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Polynomial Time Algorithm for Determining Max-Min Paths in Networks and Solving Zero Value Cyclic Games*

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Abstract

We study the max-min paths problem, which represents a game version of the shortest and the longest paths problem in a weighted directed graph. In this problem the vertex set V of the weighted directed graph $G = (V, E)$ is divided into two disjoint subsets V_A and V_B which are regarded as positional sets of two players. The players are seeking for a directed path from the given starting position v_0 to the final position v_f, where the first player intends to maximize the integral cost of the path while the second one has aim to minimize it. Polynomial-time algorithm for determining max-min path in networks is proposed and its application for solving zero value cyclic games is developed.

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1 Introduction and Problem Formulation

In this paper we consider the max-min paths problem on networks, which generalizes classical combinatorial problems of the shortest and the longest paths in weighted directed graphs. This max-min paths problem arose as an auxiliary one when searching optimal stationary

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strategies of players in cyclic games [1-3]. The main results are concerned with the existence of polynomial-time algorithms for determining max-min paths in networks and elaboration of such algorithms. The application of the proposed algorithms for studying and solving zero value cyclic games is shown.

The statement of the considered problem is the following.

Let $G = (V, E)$ be a directed graph with vertex set V, $|V| = n$, and edge set E, $|E| = m$. Assume that G contains a vertex $v_f \in V$ such that it is attainable from each vertex $v \in V$, i.e. v_f is a sink in G. On edge set E it is given a function $c : E \to R$, which assigns a cost c(e) to each edge $e \in E$. In addition the vertex set is divided into two disjoint subsets V_A and V_B ($V = V_A \bigcup V_B$, $V_A \bigcap V_B = \emptyset$), which we regard as position sets of two players.

On G we consider a game of two players. The game starts at position $v_0 \in V$. If $v_0 \in V_A$, then the move is done by the first player, otherwise it is done by the second one. The move means the passage from a position v_0 to a neighbour position v_1 through the edge $e_1 = (v_0, v_1) \in E$. After that if $v_1 \in V_A$, then the move is done by the first player, otherwise it is done by the second one and so on. As soon as the final position is reached the game is over. The game can be finite or infinite. If the final position v_f is reached in finite time, then the game is finite. In the case when the final position v_f is not reached, the game is infinite. The first player in this game has the aim to maximize $\sum_i c(e_i)$ while the second one has the aim to minimize $\sum_i c(e_i)$.

Strictly the considered game in normal form can be defined as follows. We identify the strategies s_A and s_B of players with the maps

$$s_A : u \to v \in V_G(u) \text{ for } u \in V_A;$$

$$s_B : u \to v \in V_G(u) \text{ for } u \in V_B,$$

where $V_G(u)$ represents the set of extremities of edges $e = (u, v) \in E$, i.e. $V_G(u) = \{v \in V | e = (u, v) \in E\}$. Since G is a finite graph then the set of strategies of players

$$S_A = \{s_A : u \to v \in V_G(u) \text{ for } u \in V_A\};$$

$$S_B = \{s_B : u \to v \in V_G(u) \text{ for } u \in V_B\}$$

are finite sets. The payoff function $F_{v_0}(s_A, s_B)$ on $S_A \times S_B$ is defined in the following way.

Let be in G a subgraph $G_s = (V, E_s)$ generated by edges of form $(u, s_A(u))$ for $u \in V_A$ and $(u, s_B(u))$ for $u \in V_B$. Then either a unique directed path $P_s(v_0, v_f)$ from v_0 to v_f exists in G_s or such a path does not exist in G_s. In the second case in G_s there exists a unique directed cycle C_s, which can be reached from v_0.

