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Game-Theoretic Approach for Solving Multiobjective Flow Problems on Networks*

Maria A. Fonoberova, Dmitrii D. Lozovanu

Abstract

The game-theoretic formulation of the multiobjective multicommodity flow problem is considered. The dynamic version of this problem is studied and an algorithm for its solving, based on the concept of multiobjective games, is proposed.

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1 Introduction

In this paper we consider the game-theoretic formulation of the multiobjective multicommodity flow problem. This problem consists of shipping a given set of commodities from their respective sources to their sinks through a network in order to optimize different criteria so that the total flow going through each edge does not exceed its capacity. The network is a collection of locations with directed edges identifying feasible transportation operations. The planning problem is to determine the amount to transport on each link in order to move all the cargo respecting fixed criteria.

If we associate to each commodity a player, we can regard this problem as a game problem, where players interact between them and the choices of one player influence the choices of the others. Each player has a vector utility function, components of which are such factors as

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transportation cost, speed of transit time, quality of service and others. Each player seeks to optimize his own utility function in response to the actions of the other players and all the players perform this optimization simultaneously and at the same time players are interested to preserve Nash optimality principle when they interact between them. The game theory fits perfectly in the realm of such a problem, and an equilibrium or stable operating point of the system has to be found. We study the dynamic version of the multiobjective multicommodity flow problem and use the concept of multiobjective games from [1, 2].

2 The game-theoretic approach and some preliminary results

In order to study our multiobjective multicommodity flow problem we will use the game-theoretic concept from [1, 2].

The multiobjective game with p players is denoted by $\overline{G} = (X_1, X_2, \dots, X_p, \overline{F}_1, \overline{F}_2, \dots, \overline{F}_p)$, where X_i is a set of strategies of player i, $i = \overline{1,p}$, and $\overline{F}_i = (F_i^1, F_i^2, \dots, F_i^r)$ is a vector payoff function of player i, defined on set of situations $X = X_1 \times X_2 \times \cdots \times X_p$:

$$\overline{F}_i \ : \ X_1 \times X_2 \times \cdots \times X_p \to R^r, \ i = \overline{1,p}.$$

Each component F_i^k of $\overline{F}_i$ corresponds to a partial criterion of player i and represents a real function defined on set of situations $X = X_1 \times X_2 \times \cdots \times X_p$:

$$F_i^k \ : \ X_1 \times X_2 \times \cdots \times X_p \to R^1, \ k = \overline{1,r}, \ i = \overline{1,p}.$$

We call the solution of the multiobjective game $\overline{G} = (X_1, X_2, \dots, X_p, \overline{F}_1, \overline{F}_2, \dots, \overline{F}_p)$ the Pareto-Nash equilibrium and define it in the following way.

Definition. The situation $x^* = (x_1^*, x_2^*, \dots, x_p^*) \in X$ is called Pareto-Nash equilibrium for the multiobjective game $\overline{G} = (X_1, X_2,$

$\ldots, X_p, \overline{F}_1, \overline{F}_2, \ldots, \overline{F}_p)$ if for every $i \in \{1, 2, \ldots, p\}$ the strategy x_i^* represents Pareto solution for the following multicriterion problem:

$$\max_{x_i \in X_i} \rightarrow \overline{f}^{\,i}_{x^*}(x_i) = (f^{i1}_{x^*}(x_i), f^{i2}_{x^*}(x_i), \ldots, f^{ir}_{x^*}(x_i)),\ i = \overline{1, p},$$

where

$$f^{ik}_{x^*}(x_i) = F_i^k(x_1^*, x_2^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_p^*),\ k = \overline{1, r}.$$

This definition generalizes well-known Nash equilibria notion for classical noncooperative games (single objective games) and Pareto optimum for multicriterion problems. If $r = 1$, then $\overline{G}$ becomes classical noncooperative game, where x^* represents Nash equilibria solution; in the case $p = 1$ the game $\overline{G}$ becomes Pareto multicriterion problem, where x^* is Pareto solution.

Further we formulate the main theorem which represents an extension of the Nash theorem for our multiobjective version of the game.

Theorem 1. Let $\overline{G} = (X_1, X_2, \ldots, X_p, \overline{F}_1, \overline{F}_2, \ldots, \overline{F}_p)$ be a multiobjective game, where $X_1, X_2, \ldots, X_p$ are convex compact sets and $\overline{F}_1, \overline{F}_2, \ldots, \overline{F}_p$ represent continuous vector payoff functions. Moreover, let us assume that for every $i \in \{1, 2, \ldots, p\}$ each component $F_i^k(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_p)$, $k \in \{1, 2, \ldots, r\}$, of the vector function $\overline{F}_i(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_p)$ represents a concave function with respect to x_i on X_i for fixed $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_p$. Then for multiobjective game $\overline{G} = (X_1, X_2, \ldots, X_p, \overline{F}_1, \overline{F}_2, \ldots, \overline{F}_p)$ there exists Pareto-Nash equilibria situation $x^* = (x_1^*, x_2^*, \ldots, x_p^*) \in X_1 \times X_2 \times \cdots \times X_p$.

