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Measure of stability of a Pareto optimal solution to a vector integer programming problem with fixed surcharges in the l_1 and l_∞ metrics*

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Abstract

In this paper we consider a vector integer programming problem with Pareto principle of optimality for the case where partial criteria belong to the class of separable piecewise linear functions. The limit level of the initial data's perturbations in the space of vector criteria parameters with norms l_1 and l_∞, preserved Pareto optimality of the solutions is investigated. Formulas of the quasistability radius and of strong quasistability radius of the considered problem are given as corollaries.

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1 Introduction

Because of the extensive application of discrete optimization models in economics, management and design during past decades, much attention of many specialists has been given to the study of diverse aspects of stability and other questions relating to parametric and postoptimal analisis of scalar (singlecriterion) and vector (multicriterion) discrete optimization problems. Under stability of a problem in the wide

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sense we understand the existence of a neighborhood in the space of the problem parameters such that any "perturbed" problem with parameters from this neighborhood possesses a certain kind of invariance with respect to the initial problem. Under the stability of solution we understand the property of solution to keep corresponding efficiency (optimality) under mentioned perturbations.

In this paper we consider a vector integer programming problem consisted in finding Pareto set. We suppose that all partial criteria of the problem are separable piecewise linear functions with fixed surcharges. The stability of the problem defined as the semicontinuity by Hausdorff of the optimal mapping that assigns the Pareto function of choice, was investigated earlier [1]. Lower and upper bounds of stability radius of the problem in the l_∞ metrics were obtained.

The purpose of this work is to obtain the limit level of perturbation in the space of vector criteria parameters with l_1 and l_∞ metrics preserving Pareto optimality (efficiency) of a given solution. Formulas of the quasistability and strong quasistability radii of the problem were also obtained.

2 Base definitions and properties

Let m be the number of criteria, n be the number of variables, $C = [c_{ij}] \in \mathbf{R}^{m\times n}$, $D = [d_{ij}] \in \mathbf{R}^{m\times n}$, X be a finite subset of $\mathbf{Z}^n_+ = \{x \in \mathbf{Z}^n : x_j \geq 0,\ j \in N_n\}$, $N_n = \{1, 2, \dots, n\}$, where $|X| > 1$.

We define the vector criterion on the set of (feasible) solutions X

$$f(x) = (f_1(x), f_2(x), \dots, f_m(x)) \to \min_{x\in X},$$

The components (partial criteria) are piecewise linear discontinuous functions with fixed surcharges

$$f_i(x) = \sum_{j=1}^{n} c_{ij}(x_j), \qquad i \in N_m,$$

where

$$c_{ij}(x_j) = \begin{cases} c_{ij}x_j + d_{ij}, & \text{if } x_j > 0, \\ 0, & \text{if } x_j = 0. \end{cases}$$

For any integer vector $x \in X$ we define a boolean vector $\tilde{x} \in \mathbf{E}^n = \{0,1\}^n$ with the components

$$\tilde{x}_j = \begin{cases} 1, & \text{if } x_j > 0, \\ 0, & \text{if } x_j = 0. \end{cases}$$

Then partial criteria are linear functions:

$$f_i(x) = C_i x + D_i \tilde{x}, \qquad i \in N_m,$$

where $x = (x_1, x_2, \dots, x_n)^T$, $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)^T$, and the subscript at the matrix points to the corresponding row of the matrix. For example, $C_i = (c_{i1}, c_{i2}, \dots, c_{in})$.

Changing the elements of the pair (C, D), we obtain different vector criteria. Therefore the pair (C, D) can be used for indexing a vector criterion. Its partial criteria are denoted by $f_i(x, C_i, D_i)$.

