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A generalization of the chromatic polynomial of a cycle

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Abstract

We prove that if an edge of a cycle on n vertices is extended by adding k vertices, then the the chromatic polynomial of such generalized cycle is:

$$P(H_k, \lambda) = (\lambda - 1)^n \sum_{i=0}^{k} \lambda^i + (-1)^n(\lambda - 1).$$

1 Introduction

We consider simple finite graphs and assume that the basic definitions from graph and hypergraph theory (see, for example, [1, 3, 4]) are familiar to the reader.

Proper coloring of a graph $G = (V, \mathcal{E})$, is a mapping $f : V(G) \rightarrow \{1, 2, \ldots, \lambda\}$ which is defined as an assignment of distinct colors from a finite set of colors $[\lambda]$ to the *vertices* of G in such a way that *adjacent vertices* have different colors. Such notion has been extended in 1966 by P. Erdös and A. Hajnal to the coloring of a hypergraph [2]. Thus, in general case, the *proper coloring* of a hypergraph $H = (V, \mathcal{E})$ is the labelling of the vertices of H in such a way that every *hyperedge* $E \in \mathcal{E}$ has at least two vertices of distinct colors.

The function $P(H, \lambda)$ counts the mappings $f : V(H) \rightarrow [\lambda]$ that properly color H using colors from the set $[\lambda] = \{1, 2, \ldots, \lambda\}$. Thus, we define *the chromatic polynomial* of a hypergraph H as the number of all proper colorings of H using at most λ colors [3].

Let $C_n = (V, \mathcal{E})$ be a cycle on n vertices, $n \geq 3$, where $V = \{v_1, v_2, \dots, v_n\}$. Consider an edge $E = \{v_1, v_2\}$ of C_n. We sequentially increase the size of E by adding k pendant vertices (a vertex is called *pendant* if its degree is one) from the set $S_k = \{x_1, x_2, x_3, \dots, x_k\}$, $k \geq 1$. Notice that E becomes a hyperedge E', containing $k + 2 \geq 3$ vertices. We compute the chromatic polynomial of the obtained hypergraph $H_k = (V \cup S_k, \mathcal{E}')$, where k is the number of pendant vertices added.

2 Proof of the formula

Theorem 1. *The chromatic polynomial of the hypergraph H_k has the following form:*

$$P(H_k, \lambda) = (\lambda - 1)^n \sum_{i=0}^{k} \lambda^i + (-1)^n (\lambda - 1).$$

Proof. Induction on the number of pendant vertices k. Observe that

$$P(H_0, \lambda) = (\lambda - 1)^n \lambda^0 + (-1)^n (\lambda - 1) = (\lambda - 1)^n + (-1)^n (\lambda - 1) = P(C_n, \lambda)$$

what is the chromatic polynomial of any cycle on n vertices, see [4, p.229].

The idea of proof consists in the following procedure: we apply to H_k, $k \geq 1$, the connection-contraction algorithm which is a special case of the splitting-contraction algorithm for mixed hypergraphs, see [3, p.30]. In any proper coloring of H, the vertices v_1, and x_1 either have different colors or have the same color. In the first case, we connect x_1 and v_1 by an edge; in the second case, we contract the edge $\{x_1, v_1\}$ and in this way identify the vertices x_1 and v_1. After removing of an exterior hyperedge containing vertices x_1, v_1, we obtain two graphs and some isolated vertices and compute the chromatic polynomial as a sum of two chromatic polynomials of the respective graphs.

Consider the case $k = 1$. We obtain that

$$P(H_1, \lambda) = P(T_{n+1}, \lambda) + P(H_0, \lambda),$$

where T_n is a tree on n vertices; it is well known that

$$P(T_n, \lambda) = \lambda(\lambda - 1)^{n-1}.$$

Since $P(H_0, \lambda) = P(C_n, \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1)$ we obtain

$$P(H_1, \lambda) = \lambda(\lambda - 1)^n + (\lambda - 1)^n + (-1)^n(\lambda - 1) =$$

$$= (\lambda - 1)^n(\lambda + 1) + (-1)^n(\lambda - 1).$$

Consider the case $k = 2$. Using the same procedure we obtain a tree, a cycle and one isolated vertex. Therefore

$$P(H_2, \lambda) = P(T_{n+1}, \lambda)\lambda + P(H_1, \lambda).$$

Notice that the chromatic polynomial of the independent vertex set $P(S_k, \lambda) = \lambda^k$ because each isolated vertex can be assigned λ colors. Using $P(H_1, \lambda) = (\lambda - 1)^n(\lambda + 1) + (-1)^n(\lambda - 1)$ we establish the following equality:

$$P(H_2, \lambda) = \lambda(\lambda - 1)^n\lambda + (\lambda - 1)^n(\lambda + 1) + (-1)^n(\lambda - 1) =$$

$$= (\lambda - 1)^n(\lambda^2 + \lambda + 1) + (-1)^n(\lambda - 1).$$

Let us assume that our formula for the chromatic polynomial of $P(H_j, \lambda)$ is true for any number $j \geq 1$ of pendant vertices. We now prove that

$$P(H_{j+1}, \lambda) = (\lambda - 1)^n(\lambda^{j+1} + \lambda^j + \ldots + \lambda^1 + \lambda^0) + (-1)^n(\lambda - 1).$$

Consider $j + 1$ number of pendant vertices from the set $S_{j+1} = \{x_1, x_2, \ldots, x_j, x_{j+1}\}$ added to the edge $E = \{v_1, v_2\}$ of the cycle $C_n = (V, \mathcal{E})$. The edge $E = \{v_1, v_2\}$ becomes a hyperedge $E' = \{v_1, v_2, x_1, x_2, \ldots, x_j, x_{j+1}\} \in \mathcal{E}'$ of the new graph $H_{j+1} = (V \cup S_{j+1}, \mathcal{E}')$. Applying the algorithm as described in the previous cases to H_{j+1} yields the following chromatic polynomial equality:

$$P(H_{j+1}, \lambda) = P(T_{n+1}, \lambda)P(S_j, \lambda) + P(H_j, \lambda).$$

By the induction hypothesis,

$$P(H_j,\lambda) = (\lambda-1)^n(\lambda^j+\lambda^{j-1}+\ldots+\lambda^1+\lambda^0)+(-1)^n(\lambda-1);$$

also, $P(S_j,\lambda)=\lambda^j$. Therefore the following equality holds:

$$P(H_{j+1},\lambda) =$$

$$= \lambda(\lambda-1)^n\lambda^j+(\lambda-1)^n(\lambda^j+\lambda^{j-1}+\ldots+\lambda^1+\lambda^0)+(-1)^n(\lambda-1) =$$

$$= (\lambda-1)^n(\lambda^{j+1}+\lambda^j+\ldots+\lambda^1+\lambda^0)+(-1)^n(\lambda-1).$$

Consequently,

$$P(H_k,\lambda) = (\lambda-1)^n\sum_{i=0}^{k}\lambda^i+(-1)^n(\lambda-1)$$

holds for any number $k \geq 1$ of pendant vertices added to an edge of C_n.

□

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