For given s_A and s_B we set

$$F_{v_0}(s_A, s_B) = \sum_{e \in E(P_s(v_0, v_f))} c(e),$$

if in G_s there exists a directed path $P_s(v_0, v_f)$ from v_0 to v_f, where $E(P_s(v_0, v_f))$ is a set of edges of the directed path $P_s(v_0, v_f)$. If in G there are no directed paths from v_0 to v_f, then we define $F_{v_0}(s_A, s_B)$ as follows. Let $P'_s(v_0, u_0)$ be a directed path, which connects the vertex v_0 with the cycle C_s and $P_s(v_0, u_0)$ has no other common vertices with C_s except u_0. Then we put

$$F_{v_0}(s_A, s_B) = \begin{cases} +\infty, & \text{if } \sum\limits_{e \in E(C_s)} c(e) > 0; \\ \sum\limits_{e \in E(P'_s(v_0, u_0))} c(e), & \text{if } \sum\limits_{e \in E(C_s)} c(e) = 0; \\ -\infty, & \text{if } \sum\limits_{e \in E(C_s)} c(e) < 0. \end{cases}$$

This game is related to zero-sum positional games of two players and it is determined by the graph G with the sink vertex v_f, the partition $V = V_A \bigcup V_B$, the cost function $c : E \to R$ and the starting position v_0. We denote the network, which determines this game, by (G, V_A, V_B, c).

In [4] it is shown that if G does not contain directed cycles, then for every $v \in V$ the following equality holds

$$p(v) = \max_{s_A \in S_A} \min_{s_B \in S_B} F_v(s_A, s_B) = \min_{s_B \in S_B} \max_{s_A \in S_A} F_v(s_A, s_B), \tag{1}$$

which means the existence of optimal strategies of players in the considered game. Moreover, in [4] it is shown that in G there exists a tree $T^* = (V, E^*)$ with sink vertex v_f, which gives the optimal strategies of players in the game for an arbitrary starting position $v_0 \in V$. The strategies of players are obtained by fixing

$$s_A^*(u) = v, \text{ if } (u, v) \in E^* \text{ and } u \in V_A \setminus \{v_f\};$$

$$s_B^*(u) = v, \text{ if } (u, v) \in E^* \text{ and } u \in V_B \setminus \{v_f\}.$$

In general case for an arbitrary graph G equality (1) may fail to hold. Therefore we formulate necessary and sufficient conditions for the existence of optimal strategies of players in this game and a polynomial-time algorithm for determining the tree of max-min paths from every $v \in V$ to v_f. Furthermore we show that our max-min paths problem on the network can be regarded as an zero value ergodic cycle game. Therefore the proposed algorithm can be used for solving such games.

The formulated game on network (G, V_A, V_B, c) in [4] is named the dynamic c-game. Some preliminary results related to this problem have been obtained in [4-7]. More general models of positional games on networks with p players have been studied in [8,9].

2 Algorithm for solving the problem on acyclic networks

The formulated problem for acyclic networks has been studied in [4].

Let $G = (V, E)$ be a finite directed graph without directed cycles and given sink vertex v_f. The partition $V = V_A \bigcup V_B$ $(V_A \bigcap V_B = \emptyset)$ of vertex set of G is given and the cost function $c : E \rightarrow R$ on edges is defined. We consider the dynamic c-game on G with given starting position $v \in V$.

It is easy to observe that for fixed strategies of players $s_A \in S_A$ and $s_B \in S_B$ the subgraph $G_s = (V, E_s)$ has a structure of directed tree with sink vertex $v_f \in V$. This means that the value $F_{v_0}(s_A, s_B)$ is determined uniquely by the sum of edge costs of the unique directed path $P_s(v_0, v_f)$ from v_0 to v_f. In [5,6] it is proved that for acyclic c-game on network (G, V_A, V_B, c) there exist the strategies of players s_A^*, s_B^* such that

$$p(v) = F_v(s_A^*, s_B^*) = \max_{s_A \in S_A} \min_{s_B \in S_B} F_v(s_A, s_B) = \\ = \min_{s_B \in S_B} \max_{s_A \in S_A} F_v(s_A, s_B) \qquad (2)$$

and s_A^*, s_B^* do not depend on starting position $v \in V$, i.e. (2) holds for every $v \in V$.