Further under the vector (m-criteria) problem $Z^m(C, D)$ with fixed surcharges we understand the problem of finding of the Pareto set (the set of efficient solutions)

$$P^m(C, D) = \{x \in X : P^m(x, C, D) = \emptyset\},$$

where

$$P^m(x, C, D) = \{x' \in X : g(x, x', C, D) \leq 0_{(m)},\ g(x, x', C, D) \neq 0_{(m)}\},$$

$$g(x, x', C, D) = (g_1, g_2, \dots, g_m),$$

$$g_i = g_i(x, x', C_i, D_i) = f_i(x', C_i, D_i) - f_i(x, C_i, D_i), \qquad i \in N_m,$$

$$0_{(m)} = (0, 0, \dots, 0) \in \mathbf{R}^m.$$

While the set X is finite, the Pareto set $P^m(C, D)$ is nonempty for any matrices $C, D \in \mathbf{R}^{m \times n}$ and for any natural number $m \geq 1$.

Further we need the following evident statements.

Property 1. *Let $x \in X$, $x' \in P^m(x, C, D)$. Then for any index $i \in N_m$ the following inequality is valid*

$$g_i(x, x', C_i, D_i) \leq 0.$$

Property 2. *The solution x is efficient if for any solution $x' \neq x$ there exists an index $i \in N_m$, such that*

$$g_i(x, x', C_i, D_i) > 0.$$

Note, that the problem in the scalar case ($m = 1$) can be understood as the problem of piecewise linear concave programming with separable discontinuous function [2, 3]. It is obvious that in another particular case, where D is the null matrix, the problem $Z^m(C, D)$ changes into the m-criteria integer linear programming problem, different stability types of which were investigated in [4].

For any natural number p we define two metrics l_1 and l_∞ in the space $\mathbf{R}^p$, i.e. under metrics of a vector $y = (y_1, y_2, \ldots, y_p)$ we understand correspondingly the numbers

$$||y||_1 = \sum_{i=1}^{p} |y_i|, \qquad ||y||_\infty = \max\{|y_i| : \ i \in N_p\}.$$

Under the norm of a matrix we understand the norm of the vector composed from its elements.

The following properties are obvious for any index $i \in N_m$.

Property 3. $g_i(x, x', C_i, D_i) \leq (||C_i||_1 + ||D_i||_1)||x - x'||_\infty$.

Property 4. $g_i(x, x', C_i, D_i) \leq ||C_i||_\infty ||x - x'||_1 + ||D_i||_\infty ||\tilde{x} - \tilde{x}'||_1$.

Let $\varepsilon > 0$. According to the selected metric (l_1 or l_∞) in the parameter space $\mathbf{R}^{m\times n} \times \mathbf{R}^{m\times n}$ we perturb the elements of the pair (C, D) by addition this pair with a pairs (C', D') from the set

$$\Omega_1(\varepsilon) = \{(C', D') \in \mathbf{R}^{m\times n} \times \mathbf{R}^{m\times n} : ||C'||_1 + ||D'||_1 < \varepsilon\},$$

if the metric is l_1, and from the set

$$\Omega_\infty(\varepsilon) = \{(C', D') \in \mathbf{R}^{m\times n} \times \mathbf{R}^{m\times n} : ||C'||_\infty < \varepsilon, ||D'||_\infty < \varepsilon\},$$

if the metric is l_∞.

The problem $Z^m(C+C', D+D')$, obtained from $Z^m(C, D)$ by such addition, is called perturbed. The pair (C', D') is called perturbing.

By analogy with [5–8], under stability radius of the efficient solution $x \in P^m(C, D)$ of the problem $Z^m(C, D)$ we understand the number

$$\rho^m(x, C, D) = \begin{cases} \sup \Xi, & \text{if} \quad \Xi \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\Xi = \{\varepsilon > 0 : \ \forall \ (C', D') \in \Omega_k(\varepsilon) \quad (x \in P^m(C + C', D + D'))\}.$$

Here $k = 1$ or $k = \infty$ according to the above mentioned notation.

Hence, the stability radius of the efficient solution $x \in P^m(C, D)$ is the maximum level of perturbations of the vector criterion parameters in space $\mathbf{R}^{m \times n}$ (with one of the norms), which keep the efficiency of the solution x.

