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About a family of C^2 splines with one free generating function

Igor Verlan

Abstract

The problem of interpolation of discrete set of data on the interval $[a, b]$ representing the function f is investigated. A family of C^2 splines with one free generating function is introduced in order to solve this problem. Cubic C^2 splines belong to this family. The required conditions which must satisfy the generating function in order to obtain explicit interpolants are presented and examples of generating functions are given.

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1 Introduction

Let us assume that the mesh $\Delta : a = x_0 < x_1 < ... < x_n = b$ is given on the interval $[a, b]$ and $f_i = f(x_i)$, $i = 0(1)n$, are the corresponding data points. The problem of the construction of a interpolation function S, such that interpolation conditions $S(x_i) = f_i$, $i = 0(1)n$, are hold and $S \in C^2[a, b]$ is considered.

It is well known (e.g. [1]-[3]) that C^2 cubic splines as well as different types of generalized cubic splines [5] may be used to solve this problem. In present work the family of splines, which includes many well known types of splines, is introduced. This family allows to generate new types of splines.

2 A family of C^2 splines with one free generating function

Let us define splines on the interval $[x_i, x_{i+1}]$ as follows:

$$\begin{aligned} S(x) &= f_i + (f_{i+1} - f_i)t + \\ &+ \{h_i^2 M_i[2(\nu(0,p_i) - \nu(t,p_i)) + 2(\nu(1,p_i) - \nu(0,p_i))t - \\ &- \nu^{"}(1,p_i)t(1-t)] + h_i^2 M_{i+1}[2(\nu(t,p_i) - \nu(0,p_i)) - \\ &- 2(\nu(1,p_i) - \nu(0,p_i))t + \nu^{"}(0,p_i)t(1-t)]\}/ \\ &/ \ (2(\nu^{"}(1,p_i) - \nu^{"}(0,p_i))) \end{aligned} \tag{1}$$

where

$$t = (x - x_i)/h_i, h_i = x_{i+1} - x_i, S"(x_i) = M_i.$$

The function ν, which in the sequel will be called generating function for the spline (1), must satisfy the conditions

$$\nu(t,p) \in C^2[0,1], \nu^{"}(1,p) \neq \nu^{"}(0,p) \tag{2}$$

where p is a vector of free parameters.

From (1) the following formulas for the first and second derivatives of the spline are obtained:

$$\begin{aligned} S^{'}(x) &= (f_{i+1} - f_i)/h_i + \{h_i M_i(-2\nu^{'}(t,p_i) + 2(\nu(1,p_i) - \nu(0,p_i)) - \\ &- \nu^{"}(1,p_i)(1-2t)) + h_i M_{i+1}(2\nu^{'}(t,p_i) - \\ &- 2(\nu(1,p_i) - \nu(0,p_i)) + \nu^{"}(0,p_i) - 2t\nu^{"}(0,p_i))\}/ \\ &/ \ (2(\nu^{"}(1,p_i) - \nu^{"}(0,p_i)) \end{aligned} \tag{3}$$

$$S^{"}(x) = \frac{[M_i(\nu^{"}(1,p_i) - \nu^{"}(t,p_i)) + M_{i+1}(\nu^{"}(t,p_i) - \nu^{"}(0,p_i))]}{(\nu^{"}(1,p_i) - \nu^{"}(0,p_i))} \tag{4}$$

From (1) and (3) taking into account (2) it follows immediately that the spline and the second derivative are continuous. From the requirement of the continuity of the first derivative of the spline at the knots of mesh Δ we obtain the following system of linear algebraic equations:

$$c_iM_{i-1} + a_iM_i + b_iM_{i+1} = \delta_i^{(1)} - \delta_{i-1}^{(1)}, \quad i = 1(1)n-1, \tag{5}$$

where

$$c_i = \frac{h_{i-1}(-2\nu'(1,p_{i-1}) + 2\nu(1,p_{i-1}) - 2\nu(0,p_{i-1}) + \nu''(1,p_{i-1}))}{(2(\nu''(1,p_{i-1}) - \nu''(0,p_{i-1})))}$$

$$\begin{aligned} a_i &= \frac{h_{i-1}(2\nu'(1,p_{i-1}) - 2\nu(1,p_{i-1}) + 2\nu(0,p_{i-1}) - \nu''(0,p_{i-1}))}{(2(\nu''(1,p_{i-1}) - \nu''(0,p_{i-1})))} - \\ &- \frac{h_i(-2\nu'(0,p_i) + 2\nu(1,p_i) - 2\nu(0,p_i) - \nu''(1,p_i))}{(2(\nu''(1,p_i) - \nu''(0,p_i)))} \end{aligned}$$

$$b_i = \frac{h_i(-2\nu'(0,p_1) + 2\nu(1,p_i) - 2\nu(0,p_i) + \nu''(0,p_i))}{(2(\nu''(1,p_i) - \nu''(0,p_i)))}$$

$$\delta_i^{(1)} = (f_{i+1} - f_i)/h_i.$$

The system presented above is the undetermined one. Since (5) provides only $n-1$ linear equations in $n+1$ parameters M_i, it follows that two additional linearly independent conditions are always needed. In the present paper we shall consider that $f''(a) = f_0''$ and $f''(b) = f_n''$ are available, therefore $M_0 = f_0''$ and $M_n = f_n''$ is an obvious choice as end conditions.

It should be mentioned that in the case when $\nu(t,p) = t^3$ the splines (1) represent the well known C^2 cubic splines. As another examples of generating functions the following ones may be given: 1) $\nu(t,p) = t^n$, $n \geq 3$ and 2) $\nu(t,p) = t^3/(1+p(1-t))$, where p is a free parameter.

3 A subfamily of C^2 explicit splines with one free generating function

In many practical problems the scheme of explicit interpolation represents a special interest. From family (1) we can obtain explicit interpolation schemes. In order to get an explicit interpolant the generating function has to satisfy the following conditions:

$$-2\nu'(1,p) + 2\nu(1,p) - 2\nu(0,p) + \nu''(1,p) = 0 \qquad (6)$$

$$-2\nu'(0,p) + 2\nu(1,p) - 2\nu(0,p) + \nu''(0,p) = 0 \qquad (7)$$

It is easy to prove that conditions (6) and (7) are satisfied, for example, by the following functions:

$$\nu(t) = t^2 + 6t^3 - 10t^4 + 4t^5,$$

$$\nu(t) = t^2/(2t^2 - 2t + 1),$$

which were proposed in [4].

In this case $c_i = 0$ and $b_i = 0$ in the system of linear algebraic equations (5). As a result we get

$$M_i = (\delta_i^{(1)} - \delta_{i-1}^{(1)})/a_i, \quad i = 1(1)n-1,$$

and (1) becomes the explicit one.

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Igor Verlan,

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Department of Applied Mathematics, Moldova State University
60 Mateevich str.
Chişinău, MD2009, Moldova

Institute of Mathematics and Computer Science
5 Academiei str.
Chişinău MD2028, Moldova
E–mail: *iverlan@rambler.ru*