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Note about the upper chromatic number of mixed hypertrees

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Abstract

A mixed hypergraph is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where X is the vertex set and each of $\mathcal{C}$, $\mathcal{D}$ is a family of subsets of X, the $\mathcal{C}$-edges and $\mathcal{D}$-edges, respectively. A proper k-coloring of $\mathcal{H}$ is a mapping $c : X \to [k]$ such that each $\mathcal{C}$-edge has two vertices with a common color and each $\mathcal{D}$-edge has two vertices with distinct colors. Upper chromatic number is the maximum number of colors that can be used in a proper coloring. A mixed hypergraph $\mathcal{H}$ is called *a mixed hypertree* if there exists a host tree on the vertex set X such that every edge ($\mathcal{C}$- or $\mathcal{D}$-) induces a connected subtree of this tree.

We show that if a mixed hypertree can be decomposed into interval mixed hypergraphs then the upper chromatic number can be computed using the same formula.

1 Introduction

In this paper, we use the terminology of [1, 2, 3, 4, 5, 6, 7]. A mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ (each element of $\mathcal{C} \cup \mathcal{D}$ is of size at least 2) is said to be a *mixed hypertree* if there exists a host tree $T = (X, F)$ such that every $C \in \mathcal{C}$ and every $D \in \mathcal{D}$ induces a subtree in T [7]. An *interval mixed hypergraph* represents a special case of a mixed hypertree, namely, when the host graph is simply a path. In a mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, a subfamily $\{C_i\}$ of $\mathcal{C}$-edges is said to be a *sieve*, if for any $x, y \in X$ and any j, k, $j \neq k$, the following implication holds (see [1, 7]):

$$\{x, y\} \subseteq C_j \cap C_k \Rightarrow \{x, y\} = D \in \mathcal{D} \text{ for some } D \in \mathcal{D}.$$

In other words, a sieve represents a subfamily of $\mathcal{C}$-edges with the property that if two $\mathcal{C}$-edges intersect, then every pair of vertices from the intersection forms a $\mathcal{D}$-edge. It appears that sieves play a role in estimating the upper chromatic number. The maximum cardinality of a sieve is the *sieve number* of $\mathcal{H}$ and is denoted by $s(\mathcal{H})$. It is proved (see [1, 7]) that if $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is a reduced interval mixed hypergraph (the size of each $\mathcal{C}$-edge is at least 3, the size of each $\mathcal{D}$-edge is at least 2, and no included edges of any type), then

$$\overline{\chi}(\mathcal{H}) = |X| - s(\mathcal{H}).$$

A $\mathcal{C}$-edge is called *redundant* if it contains no other $\mathcal{C}$-edges and after its removal no new coloring appears. This property never happens in classic graph or hypergraph coloring. We call a mixed hypertree $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ *simple* if it is reduced and has no redundant $\mathcal{C}$-edges. In order to generalize the result for the upper chromatic number from interval mixed hypergraphs to mixed hypertrees, it is necessary to investigate the following question: if $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is a simple mixed hypertree with $|X| = n$ and sieve number s, when is $\overline{\chi}(\mathcal{H}) = n - s$?

The equality holds for many classes of mixed hypertrees. In this paper, we prove it for mixed hypertrees having uniquely colorable separator with the respective subgraphs being interval mixed hypergraphs. However, for general mixed hypertrees it is not true. We exhibit that the difference between $\overline{\chi}(\mathcal{H})$ and $n - s$ can actually be made arbitrarily large.

2 Results

Theorem 1. *If $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is a simple mixed hypertree that can be decomposed into the union of interval mixed hypergraphs $T_1, T_2, \ldots, T_k$, where $k \geq 1$, so that if any two of these hypergraphs meet, they meet only at a single vertex, then we have*

$$\overline{\chi}(\mathcal{H}) = |X(\mathcal{H})| - s(\mathcal{H}).$$

Before we prove the theorem, we establish a lemma regarding the sieve number for simple mixed hypertrees as described in the statement of the theorem.

Lemma 1. *If $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is a simple mixed hypertree that can be decomposed into the union of interval mixed hypergraphs $T_1, T_2, \ldots, T_k$, where $k \geq 1$, so that if any two of these hypergraphs meet, they meet only at a single vertex, then*

$$s(\mathcal{H}) = s(\mathcal{T}_1) + s(\mathcal{T}_2) + \cdots + s(\mathcal{T}_k).$$

Proof. Note that hyperedges in different maximum sieves in (say) T_i, T_j, where $i \neq j$, will also be in a maximum sieve of $\mathcal{H}$. For, if they intersect, they may only intersect at a single point. □