3 Lemmas

By definition, put

$$g_i^+(x, x', C_i, D_i) = \max\{0, g_i(x, x', C_i, D_i)\}.$$

Lemma 1. *If the inequality*

$$g_i(x, x', C_i + C'_i, D_i + D'_i) \leq 0, \tag{1}$$

holds for any index $i \in N_m$, then

$$g_i^+(x, x', C_i, D_i) \leq (||C'_i||_1 + ||D'_i||_1)||x - x'||_\infty. \tag{2}$$

Actually, when $g_i(x, x', C_i, D_i) \leq 0$ inequality (2) is evident. If $g_i(x, x', C_i, D_i) > 0$, then taking into account condition (1), linearity of the function $g_i(x, x', C_i, D_i)$ and property 3 we deduce

$$g_i^+(x, x', C_i, D_i) = g_i(x, x', C_i, D_i) =$$

$$= g_i(x, x', C_i + C'_i, D_i + D'_i) - g_i(x, x', C'_i, D'_i) \leq$$

$$\leq -g_i(x, x', C'_i, D'_i) \leq (||C'_i||_1 + ||D'_i||_1)||x - x'||_\infty.$$

Lemma 1 is proved.

Lemma 2. *Let $x, x' \in X$, $x \neq x'$. For any φ, satisfying the inequalities*

$$0 < \varphi||x - x'||_\infty \leq \sum_{i \in N_m} g_i^+(x, x', C_i, D_i), \tag{3}$$

and for any perturbing pair $(C', D') \in \Omega_1(\varphi)$ the following ratio is valid

$$x' \notin P^m(x, C + C', D + D').$$

Proof. Suppose the opposite, i.e. there exists perturbing pair $(C', D') \in \Omega_1(\varphi)$, such that $x' \in P^m(x, C+C', D+D')$. Then according to property 1 for any index $i \in N_m$ the inequality (1) is true. Therefore based on lemma 1 the inequality (2) is true. Hence, using inclusion $(C', D') \in \Omega_1(\varphi)$, we conclude

$$\sum_{i \in N_m} g_i^+(x, x', C_i, D_i) \leq \sum_{i \in N_m} (||C'_i||_1 + ||D'_i||_1)||x - x'||_\infty \leq$$

$$\leq (||C'||_1 + ||D'||_1)||x - x'||_\infty < \varphi||x - x'||_\infty,$$

that contradicts to condition (3).

Lemma 2 is proved.

Lemma 3. *Let $x, x' \in X$, $x \neq x'$. For any number ε such that*

$$\varepsilon > \sum_{i \in N_m} \eta_i,$$

where

$$\eta_i||x - x'||_\infty > g_i^+(x, x', C_i, D_i), \qquad i \in N_m, \tag{4}$$

there exists perturbing pair $(C', D') \in \Omega_1(\varepsilon)$, such that $x' \in P^m(x, C + C', D + D')$.

Proof. At first note, that any number ε, under conditions of the lemma, is positive since all the numbers $\eta_i, i \in N_m$. Obviously that for proving our lemma it suffices to show the perturbing pair $(C', D') \in \Omega_1(\varepsilon)$, such that following inequalities

$$g_i(x, x', C_i + C'_i, D_i + D'_i) < 0, \qquad i \in N_m \tag{5}$$

are fulfilled. Let

$$q = \arg\max\{|x_j - x'_j| : \ j \in N_n\}$$

and define the elements of the perturbing pair (C', D') by formulas

$$c'_{ij} = \begin{cases} \eta_i \cdot \mathrm{sign}(x_q - x'_q), & \text{if } i \in N_m, \ j = q, \\ 0 & \text{otherwise,} \end{cases}$$

$$d'_{ij} = 0, \qquad i \in N_m, \ \ j \in N_n.$$

It is easy to see that $(C', D') \in \Omega_1(\varepsilon)$. By virtue of construction of the pair (C', D') the equalities

$$g_i(x, x', C'_i, D'_i) = -\eta_i ||x - x'||_\infty, \qquad i \in N_m$$

are true. Thus, taking into account the linearity of $g_i(x, x', C_i, D_i)$ and ratios (4), we make sure of correctness of inequality (5):

$$g_i(x, x', C_i + C'_i, D_i + D'_i) = g_i(x, x', C_i, D_i) + g_i(x, x', C'_i, D'_i) =$$

$$= g_i(x, x', C_i, D_i) - \eta_i ||x - x'||_\infty \leq$$

$$\leq g_i^+(x, x', C_i, D_i) - \eta_i ||x - x'||_\infty < 0, \qquad i \in N_m.$$

Lemma 3 is proved.