The equality (2) is evident in the case when $\mathrm{ext}(c, u) = 0$, $\forall u \in V \setminus \{v_f\}$, where

$$\mathrm{ext}(c, u) = \begin{cases} \max\limits_{v \in V_G(u)} \{c(u, v)\}, u \in V_A; \\ \min\limits_{v \in V_G(u)} \{c(u, v)\}, u \in V_B. \end{cases}$$

In this case $p(u) = 0$, $\forall u \in U$ and the optimal strategies of players can be obtained by fixing the maps $s_A^* : V_A \setminus \{v_f\} \to V$ and $s_B^* : V_B \setminus \{v_f\} \to V$ such that $s_A^* \in \mathrm{VEXT}(c, u)$ for $u \in V_A \setminus \{v_f\}$ and $s_B^* \in \mathrm{VEXT}(c, u)$ for $u \in V_B \setminus \{v_f\}$, where

$$\mathrm{VEXT}(c, u) = \{v \in V_G(u) | c(u, v) = \mathrm{ext}(c, u)\}.$$

If the network (G, V_A, V_B, c) has the property $\mathrm{ext}(c, u) = 0$, $\forall u \in V \setminus \{v_f\}$, then it is named the network in canonic form. So, for the acyclic c-game on network in canonic form equality (2) holds and $p(v) = 0$, $\forall v \in V$.

In general case equality (2) can be proved using properties of the potential transformation $c'(u, v) = c(u, v) + \varepsilon(v) - \varepsilon(u)$ on edges $e = (u, v)$ of the network, where $\varepsilon : V \to R$ is an arbitrary real function on V (the potential transformation for positional games has been introduced in [2]). The fact is that such transformation of the costs on edges of the acyclic network in c-game does not change the optimal strategies of players, although values $p(v)$ of positions $v \in V$ are changed by $p(v) + \varepsilon(v_f) - \varepsilon(v)$. It means that for an arbitrary function $\varepsilon : V \to R$ the optimal strategies of the players in acyclic c-games on the networks

(G, V_A, V_B, c) and (G, V_A, V_B, c') are the same. Using such property in [4,5] the following theorem is proved.

Theorem 1. *For an arbitrary acyclic network (G, V_A, V_B, c) with a sink vertex v_f there exists a function $\varepsilon : V \to R$ which determines the potential transformation $c'(u, v) = c(u, v) + \varepsilon(v) - \varepsilon(u)$ on edges $e = (u, v)$ such that the network (G, V_A, V_B, c) has the canonic form. The values $\varepsilon(v)$, $v \in V$, which determine $\varepsilon : V \to R$, can be found by using the following recursive formula*

$$\varepsilon(v_f) = 0$$

$$\varepsilon(u) = \begin{cases} \max\limits_{v \in V_G(u)} \{c(u, v) + \varepsilon(v)\} \text{ for } u \in V_A \setminus \{v_f\}; \\ \min\limits_{v \in V_G(u)} \{c(u, v) + \varepsilon(v)\} \text{ for } u \in V_B \setminus \{v_f\}. \end{cases} \qquad (3)$$

On the basis of this theorem the following algorithm for determining optimal strategies of players in c-game is proposed in [4,5].

Algorithm 1.

1. Find the values $\varepsilon(u)$, $u \in V$, according to recursive formula (3) and the corresponding potential transformation $c'(u, v) = c(u, v) + \varepsilon(v) - \varepsilon(u)$ on edges $(u, v) \in E$.

2. Fix arbitrary maps $s_A^* : V_A \setminus \{v_f\} \to V$ and $s_B^*(u) \in \text{VEXT}(c', u)$ for $u \in V_B \setminus \{v_f\}$. $\square$

Remark 1. *The values $\varepsilon(u)$, $u \in V$, represent the values of the acyclic c-game on (G, V_A, V_B, c) with starting position u, i.e. $\varepsilon(u) = p(u)$, $\forall u \in V$. Algorithm 1 needs $O(n^2)$ elementary operations because the tabulation of the values $\varepsilon(u)$, $u \in V$, using formula (3) for acyclic networks needs such number of operations.*

3 The main results for the problem on an arbitrary network

First of all we give an example which shows that equality (1) may fails to hold. In fig.1 it is given the network with starting position $v_0 = 1$ and final position $v_f = 4$, where positions of the first player are represented by cycles and positions of the second player are represented by squares; values of cost functions on edges are given alongside them.

