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An Automatic Proof of Euler's Formula

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Abstract

In this information age, everything is digitalized. The encoding of functions and the automatic proof of functions are important. This paper will discuss the automatic calculation for Taylor expansion coefficients, as an example, it can be applied to prove Euler's formula automatically.

Keywords: function, coefficient, automatic proof.

1 Introduction

The expansion of Taylor series is a very old topic in both pure and applied mathematics that plays a crucial role in both fundamental theory and applications. Computer algebra systems provide an interactive environment to assist in solving many mathematical problems.

One way to define an analytic function $f(z)$ is in terms of its Taylor series expansion at $z = 0$,

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

Quite a few theorems exist about how to find the coefficient a_n of a general term $a_n z^n$ in the expansion, which we shall denote $[z^n]f(z)$. Under some conditions, we have Taylor's formula [1]:

$$a_n = [z^n]f(z) = \frac{f^{(n)}(0)}{n!}.$$

This is a very nice formula and can be quite useful in finding a specific term such as $[z^3]f(z)$. However, for an arbitrary number n (usually considered to be very large), we cannot use the formula directly to determine $[z^n]f(z)$.

Ravenscroft implemented a Maple package called `genfunc` that can calculate $[z^n]f(z)$ for any rational function $f(z)$ [2]. Rational functions, however, are very well structured and easy to handle. As shown by Ravenscroft, every nontrivial rational generating function $F(z)$ encodes a sequence that is defined by a homogeneous linear recurrence with constant coefficients [3]. So finding $[z^n]f(z)$ reduces to solving a linear homogeneous recurrence with constant coefficients which, in turn, reduces to solving a corresponding polynomial equation.

If $f(z)$ is not a rational function, it is difficult in practice to calculate $[z^n]f(z)$. In many cases, an exact expansion of $[z^n]f(z)$ is impossible to find or too complicated to be of practical value. In such instances, we often have to settle for an asymptotic representation of $[z^n]f(z)$. Sadly and perhaps surprisingly, as Bruno Salvy stated a decade ago, "Current symbolic computation systems generally lack facilities for manipulating asymptotic expansion computations of a form more complex than the first terms of Taylor series or Puiseux expansions (involving fractional powers)[4]". This situation has not changed significantly since then.

This work is to provide an approach to calculate Taylor coefficients of functions, as an example, it can be applied to prove Euler's formula automatically.

2 Laplace Method

Assume that the function f is defined for $0 \leq t < \infty$. We write the Laplace transform as

$$F(s) = \mathrm{Ł}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt.$$

We shall refer to $f(t)$ as the original function and to $F(s)$ as the Laplace transform of the function $f(t)$. We also refer to $f(t)$ as the inverse Laplace transform of $F(s)$. The symbol Ł denotes the Laplace transformation. The function e^{-st} is called the kernel of the transformation. In our work, we think of s as a real variable. If the integral converges for all s greater than some s_0, then $F(s)$ is well defined and we say that the transform exists.

Now, let us look at some examples:

Example. Compute the Laplace transform of $f(t) = e^{2t}$.

$$\begin{aligned} \int_0^\infty e^{-st} f(t)dt &= \int_0^\infty e^{-st} e^{2t} dt = \int_0^\infty e^{-(s-2)t} dt \\ &= \lim_{b\to\infty} \int_0^b e^{-(s-2)t} dt = \lim_{b\to\infty} -\left[\frac{e^{-(s-2)t}}{s-2} \bigg|_0^b \right] \\ &= \lim_{b\to\infty} \left[\frac{1}{s-2} - \frac{e^{-(s-2)b}}{s-2} \right]. \end{aligned}$$

This limit exists only when $s > 2$. Hence,

$$\int_0^\infty e^{-st} f(t)dt = \frac{1}{s-2}, \qquad s > 2. \qquad \diamondsuit$$

Now, let us consider the integral

$$\hat{f}(x) = \frac{1}{x} \int_0^\infty e^{-t/x} f(t)dt.$$

This is just the Laplace transform in which the variable x of the generating function has been replaced by its reciprocal.

3 Expansion Theory

We present the main theorem for our work based on Laplace transformation. See [5] for a proof.

Theorem 3.1. If

1. $f(t)$ is bounded and continuous for $0 < t < \infty$,
2. $\hat{f}(x) = \frac{1}{x} \int_0^\infty e^{-t/x} f(t)dt$, and
3. $\hat{f}(x) = \sum_{n=0}^\infty a_n x^n, \qquad 0 < x < \rho,$

then we have

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \qquad 0 < x < \infty.$$

This theorem can serve as an alternate way to calculate the general term of a Taylor series expansion. Let us look at several examples.

Example. Consider $f(t) = \sin(t)$.

$$\begin{aligned}
\hat{f}(x) &= \frac{1}{x}\int_0^\infty e^{-t/x}\sin(t)dt \\
&= -\int_0^\infty \sin(t)de^{-t/x} \\
&= -\lim_{b\to\infty}\left[e^{-t/x}\sin(t)\Big|_0^b\right] + \int_0^\infty e^{-t/x}\cos(t)dt \\
&= -\lim_{b\to\infty}\left[xe^{-t/x}\cos(t)\Big|_0^b\right] - x\int_0^\infty e^{-t/x}\sin(t)dt \\
&= x - x^2\hat{f}(x),
\end{aligned}$$

so we have

$$\hat{f}(x) = \frac{x}{x^2+1} = x - x^3 + x^5 - \cdots, \qquad 0 < x < 1.$$

By Theorem 3.1,

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$

is the series expansion for $\sin(x)$, as predicted. ♢

4 Automatic Calculation

Based on the above discussion, we can implement a procedure in Maple called "coefficient" as following:

```
with(inttrans);
with(genfunc);
```

```
coefficient := proc (y, x)
   local tem1, tem2;
   tem1 :=laplace(y, x, s);
   tem1 := subs (s = 1/t, tem1);
   tem1 := tem1/t;
   tem2 := rgf_expand(tem1, t, n));
   tem2 := (simplify(tem2))/n!;
   return tem2
end
```

Example 1. Consider $f(t) = \sin(t)$. By applying the above "coefficient" procedure in Maple, we have an answer

$$[t^n]f(t) = \sin(n\pi/2)/n!.$$

Example 2. Consider $f(t) = \cos(at)\sin(bt)$, where a and b are nonzero real constants ($a \neq b$). Apply the "coefficient" procedure in Maple, we get an answer equivalent to

$$[t^n]f(t) = \frac{1}{4(a-b)n!}((I(a-b))^n + (-I(a-b))^n) - \frac{1}{4(a+b)n!}((I(a+b))^n + (-I(a+b))^n).$$

where I is the imaginary number such that $I^2 = -1$.

Example 3. Consider $f(t) = e^{Ix} - \cos(x) - I\sin(x)$. By applying the above "coefficient" procedure in Maple, we have an answer

$$[t^n]f(t) = 0.$$

Since $f(x)$ is analytic, and all its Taylor's expansion coefficients are 0, we proved the Euler's formula $e^{Ix} = \cos(x) + I\sin(x)$.

5 Conclusion

This method provides a way to calculate the general coefficients of Taylor's expansion. It works for all the functions such that their Laplace transforms are rational. There is a wide range of functions satisfying such a condition, including the examples above, e^x, $\sin^k(z)$, $\cos^k(z)$, etc, where k is an natural number.

More advanced algorithms were developed in [6]. The algorithms developed in [6] can be used to calculate the coefficients for a much wide range of functions beyond rational functions, and return exact solutions. This paper provides an alternative solution with simpler implementation.

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