Lemma 4. *If for any index $i \in N_m$ the inequality*

$$g_i(x, x', C_i, D_i) > \max\{||C'_i||_\infty, ||D'_i||_\infty\}(||x - x'||_1 + ||\tilde{x} - \tilde{x}'||_1),$$

holds, then

$$g_i(x, x', C_i + C'_i, D_i + D'_i) > 0.$$

Proof, from property 4, combining it with the linearity of the function $g_i(x, x', C_i, D_i)$, and condition of the lemma we obtain

$$g_i(x, x', C_i + C'_i, D_i + D'_i) = g_i(x, x', C_i, D_i) + g_i(x, x', C'_i, D'_i) \geq$$
$$\geq g_i(x, x', C_i, D_i) - ||C'_i||_\infty ||x - x'||_1 - ||D'_i||_\infty ||\tilde{x} - \tilde{x}'||_1 \geq$$
$$\geq g_i(x, x', C_i, D_i) - \max\{||C'_i||_\infty, ||D'_i||_\infty\}(||x - x'||_1 + ||\tilde{x} - \tilde{x}'||_1) > 0.$$

Lemma 4 is proved.

Lemma 5. *Let $x, x' \in X$, $x \neq x'$. For any number ε such that*

$$\varepsilon(||x-x'||_1+||\tilde{x}-\tilde{x}'||_1) > \max\{g_i(x, x', C_i, D_i) : \ i \in N_m\}, \quad \varepsilon > 0 \quad (6)$$

there exists a perturbing pair $(C', D') \in \Omega_\infty(\varepsilon)$, such that $x' \in P^m(x, C + C', D + D')$.

Proof. Obviously that for proving our lemma it suffices to show the perturbing pair $(C', D') \in \Omega_\infty(\varepsilon)$, such that following inequalities

$$g_i(x, x', C_i + C'_i, D_i + D'_i) < 0, \qquad i \in N_m \qquad (7)$$

are true.

By virtue of (6) there exists a number α, such that

$$0 < \alpha < \varepsilon,$$
$$\alpha(||x - x'||_1 + ||\tilde{x} - \tilde{x}'||_1) > \max\{g_i(x, x', C_i, D_i) : \ i \in N_m\}. \qquad (8)$$

We assign the elements of the perturbing pair $(C', D') \in \Omega_\infty(\varepsilon)$ by the rule:

$$c'_{ij} = \alpha \cdot \mathrm{sign}(x_j - x'_j), \quad d'_{ij} = \alpha \cdot \mathrm{sign}(\tilde{x}_j - \tilde{x}'_j), \qquad i \in N_m, j \in N_n.$$

Then taking into account the linearity of the function $g_i(x, x', C_i, D_i)$, combining it with evident inequalities

$$g_i(x, x', C'_i, D'_i) = -\alpha(||x - x'||_1 + ||\tilde{x} - \tilde{x}'||_1), \qquad i \in N_m,$$

and inequality (8), we obtain:

$$g_i(x, x', C_i + C'_i, D_i + D'_i) = g_i(x, x', C_i, D_i) + g_i(x, x', C'_i, D'_i) =$$
$$= g_i(x, x', C_i, D_i) - \alpha(||x - x'||_1 + ||\tilde{x} - \tilde{x}'||_1) < 0, \qquad i \in N_m.$$

Lemma 5 is proved.

4 Formulas of the stability radius of an efficient solution

Theorem. *For any number $m \geq 1$ the stability radius $\rho^m(x, C, D)$ of any efficient solution $x \in P^m(C, D)$ of the problem $Z^m(C, D)$ is expressed by the formula*

$$\rho^m(x, C, D) =$$

$$= \min_{x' \in X \setminus \{x\}} \begin{cases} \sum\limits_{i \in N_m} \dfrac{g_i^+(x, x', C_i, D_i)}{||x - x'||_\infty}, & \text{if the metric is } l_1, \\ \max\limits_{i \in N_m} \dfrac{g_i(x, x', C_i, D_i)}{||x - x'||_1 + ||\tilde{x} - \tilde{x}'||_1}, & \text{if the metric is } l_\infty. \end{cases} \quad (9)$$

Proof. It is evident that in the right part of the equation (9) we have non-negative numbers.