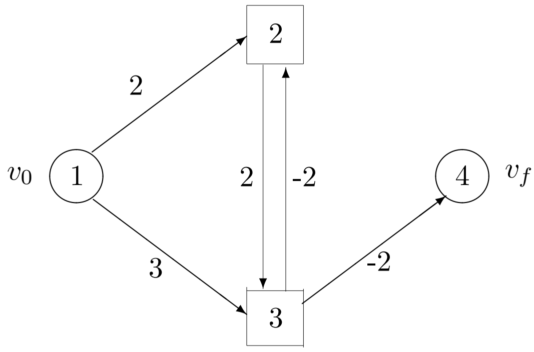

Fig. 1

It is easy to observe that

$$\max_{s_A \in S_A} \min_{s_B \in S_B} F_{12}(s_A, s_B) = 1, \quad \min_{s_B \in S_B} \max_{s_A \in S_A} F_{12}(s_A, s_B) = 2.$$

The following theorem gives conditions for the existence of settle values $p(v)$ for each $v \in V$ in the c-game.

Theorem 2. *Let (G, V_A, V_B, c) be an arbitrary network with sink vertex $v_f \in V$. Moreover let us consider that $\sum_{e \in E(C_s)} c(e) \neq 0$ for every directed cycle C_s from G_s. Then for c-game on (G, V_A, V_B, c) condition (1) holds if and only if there exists a function $\varepsilon : V \to R$, which determines a potential transformation $c'(u, v) = c(u, v) + \varepsilon(v) - \varepsilon(u)$ on edges $(u, v) \in E$ such that* $\operatorname{ext}(c', u) = 0$, *$\forall v \in V$. If $\sum_{e \in E(C_s)} c(e) \neq 0$ for every directed cycle and in G there exists the potential transformation $c'(u, v) = c(u, v) + \varepsilon(v) - \varepsilon(u)$ on edges $(u, v) \in E$, then $\varepsilon(v) = p(v)$, $\forall v \in V$.*

Proof. $\Longrightarrow$ Let us consider that $\sum_{e\in E(C_s)} c(e) \neq 0$ for every directed cycle C_s in G and condition (1) holds for every $v \in V$. Moreover, we consider that $p(v)$ is a finite value for every $v \in V$. Taking into account that the potential transformation does not change the cost of cycles, we have that such transformation does not change optimal strategies of players although values $p(v)$ of positions $v \in V$ are changed by $p(v) - \varepsilon(v) + \varepsilon(v_f)$. It is easy to observe that if we put $\varepsilon(v) = p(v)$ for $v \in V$, then the function $\varepsilon : E \to R$ determines the potential transformation $c'(u,v) = c(u,v) + \varepsilon(v) - \varepsilon(u)$ on edges $(u,v) \in E$ such that $\mathrm{ext}(c', u) = 0$, $\forall v \in V$.

$\Longleftarrow$ Let us consider that there exists a potential transformation $c'(u,v) = c(u,v) + \varepsilon(v) - \varepsilon(u)$ on edges $(u,v) \in E$ such that $\mathrm{ext}(c', u) = 0$, $\forall v \in V$. The value $p(v)$ of the game after the potential transformation is zero for every $v \in V$ and optimal strategies of players can be found by fixing s_A^* and s_B^* such that $s_A^*(u) \in \mathrm{VEXT}(c', u)$ for $u \in V_A \setminus \{v_f\}$ and $s_B^*(u) \in \mathrm{VEXT}(c', u)$ for $u \in V_B \setminus \{v_f\}$. Since the potential transformation does not change optimal strategies of players we put $p(v) = \varepsilon(v) - \varepsilon(v_f)$ and obtain (1). □

Corollary 1. *The values $p(v)$, $v \in V$, can be found as follows $p(v) = \varepsilon(v) - \varepsilon(v_f)$, i.e. the difference $\varepsilon(v) - \varepsilon(v_f)$ is equal to the cost of the max-min path from v to v_f. If $\varepsilon(v_f) = 0$, then $p(v) = \varepsilon(v)$, $\forall v \in V$.*

Corollary 2. *If for every directed cycle C_s in G the condition $\sum_e c(e) \neq 0$ holds then the existence of the potential transformation $c'(u,v) = c(u,v) + \varepsilon(v) - \varepsilon(u)$ on edges $(u,v) \in E$ such that*

$$\mathrm{ext}(c', u) = 0, \forall v \in V \tag{4}$$

represents necessary and sufficient conditions for validity of equality (1) for every $u \in V$. In the case when in G there exists cycle C_s with $\sum_{e\in E(C_s)} c(e) = 0$ condition (4) becomes only necessary one for validity (1) for every $v \in V$.