The proof of Theorem 1 is given in [2].

So, if conditions of Theorem 1 are satisfied then Pareto-Nash equilibria solution for multiobjective game can be found by using the following algorithm.

Algorithm

1. Fix an arbitrary set of real numbers $\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1r}, \alpha_{21}, \alpha_{22}, \ldots, \alpha_{2r}, \ldots, \alpha_{p1}, \alpha_{p2}, \ldots, \alpha_{pr}$, which satisfy condition

$$\begin{cases} \sum_{k=1}^{r} \alpha_{ik} = 1, & i = \overline{1,p}; \\ \alpha_{ik} > 0, & k = \overline{1,r},\ i = \overline{1,p}; \end{cases}$$

2. Form the single objective game $G = (X_1, X_2, \ldots, X_p, f_1, f_2, \ldots, f_p)$, where

$$f_i(x_1, x_2, \ldots, x_p) = \sum_{k=1}^{r} \alpha_{ik} F_i^k(x_1, x_2, \ldots, x_p),\ i = \overline{1,p};$$

3. Find Nash equilibria $x^* = (x_1^*, x_2^*, \ldots, x_p^*)$ for noncooperative game $G = (X_1, X_2, \ldots, X_p, f_1, f_2, \ldots, f_p)$ and fix x^* as Pareto-Nash equilibria solution for multiobjective game $\overline{G} = (X_1, X_2, \ldots, X_p, \overline{F}_1, \overline{F}_2, \ldots, \overline{F}_p)$.

3 The multiobjective multicommodity flow problem

3.1 The static model

We consider a network $N = (V, E, K, c, d, \varphi)$ that contains a directed graph $G = (V, E)$, where V is a set of vertexes, E is a set of edges, and $K = \{1, 2, \ldots, p\}$ is a set of commodities that must be routed through the same network. Each edge $e \in E$ has a nonnegative capacity c_i^e which bounds the amount of flow of commodity i allowed on arc e. There is a throughput demand d_i^v defined on vertexes for each commodity in the network. To model transit costs we define the cost function φ: $E \times R_+ \to R_+$. In such a way the following restrictions have to be verified for the flow x_i^e of commodity i sent on edge e:

$$\sum_{e \in E^+(v)} x_i^e - \sum_{e \in E^-(v)} x_i^e = d_i^v,\ \forall\, v \in V,\ \ \forall\, i \in K;$$

$$0 \le x_i^e \le c_i^e, \ \ \forall\, e \in E, \ \ \forall\, i \in K;$$

where $E^+(v) = \{(u,v)\,|\,(u,v) \in E\}$, $\ E^-(v) = \{(v,u)\,|\,(v,u) \in E\}$.

The flow has to be shipped through the network in such a way to optimize the vector utility function $\overline{F}_i = (F_i^1, F_i^2, \dots, F_i^r)$ for every $i \in K$:

$$\overline{F}_i \ : \ X_1 \times X_2 \times \cdots \times X_p \to R^r,$$

$$F_i^k \ : \ X_1 \times X_2 \times \cdots \times X_p \to R^1, \ k = \overline{1,r},$$

where X_i is a set of flows of commodity i, r is a number of criteria.

3.2 The dynamic model

The static flow can not properly consider the evolution of the system under study over time. The time is an essential component, either because the flows take time to pass from one location to another, or because the structure of network changes over time. To tackle this problem, we use dynamic network flow models instead of the static ones.

We consider the discrete time model, in which all times are integral and bounded by time horizon T, which defines the makespan $\mathbb{T} = \{0, 1, \dots, T\}$ of time moments we consider. Time is measured in discrete steps, so that if one unit of flow leaves node u at time t on arc $e = (u,v)$, one unit of flow arrives at node v at time $t + \tau^e$, where τ^e is the transit time of arc e. Each commodity has its own time-interval $\mathbb{T}_i \subset \mathbb{T}$.

A dynamic network $N = (V, E, K, c, \tau, d, \varphi)$ consists of a directed graph $G = (V, E)$, a set $K = \{1, 2, \dots, p\}$ of commodities that must be routed through the same network within the makespan $\mathbb{T}$, capacity function c: $E \times K \times \mathbb{T} \to R_+$, transit time function τ: $E \to R_+$, demand function d: $V \times K \times \mathbb{T} \to R$ and cost function φ: $E \times R_+ \times \mathbb{T} \to R_+$. In such a way, the following restrictions have to be verified for the flow $x_i^e(t)$ of commodity i sent on link e at time $t \in \mathbb{T}$:

$$\sum_{\substack{e \in E^+(v) \\ t - \tau_e \ge 0}} x_i^e(t - \tau_e) - \sum_{e \in E^-(v)} x_i^e(t) = d_i^v(t), \ \forall\, t \in \mathbb{T}_i, \ \forall\, v \in V, \ \forall\, i \in K;$$

$$0 \le x_i^e(t) \le c_i^e(t), \quad \forall\, t \in \mathbb{T}_i,\ \forall\, e \in E,\ \forall\, i \in K;$$

$$x_i^e(t) = 0,\ \forall\, e \in E,\ t = \overline{T - \tau_e + 1, T},\ \forall\, i \in K.$$