1. The case of l_1 metric. First let us prove the inequality $\rho^m(x, C, D) \geq \varphi_1$. We see that it suffices to consider the case $\varphi_1 > 0$. By definition of the number φ_1, for any solution $x' \neq x$ inequalities (3) are correct. Hence based on lemma 2 for any perturbing pair $(C', D') \in \Omega_1(\varphi_1)$ solution $x' \notin P^m(x, C + C', D + D')$. Therefore the set $P^m(x, C + C', D + D') = \emptyset$. Thus for any perturbing pair $(C', D') \in \Omega_1(\varphi_1)$ solution $x \in P^m(C + C', D + D')$. Consequently $\rho^m(x, C, D) \geq \varphi_1$.

Now we show that $\rho^m(x, C, D) \leq \varphi_1$. Let $\varepsilon > \varphi_1$. According to definition of the number φ_1 there exists a solution $x^* \neq x$, such that

$$\varphi_1 ||x - x^*||_\infty = \sum_{i \in N_m} g_i^+(x, x^*, C_i, D_i).$$

Therefore there exists such positive numbers η_i that

$$\eta_i ||x - x^*||_\infty > g_i^+(x, x^*, C_i, D_i), \qquad i \in N_m,$$

$$\varepsilon > \sum_{i \in N_m} \eta_i > \varphi_1.$$

Hence by lemma 3 there exists a perturbing pair $(C', D') \in \Omega_1(\varepsilon)$ such that $x^* \in P^m(x, C + C', D + D')$, i.e. the solution $x \notin P^m(C + C', D +$

D'). Hence for any number $\varepsilon > \varphi_1$ the inequality $\rho^m(x, C, D) < \varepsilon$ holds. Thus $\rho^m(x, C, D) \leq \varphi_1$.

2. The case of l_∞ metric. First prove the inequality $\rho^m(x, C, D) \geq \varphi_\infty$. Without loss of generality it can be assumed that $\varphi_\infty > 0$. By definition of value φ_∞ for any perturbing pair $(C', D') \in \Omega_\infty(\varphi_\infty)$ and any solution $x' \neq x$ there exists index $i \in N_m$, such that

$$\frac{g_i(x, x', C_i, D_i)}{||x - x'||_1 + ||\tilde{x} - \tilde{x}'||_1} \geq \varphi_\infty > \max\{||C'||_\infty, ||D'||_\infty\}.$$

From this according to lemma 4 we have

$$g_i(x, x', C_i + C'_i, D_i + D'_i) > 0.$$

Thus, taking into account property 2, solution x belongs to the Pareto set of the perturbed problem $Z^m(C + C', D + D')$. Consequently $\rho^m(x, C, D) \geq \varphi_\infty$.

Further we prove that $\rho^m(x, C, D) \leq \varphi_\infty$. According to the definition of number φ_∞ there exists a solution $x^* \neq x$, such that

$$\varphi_\infty(||x - x^*||_1 + ||\tilde{x} - \tilde{x}^*||_1) = \max\{g_i(x, x^*, C_i, D_i) : \ i \in N_m\}.$$

Then for $\varepsilon > \varphi_\infty$ inequality (6) is fulfilled. Hence based on lemma 5 there exists a perturbing pair $(C', D') \in \Omega_\infty(\varepsilon)$, such that $x^* \in P^m(x, C + C', D + D')$, i.e. the solution $x \notin P^m(C + C', D + D')$. Thereby it is proved that for any number $\varepsilon > \varphi_\infty$ the inequality $\rho^m(x, C, D) < \varepsilon$ is valid. Hence $\rho^m(x, C, D) \leq \varphi_\infty$.

Theorem is proved.