Corollary 3. *If in c-game there exist the strategies s_A^* and s_B^*, for which (1) holds for every $v \in V$ and these strategies generate in G*

a tree $T_{s^*} = (V, E_{s^*})$ with sink vertex v_f, then there exists the potential transformation $c'(u,v) = c(u,v) + \varepsilon(v) - \varepsilon(u)$ on edges $(u,v) \in E$ such that the graph $G^0 = (V, E^0)$, generated by the set of edges $E^0 = \{(u,v) \in E \mid c'(u,v) = 0\}$, contains the tree T_{s^*} as a subgraph.

Taking into account the mentioned above results we may propose the following algorithm for determining the optimal strategies of players in c-game based on the constructing of the tree of min-max paths.

Algorithm 2.

Preliminary step (step 0) Set $V^* = \{v_f\}$, $\varepsilon(v_f) = 0$.
General step (step k) Find the set of vertices

$$V' = \{u \in V \setminus V^* \mid (u,v) \in E, v \in V^*\}.$$

For each $u \in V'$ we calculate

$$\varepsilon(u) = \begin{cases} \max\limits_{v \in O_{V^*}(u)} \{\varepsilon(v) + c(u,v)\}, \ u \in V_A \bigcap V'; \\ \min\limits_{v \in O_{V^*}(u)} \{\varepsilon(v) + c(u,v)\}, \ u \in V_B \bigcap V', \end{cases} \tag{5}$$

where $O_{V^*}(u) = \{v \in V^* \mid (u,v) \in E\}$. Then in $V^* \bigcup V'$ we find the subset

$$U^k = \left\{ u \in V^* \bigcup V' \;\middle|\; \operatorname*{extr}_{v \in O_{V^* \cup V'}(u)} \{\varepsilon(v) - \varepsilon(u) + c(u,v) = 0\} \right\}$$

and change V^* by U^k, i.e. $V^* = U^k$. After that we check if $V^* = V$. If $V^* \neq V$, then go to the next step. If $V^* = V$, then define the potential transformation $c'(u,v) = c(u,v) + \varepsilon(v) - \varepsilon(u)$ on edges $(u,v) \in E$ and find the graph $G^0 = (V, E^0)$, generated by the set of edges $E^0 = \{(u,v) \in E | c'(u,v) = 0\}$. In G^0 fix an arbitrary tree $T^* = (V, E^*)$, which determines the optimal strategies of players as follows:

$$s_A^*(u) = v, \text{ if } (u,v) \in E^* \text{ and } u \in V_A \setminus \{v_f\};$$

$$s^*_B(u) = v, \text{ if } (u, v) \in E^* \text{ and } u \in V_B \setminus \{v_f\}.$$

Now let us show that this algorithm finds the tree of max-min paths $T^* = (V, E^*)$ if such tree exists in G. $\square$

Denote by V^i the subset of V, where $v \in V^i$ if in T^* there exists the directed path $P_T(v, v_0)$ from v to v_0 which contains i edges, i. e. $V^i = \{v \in V \mid |P_{T^*}(v, v_0)| = i\}$. So, $V = V^0 \bigcup V^1 \bigcup V^2 \bigcup \cdots \bigcup V^r$ $(V^i \bigcap V^j = \emptyset)$, where $V^0 = \{v_f\}$ and V^i, $i \in \{1, 2, \ldots, r\}$, represents the level i of vertex set of T^*. If in G there exists several max-min trees $T^*_1 = (V, E^*_1)$, $T^*_2 = (V, E^*_2)$, $\ldots$, $T^*_q = (V, E^*_q)$ then we will select the one which has number of levels $r = \min_{1 \le i \le q} \{r_i\}$.

Theorem 3. *If in G there exists a tree of max-min path $T^* = (V, E^*)$ with sink vertex v_f then Algorithm 2 finds it using $k = r$ iterations. The running time of the algorithm is $O(n^3)$.*

Proof. We prove the theorem by using the induction principle on number of levels of max-min tree. If $r = 1$ the theorem is evident. Assume that the theorem is true for any $r \le p$ and let us show that it is true for $r = p + 1$.