Proof. We prove the theorem by induction on the number k of interval mixed hypergraphs. For $k = 1$, the mixed hypertree is an interval mixed hypergraph, in which case it is already known the result holds. Let us assume the result is true if there are $k = m$ interval mixed hypergraphs, and establish that it is true when there are $m + 1$ interval mixed hypergraphs. Remove any one of the $(m + 1)$-many interval mixed hypergraphs $\mathcal{T}_i$ from $\mathcal{H}$ (except for the intersecting vertex), and consider the resulting mixed hypertree $\mathcal{H}'$. Note that $\mathcal{H}'$ is the union of m-many interval mixed hypergraphs, and so by the inductive hypothesis and with obvious notation we have

$$\overline{\chi}(\mathcal{H}') = |X'| - s(\mathcal{H}').$$

Now, for the removed interval mixed hypergraph $\mathcal{T}_i$ we have

$$\overline{\chi}(\mathcal{T}_i) = |X_i| - s(\mathcal{T}_i),$$

since it is an interval mixed hypergraph.

Now, we note that $|X(\mathcal{H})| = |X(\mathcal{H}')| + |X(\mathcal{T}_i)| - 1$. Let v be the common vertex. Before we re-insert $\mathcal{T}_i$ back into $\mathcal{H}'$ to form the original mixed hypertree $\mathcal{H}$, we re-color v (in $\mathcal{T}_i$) and all vertices in $\mathcal{T}_i$ having that color (in the maximum coloring) in the same color that v has in a maximum coloring of $\mathcal{H}'$. Thus, we have the following:

$$\begin{aligned}\overline{\chi}(\mathcal{H}) &= \overline{\chi}(\mathcal{H}') + \overline{\chi}(\mathcal{T}_i) - 1\\ &= |X(\mathcal{H}')| - s(\mathcal{H}') + |X(\mathcal{T}_i)| - s(\mathcal{T}_i) - 1\\ &= (|X(\mathcal{H}')| + |X(\mathcal{T}_i)| - 1) - (s(\mathcal{H}') + s(\mathcal{T}_i))\\ &= |X(\mathcal{H})| - s(\mathcal{H}).\end{aligned}$$

In the displayed equations above, the only item we have not yet established is the first equality. To prove this, first note that giving both $\mathcal{H}'$ and $\mathcal{T}_i$ a maximum coloring, then merging back these mixed hypertrees with the necessary adjustment in the coloring common vertex v will give a proper coloring of $\mathcal{H}$; thus, $\overline{\chi}(\mathcal{H}) \geq \overline{\chi}(\mathcal{H}') + \overline{\chi}(\mathcal{T}_i)$.

On the other hand, suppose we have a maximum coloring of $\mathcal{H}$ with $\overline{\chi}(\mathcal{H})$ colors. Now decompose $\mathcal{H}$ into $\mathcal{H}'$ and $\mathcal{T}_i$, preserving the given maximum coloring. Then this is a coloring of $\mathcal{H}'$ and of $\mathcal{T}_i$; thus, we have $\overline{\chi}(\mathcal{H}) \leq \overline{\chi}(\mathcal{H}') + \overline{\chi}(\mathcal{T}_i) - 1$.

□

3 Examples

First, we exhibit a simple mixed hypertree $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ in which the difference between $\overline{\chi}(\mathcal{H})$ and $n - s$ is 1, where $n = |X|$ and $s = s(\mathcal{H})$.

Example 1. Consider the mixed hypertree $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where $X = \{x_0, x_1, \ldots, x_5\}$,

$$\mathcal{C} = \{\{x_0, x_1, x_2\}, \{x_0, x_2, x_3\}, \{x_0, x_3, x_4\}, \{x_0, x_4, x_5\}, \{x_0, x_5, x_1\}\},$$

and $\mathcal{D} = \emptyset$. Here, $\bar{\chi}(\mathcal{H}) = 3$, $|X| = n = 6$, and the sieve number $s = 2$. Thus,

$$\bar{\chi}(\mathcal{H}) = 3 \neq 4 = n - s.$$

This example can also be generalized to similar mixed hypergraphs with an odd number of such "satellite vertices" (the vertices that are not the central vertex) as to make the difference between $\overline{\chi}(\mathcal{H})$ and $|X| - s$ as large as desired. Here is how: Start with the mixed hypertree $\mathcal{H}$ in example 1 above. Then, pick any of the satellite vertices and make it the satellite vertex of an added-on copy of $\mathcal{H}$. The resulting mixed hypertree – call it $\mathcal{H}_1$ – will have $\overline{\chi}(\mathcal{H})$ - ($|X| - s) = 2$. To make the difference equal to 3, one could pick one of the new satellite vertices from $\mathcal{H}_1$ and make it the satellite vertex of an added-on copy of $\mathcal{H}$. Continuing in this fashion, one can see how to make the difference arbitrarily large.

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