5 Corollaries

Under the quasistability of a vector problem of discrete optimization we usually understand [4, 9–12] the Hausdorff lower semicontinuity of the set-valued (point-to-set) mapping determining the Pareto choice function. In other words, the problem $Z^m(C, D)$ is called quasistability if there exists a number $\varepsilon > 0$, such that for any perturbing pair $(C', D') \in \Omega(\varepsilon)$ the following conclusion is valid

$$P^m(C, D) \subseteq P^m(C + C', D + D').$$

Therefore quasistability radius of the problem is determined as follows:

$$\rho_1^m(C,D) = \begin{cases} \sup \Phi_1, & \text{if } \Phi_1 \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\Phi_1 = \{\varepsilon > 0 : \ \forall \, (C', D') \in \Omega_k(\varepsilon) \quad (P^m(C,D) \subseteq P^m(C+C', D+D'))\},$$

$\Omega_k(\varepsilon)$ is the above mentioned set of perturbing pairs (C', D'). In other words, quasistability radius of the problem $Z^m(C,D)$ is the maximum level of perturbations of matrices C and D in the space of parameters of the vector criterion with the corresponding norm such that Pareto set can only expand.

Directly from the theorem we obtain

Corollary 1. *For any $m \geq 1$ for quasistability radius $\rho_1^m(C,D)$ of the problem $Z^m(C,D)$ the following formula is valid*

$$\rho_1^m(C,D) =$$

$$= \min_{x \in P^m(C,D)} \min_{x' \in X \setminus \{x\}} \begin{cases} \sum\limits_{i \in N_m} \dfrac{g_i^+(x,x',C_i,D_i)}{||x - x'||_\infty}, & \textit{if the metric is } l_1, \\ \max\limits_{i \in N_m} \dfrac{g_i(x,x',C_i,D_i)}{||x-x'||_1 + ||\tilde{x} - \tilde{x}'||_1}, & \textit{if the metric is } l_\infty. \end{cases}$$

Therefore next corollary is true.

Corollary 2. *For any $m \geq 1$ the problem $Z^m(C,D)$ is quasistable if and only if $P^m(C,D) = S^m(C,D)$.*

Here $S^m(C,D)$ is the traditional Smale set, i.e. the set of strictly efficient solutions of the problem $Z^m(C,D)$, which is a subset of the Pareto set and defined in the following way [13]:

$$S^m(C,D) = \{x \in X : \ S^m(x,C,D) = \emptyset\},$$

where

$$S^m(x,C,D) = \{x' \in X \setminus \{x\} : \quad g(x,x',C,D) \leq 0_{(m)}\}.$$

When we relax the demand of preservation of all the Pareto set in definition of the quasistability of problem $Z^m(C,D)$, we get the concept of the strong quasistability. This type of the stability means the existence of neighborhood of vector criterion parameters such that although disappearance of the old effective solutions is possible but there exists at least one pareto-optimal solution of initial problem, that preserves its efficiency under small perturbations of parameters. In other words there exists at least one stable Pareto optimum. Thus under the strong quasistability radius of the problem $Z^m(C,D)$ we understand the number

$$\rho_2^m(C,D) = \begin{cases} \sup \Phi_2, & \text{if } \Phi_2 \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\Phi_2 = \{\varepsilon > 0 : \ \exists\, x \in P^m(C,D) \ \ \forall\, (C',D') \in \Omega_k(\varepsilon)$$

$$(x \in P^m(C+C', D+D'))\}.$$

Directly from the theorem we obtain

Corollary 3. *For strong quasistability radius $\rho_2^m(C,D)$, $m \geq 1$ of the problem $Z^m(C,D)$ the following formula is valid*

$$\rho_2^m(C,D) =$$

$$= \max_{x \in P^m(C,D)} \min_{x' \in X \setminus \{x\}} \begin{cases} \sum\limits_{i \in N_m} \dfrac{g_i^+(x,x',C_i,D_i)}{||x-x'||_\infty}, & \textit{if the metric is } l_1, \\ \max\limits_{i \in N_m} \dfrac{g_i(x,x',C_i,D_i)}{||x-x'||_1 + ||\tilde{x}-\tilde{x}'||_1}, & \textit{if the metric is } l_\infty. \end{cases}$$

Hence we obtain

Corollary 4. *For any $m \geq 1$ the problem $Z^m(C,D)$ is strongly quasistable problem if and only if $S^m(C,D) \neq \emptyset$.*

The next two statements follow from corollaries 3 and 4.