Denote by $V^0, V^1, \ldots, V^r$ the level sets of the tree $T^* = (V, E^*)$, $V = V^0 \bigcup V^1 \bigcup V^2 \bigcup \cdots \bigcup V^r$ $(V^i \bigcap V^j = \emptyset)$. It is easy to observe that if we delete from T^* the vertex set V^r and corresponding pendant edges $e = (u, v)$, $v \in V^r$, then we obtain a tree $\overline{T}^* = (\overline{V}, \overline{E}^*)$, $\overline{V} = V \setminus V^r$. This tree $\overline{T}^*$ represents the tree of max-min paths for the subgraph $\overline{G} = (\overline{V}, \overline{E})$ of G generated by vertex set $\overline{V}$.

If we apply Algorithm 2 with respect to $\overline{G}$ then according to the induction principle we find the tree of max-min paths $\overline{T}^*$, which determines $\varepsilon : \overline{V} \to R$ and the potential transformation $\overline{c}(u, v) = c(u, v) + \varepsilon(u) - \varepsilon(v)$ on edges $(u, v) \in \overline{E}$ such that $\text{extr}(c', v) = 0$, $\forall v \in \overline{V}$. So, Algorithm 2 on $\overline{G}$ determines uniquely the values $\varepsilon(u)$ according to (5).

It is easy to observe that in G for an arbitrary vertex $u \in V^r$ calculated on the basis of formula (5) the following condition holds:

$$\varepsilon(u) = \begin{cases} \max\limits_{v \in V_G(u)} \{\varepsilon(v) + c(u,v)\}, \ u \in V^r \bigcap V_A; \\ \min\limits_{v \in V_G(u)} \{\varepsilon(v) + c(u,v)\}, \ u \in V^r \bigcap V_B. \end{cases}$$

This means that if we apply Algorithm 2 on G then after $r-1$ iterations the vertex set U^{r-1} coincides with $V \setminus V^r$. So, Algorithm 2 determines uniquely the values $\varepsilon(v)$, $v \in V$. Nevertheless here we have to note that in the process of the algorithm $V^k \subset U^k$ and V^k may differ from U^k for some $k = 1, 2, \ldots, r$.

Taking into account that at the general step of the algorithm it needs $O(n^2)$ elementary operations and $k \leq r(r \leq n)$ we obtain that the running time of the algorithm is $O(n^3)$. □

4 An application of the algorithm for solving zero value cyclic games

In this section we show that zero value ergodic cycle game can be regarded as max-min paths problem and therefore the proposed algorithm can be used for determining the optimal strategies of players in such cyclic games.

At first we remind the formulations of cyclic games and some necessary preliminary results.

4.1 Cyclic games: problem formulation

Let $G = (V, E)$ be a finite directed graph in which every vertex $u \in V$ has at least one leaving edge $e = (u, v) \in E$. On edge set E a function c: $E \rightarrow R$ is given which assigns a cost $c(e)$ to each edge $e \in E$. In addition the vertex set V is divided into two disjoint subsets V_A and V_B ($V = V_A \cup V_B$, $V_A \cap V_B = \emptyset$) which we will regard as positions sets of two players.

On G we consider the following two-person game from [1,2]. The game starts at position $v_0 \in V$. If $v_0 \in V_A$ then the move is done by first player, otherwise it is done by second one. The move means the passage from position v_0 to the neighbour position v_1 through the edge $e_1 = (v_0, v_1) \in E$. After that if $v_1 \in V_A$ then the move is done by first player, otherwise it is done by second one and so on indefinitely. The first player has the aim to maximize $\lim_{t\to\infty} \inf \frac{1}{t}\sum_{i=1}^{t} c(e_i)$ while the second player has the aim to minimize $\lim_{t\to\infty} \sup \frac{1}{t}\sum_{i=1}^{t} c(e_i)$.

In [1,2] it is proved that for this game there exists a value $p(v_0)$ such that the first player has a strategy of moves that insures $\lim_{t\to\infty} \inf \frac{1}{t}\sum_{i=1}^{t} c(e_i) \geq p(v_0)$ and the second player has a strategy of moves that insures $\lim_{t\to\infty} \sup \frac{1}{t}\sum_{i=1}^{t} c(e_i) \leq p(v_0)$. Furthermore in [1,2] it is shown that the players can achieve the value $p(v_0)$ applying the strategies of moves which do not depend on t. This means that the considered game can be formulated in the terms of stationary strategies. Such statement of the game in [2] is named cyclic game.

