relations

$$\beta_k(x,x',s) = \frac{|A_k(sx+x') + b_k(s+1)|}{||sx+x'||_1 + |s+1|} \geq$$

$$\geq \frac{|A_k x' + b_k| - |A_k x + b_k|}{||sx+x'||_1 + |s+1|} > 0, \quad s \in \{-1;1\},$$

we conclude that

$$\max\{\alpha_i(x,x') : \ i \in K(x,x')\} > 0.$$

On the basis of the theorem we have $\rho^m(A,b) > 0$, i.e. problem $Z^m(A,b)$ is quasi-stable.

We prove the **necessity** by contradiction. Let there exists a solution $x^0 \in P^m(A,b)\backslash S^m(A,b)$ under the assumption, that $Z^m(A,b)$ is quasi-stable. Then there exists a solution $x^* \neq x^0$, such that

$$|A_i x^0 + b_i| = |A_i x^* + b_i|, \quad i \in N_m.$$

Therefore

$$\forall i \in N_m \ \ \exists s \in \{-1;1\} \ \ \beta_i(x^0,x^*,s) = \frac{|A_i(sx^0+x^*) + b_i(s+1)|}{||sx^0+x^*||_1 + |s+1|} = 0.$$

From this we conclude

$$\max\{\alpha_i(x^0,x^*) : \ i \in K(x^0,x^*)\} = 0.$$

Consequently, on the basis of the theorem we obtain $\rho^m(A,b) = 0$, i.e. problem is not quasi-stable.

Corollary 1 implies

Corollary 2. *The scalar problem* $Z^1(A,b)$, *where* $A \in \mathbf{R}^n, b \in \mathbf{R}$, *is quasi-stable if and only if*

$$|P^1(A,b)| = |\mathrm{Argmin}\{|Ax+b| : \ x \in X\}| = 1.$$

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