The flow has to be shipped through the network in such a way to optimize the vector utility function $\overline{F}_i = (F_i^1, F_i^2, \dots, F_i^r)$ for every $i \in K$, where

$$\overline{F}_i \ :\ (X_1 \times \mathbb{T}_i) \times (X_2 \times \mathbb{T}_i) \times \cdots \times (X_p \times \mathbb{T}_i) \to R^r,$$

$$F_i^k \ :\ (X_1 \times \mathbb{T}_i) \times (X_2 \times \mathbb{T}_i) \times \cdots \times (X_p \times \mathbb{T}_i) \to R^1,\ k = \overline{1, r}.$$

Using the apparatus from Section 2 we reduce the considered problem to single-objective multicommodity flow problem.

4 The game formulation of the multiobjective multicommodity flow problem

In the framework of the game theory each commodity in the formulated problem is associated with a player. We consider a general model with p agents each of which wishes to optimize its own vector utility function $\overline{F}_i$, $i = \overline{1, p}$, which is defined on the set of strategies of all players. Each component F_i^k, $k = \overline{1, r}$, of the vector utility function F_i of player i corresponds to a partial criterion of player i. Control decisions are made by each player according to its own individual performance objectives and depending on the choices of the other players.

Each player competes in a Nash equilibrium manner so as to optimize his own criteria in the task of transporting of flow from its origins to its destinations. Let x_i be strategy of user i and x_{-i} be strategies of all other agents. For fixed i we say that $x^* = (x_1^*, \dots, x_p^*)$ is a Nash equilibrium if no user can improve his utility by unilateral deviation. In our problem each player has several objectives, so we use the Pareto-Nash equilibrium concept extended to networks.

In such a way, players intend to optimize their utility functions in the sense of Pareto and at the same time players are interested to preserve Nash optimality principle when they interact between them.

The cost of transportation of a given resource, the time necessary to transport it to its destination as well as the quality of the transportation play the role of the components of the vector utility function of a player in the game-theoretic formulation of the problem. If payoff functions satisfy conditions of the above theorem then for solving such a problem we apply the algorithm proposed above.

5 Applications

In real-life problems users have to make decision concerning routing as well as type and amount of resources that they wish to transport. Different sets of parameters may suit the service requirements of a user. However, the performance measures depend not only on the user's choices, but also on the decisions of other connected users, where this dependence is often described as a function of some network "state". In this setting the game paradigm and the Pareto-Nash equilibrium concept become the natural choice at the user level.

Game theoretic models are widely employed in the context of flow control, routing, virtual path bandwidth allocation and pricing in modem networking. Flow problems in multimedia applications (teleconferencing, digital libraries) over high-speed broadband networks can serve a good example of this. In a multimedia network telecommunication companies carrying different traffic types (voice, data, and video) may share the limited common network resources such as buffers or transmission lines. These companies may have different objectives of maximizing packet throughput or minimizing packet blocking probability. A Pareto-Nash equilibrium may be reached when companies achieve their objectives in such a way that no company can improve its own performance by unilaterally changing its traffic load admission and routing strategies.

The problem of providing bandwidth which will be shared by many users ([3, 4]) is one of the most important problems. As it is typical for games in such a problem the interaction among the users on their individual strategies has to be imposed. This can be done using a utility function that depends on the availability of bandwidth and other factors in the network. The problem of consumer i consists in choosing which network resource to use and how much to use it.

The game theoretic approach can be applied in a problem of power control in radio systems ([5]). In cellular radio systems power is a valuable commodity for the users, so a mobile user prefers to use less power and at the same time to obtain better quality-of-service from assigned base stations. In such a way, each mobile user wishes to optimize his own personal objectives and choices of mobile user depend on choices of other users.

At the end we want to mention that the network performance is a very important factor for the improvement of the system work, which can be achieved both during the phase, when the network parameters are sized ([6]), and during the phase of the operation of the network. In such a way, the network performance is not completely determined by the technical characteristics of the network but also is a function of system state, therefore the issue of optimal user strategies is a very actual problem.

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M.A. Fonoberova, D.D. Lozovanu,
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Institute of Mathematics and Computer Science,
5 Academiei str.
Chişinău, MD2028, Moldova.
E–mails: *mashaf83@yahoo.com*, *lozovanu@math.md*