The strategies of players in cyclic game are defined as maps

$$s_A\colon\ u \to v \in V_G(u) \ \text{ for } \ u \in V_A; \ s_B\colon\ u \to v \in V_G(u) \ \text{ for } \ u \in V_B,$$

where $V_G(u)$ represents the set of extremities of edges $e = (u, v) \in E$, i.e. $V_G(u) = \{v \in V \mid e = (u, v) \in E\}$. Since G is a finite graph then the sets of strategies of players

$$S_A = \{s_A\colon\ u \to v \in V_G(u) \ \text{ for } \ u \in V_A\};$$
$$S_B = \{s_B\colon\ u \to v \in V_G(u) \ \text{ for } \ u \in V_B\}$$

are finite sets. The payoff function $F_{v_0}\colon S_A \times S_B \to R$ in cyclic game is defined as follows.

Let $s_A \in S_A$ and $s_B \in S_B$ be fixed strategies of players. Denote by $G_s = (V, E_s)$ the subgraph of G generated by edges of form $(u, s_A(u))$

for $u \in V_A$ and $(u, s_B(u))$ for $u \in V_B$. Then G_s contains a unique directed cycle C_s which can be reached from v_0 through the edges $e \in E_s$. The value $F_{v_0}(s_A, s_B)$ we consider equal to mean edges cost of cycle C_s, i.e.

$$F_{v_0}(s_A, s_B) = \frac{1}{n(C_s)} \sum_{e \in E(C_s)} c(e),$$

where $E(C_s)$ represents the set of edges of cycle C_s and $n(C_s)$ is a number of the edges of C_s. So, the cyclic game is determined uniquely by the network (G, V_A, V_B, c) and starting position v_0. In [1,2] it is proved that there exist the strategies $s_A^* \in S_A$ and $s_B^* \in S_B$ such that

$$\overline{p}(v) = F_v(s_A^*, s_B^*) = \max_{s_A \in S_A} \min_{s_B \in S_B} F_v(s_A, s_B) = \\ = \min_{s_B \in S_B} \max_{s_A \in S_A} F_v(s_A, s_B), \ \forall\, v \in V.$$

So, the optimal strategies s_A^*, s_B^* of players in cyclic games do not depend on starting position v although for different positions $u, v \in V$ the values $\overline{p}(u)$ and $\overline{p}(v)$ may be different. It means that the positions set V can be divided into several classes $V = V^1 \cup V^2 \cup \cdots \cup V^k$ according to values of positions $\overline{p}^1, \overline{p}^2, \ldots, \overline{p}^k$, i.e. $u, v \in V^i$ if and only if $\overline{p}^i = \overline{p}(u) = \overline{p}(v)$. In the case $k = 1$ the network (G, V_A, V_B, c) is named the ergodic network [2]. In [5, 6] it is shown that every cyclic game with arbitrary network (G, V_A, V_B, c) and given starting position v_0 can be reduced to an auxiliary cyclic game on auxiliary ergodic network (G', V_A', V_B', c').

4.2 Some Preliminary Results

In [2] the following theorem is formulated and proved.

Theorem 4. *Let (G, V_A, V_B, c) be an arbitrary network with the properties described in section 1. Then there exists the value $p(v)$, $v \in V$ and the function $\varepsilon\colon V \to R$ which determine a potential transformation $c'(u, v) = c(u, v) + \varepsilon(v) - \varepsilon(u)$ for costs on edges $e = (u, v) \in E$, such that the following properties hold*

a) $\overline{p}(u) = ext(c', u)$ *for* $v \in V$,

b) $\overline{p}(u) = \overline{p}(v)$ *for* $u \in V_A \cup V_B$ *and* $v \in \mathrm{VEXT}(c', u)$,

c) $\overline{p}(u) \geq \overline{p}(v)$ *for* $u \in V_A$ *and* $v \in V_G(u)$,
d) $\overline{p}(u) \leq \overline{p}(v)$ *for* $u \in V_B$ *and* $v \in V_G(u)$,
e) $\max\limits_{e\in E} |c'(e)| \leq 2|V| \max\limits_{e\in E} |c(e)|$.