Corollary 5. *Any quasistable problem* $Z^m(C,D), m \geq 1$, *is strongly quasistable.*

Corollary 6. *For any scalar problem* $Z^1(C,D), C \in \mathbf{R}^n, D \in \mathbf{R}^n$ *the next statements are equivalent:*
a) $Z^1(C,D)$ *is quasistable,*
b) $Z^1(C,D)$ *is strongly quasistable,*
c) $Z^1(C,D)$ *has a unique optimal solution.*

6 Example

Let us give a simple example which illustrates stated results. Let us consider a two-criterion problem. $X = \{x^1, x^2, x^3, x^4, x^5\}$, where

$$x^1 = (1,0,0,0)^T, \quad x^2 = (0,1,0,0)^T, \quad x^3 = (0,0,1,0)^T,$$

$$x^4 = (2,3,1,1)^T, \quad x^5 = (2,4,1,1)^T,$$

the matrices C and D are

$$C = \begin{bmatrix} 2 & 5 & 5 & 3 \\ 7 & 1 & 1 & 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 3 & 3 & 0 \\ 2 & 1 & 1 & 4 \end{bmatrix}.$$

Then

$$f(x^1) = (3,9), \qquad f(x^2) = f(x^3) = (8,2),$$

$$f(x^4) = (34,28), \qquad f(x^5) = (39,29),$$

Pareto set consists of three solutions x^1, x^2, x^3, Smale set contains just one solution x^1. Therefore the problem is quasistable (in virtue of corollary 2), but is not strongly quasistable (in virtue of corollary 4). By the theorem it is easy to calculate

$$\rho^2(x^1, C, D) = \begin{cases} 5, & \text{if the metric is } l_1, \\ \frac{5}{4}, & \text{if the metric is } l_\infty, \end{cases} \tag{10}$$

$$\rho^2(x^2, C, D) = \rho^2(x^3, C, D) = 0 \quad \text{for any metric } (l_1 \text{ or } l_\infty).$$

Hence in virtue of corollaries 1 and 3 the quasistability and strong quasistability radii take on the forms correspondingly

$$\rho_1^2(C,D) = 0 \quad \text{for any metric } (l_1 \text{ or } l_\infty),$$

$$\rho_2^2(C,D) = \begin{cases} 5, & \text{if the metric is } l_1, \\ \frac{5}{4}, & \text{if the metric is } l_\infty. \end{cases}$$

Now consider two new another variants of the problem changing just the set of the admissible solutions X.

Variant 1. Let $X' = X \setminus \{x^3\} = \{x^1, x^2, x^4, x^5\}$. Then the Pareto set $\{x^1, x^2\}$ and the Smale set are congruent. Therefore the problem is quasistable and strong quasistable simultaneously. Using formula (9), we make sure that the value $\rho^2(x^1, C, D)$ is defined by formula (10) too.

$$\rho^2(x^2, C, D) = \begin{cases} 7, & \text{if the metric is } l_1, \\ \frac{7}{4}, & \text{if the metric is } l_\infty. \end{cases}$$

Taking into account corollaries 1 and 3, we get the quasistability and strong quasistability radii:

$$\rho_1^2(C,D) = \begin{cases} 5, & \text{if the metric is } l_1, \\ \frac{5}{4}, & \text{if the metric is } l_\infty, \end{cases}$$

$$\rho_2^2(C,D) = \begin{cases} 7, & \text{if the metric is } l_1, \\ \frac{7}{4}, & \text{if the metric is } l_\infty. \end{cases}$$

Variant 2. Let $X'' = X \setminus \{x^1\} = \{x^2, x^3, x^4, x^5\}$. Then $\{x^2, x^3\}$ is the Pareto set. The Smale set is empty. By the formula (9) we have

$$\rho^2(x^2, C, D) = \rho^2(x^3, C, D) = 0 \quad \text{for any metric.}$$

Therefore in virtue of corollaries 2 and 4, the problem is neither quasistable nor strongly quasistable.

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