The values $\overline{p}(v), v \in V$ *on network* (G, V_A, V_B, c) *are determined unequally and the optimal strategies of players can be found in the following way: fix the arbitrary strategies* $s_A^*\colon V_A \to V$ *and* $s_B^*\colon V_B \to V$ *such that* $s_A^*(u) \in \mathrm{VEXT}(c', u)$ *for* $u \in V_A$ *and* $s_B^*(u) \in \mathrm{VEXT}(c', u)$ *for* $u \in V_B$.

Further we shall use the theorem 4 in the case of the ergodic network (G, V_1, V_2, c), i.e. we shall use the following corollary.

Corollary 4. *Let* (G, V_A, V_B, c) *be an ergodic network. Then there exist the value* p *and the function* $\varepsilon\colon V \to R$ *which determines a potential transformation* $c'(u, v) = c(u, v) + \varepsilon(v) - \varepsilon(u)$ *for costs of edges* $e = (u, v) \in E$ *such that* $\overline{p} = ext(c', u)$ *for* $u \in V$. *The optimal strategies of players can be found as follows: fix arbitrary strategies* $s_A^*\colon V_A \to V$ *and* $s_B^*\colon V_B \to V$ *such that* $s_A^*(u) \in \mathrm{VEXT}(c', u)$ *for* $u \in V_A$ *and* $s_B^*(u) \in \mathrm{VEXT}(c', u)$ *for* $u \in V_B$.

4.3 The reduction of cyclic games to ergodic ones

Let us consider an arbitrary network (G, V_A, V_B, c) with given starting position $v_0 \in V$ which determines a cyclic game. In [5, 6] it is shown that this game can be reduced to a cyclic game on auxiliary ergodic network $(G', W_A, W_B, \overline{c})$, $G' = (W, F)$ in which the value $p(v_0)$ is preserving, $v_0 \in W = V \cup X \cup Y$.

The graph $G' = (W, F)$ is obtained from G if each edge $e = (u, v)$ is changed by a triple of edges $e^1 = (u, x), e^2 = (x, y), e^3 = (y, v)$ with the costs $\overline{c}(e^1) = \overline{c}(e^2) = \overline{c}(e^3) = c(e)$. Here $x \in X, y \in Y$ and $u, v \in V; W = V \cup X \cup Y$. In addition in G' each vertex x is connected with v_0 by edge (x, v_0) with the cost $\overline{c}(x, v_0) = M$ (M is a great value) and each edge (y, v_0) is connected with v_0 by edge (y, v_0) with the cost $\overline{c} = (y, v_0) = -M$. In $(G', W_A, W_B, \overline{c})$ the sets W_A and W_B are defined as follows: $W_A = V_A \cup Y$; $W_B = V_B \cup X$.

It is easy to observe that this reduction can be done in linear time.

4.4 The reduction of zero value ergodic cyclic games to max-min paths problem

Let us consider a zero value cyclic game on ergodic network (G, V_A, V_B, c), $G = (V, E)$. Then according to Theorem 8 there exists the function $\varepsilon : V \to R$ which determines the potential transformation $c'(u, v) = c(u, v) + \varepsilon(v) - \varepsilon(u)$ on edges $(u, v) \in E$ such that

$$\text{ext}(c, u) = 0, \ \forall v \in V. \tag{6}$$

This means that if v_f is a vertex of the cycle C_{s^*} determined by optimal strategies s_A^* and s_B^* then the problem of finding the function $\varepsilon : V \to R$ which determines the canonic potential transformation is equivalent to the problem of finding the values $\varepsilon(v)$, $v \in V$ in max-min paths problem on G with sink vertex v_f where $\varepsilon(v_f) = 0$.

So, in order to solve zero value cyclic game we fix each time a vertex $v \in V$ as a sink vertex $(v_f = v)$ and solve a max-min paths problem on G with sink vertex v_f. If for given $v_f = v$ the obtained function $\varepsilon : V \to R$ on the basis of Algorithm 2 determines the potential transformation which satisfies (6) then we fix s_A^* and s_B^* such that $s_A^*(u) \in \text{VEXT}(c', u)$ for $u \in V_A$ and $s_B^*(u) \in \text{VEXT}(c', u)$ for $u \in V_B$. If for given v the function $\varepsilon : V \to R$ does not satisfy (6) then we select another vertex $v \in V$ as a sink vertex and so on. This means that the optimal strategies of players in zero value ergodic cyclic games can be found in time $O(n^4)$.

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