Computer Science Journal of Moldova, vol.**12**, no.3(36), 2004

# The maximum flow in dynamic networks

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**Abstract**

The dynamic maximum flow problem that generalizes the static maximum flow problem is formulated and studied. We consider the problem on a network with capacities depending on time, fixed transit times on the arcs, and a given time horizon. The corresponding algorithm to solve this problem is proposed and some details concerning its complexity are discussed.

**Mathematics Subject Classification 2000:** 90B10, 90C35, 90C27.

**Keywords and phrases:** dynamic networks, network flow, dynamic flows, flows over time, maximum flows.

## 1 Introduction

In this paper we study the dynamic version of the maximum flow problem on networks that generalizes the well-known static one [1, 2]. This basic combinatorial optimization problem has a large implementation for many practical problems. Its solution is needed in order to solve other more complex problems, like the minimum cost circulation problem and the parametric maximum flow problem. Our dynamic model is based on the classical maximum flow problem on static networks and some generalization from [3, 4, 5, 6, 7, 8].

Despite being closer to reality, dynamic flow models have not been investigated in such a detailed form as classical flow models because of the complexity of the dynamic network flow models in comparison with the static ones. Dynamic flows are widely used to model network-structured, decision-making problems over time: problems in electronic communication, production and distribution, economic planning, cash flow, job scheduling, and transportation (see, for example, [9]). In the considered dynamic models the flow passes an arc with time and it can be delayed at nodes. The flow values on arcs and the network parameters in this problem can change with time. While very efficient solution methods exist for static flow problems, dynamic flow problems have proved to be more difficult to solve.

The object of the maximum flow problem is to send a maximum amount of flow within a given time from supply nodes to demand nodes in such a way that link capacities are not exceeded. This problem has been studied extensively in the context of static networks. In this paper, we study the maximum flow problem in dynamic networks. We assume that capacities of edges depend on time. We propose an algorithm for finding the maximum dynamic flow, which is based on reducing the dynamic problem to the classical maximum flow problem on a time-expanded network. The complexity of this algorithm depends on the complexity of the algorithm used for the maximum static flow problem.

## 2 Problem formulation

We consider a directed network $N = (V, E, u, \tau, V^-, V^+)$ with set of vertices $V = V^- \cup V^+ \cup V^0$, where $V^-, V^+$ and $V^0$ are sources, sinks and intermediate nodes, respectively, and set of edges $E$. Without loosing generality, we assume that no edges enter sources or exit sinks. Our aim is to find a maximum flow over time in the network $N$ within makespan $\mathbb{T} = \{0, 1, 2, \ldots, T\}$ while respecting the following restrictions. Each edge $e \in E$ has a nonnegative time-varying capacity $u_e$ which bounds the amount of flow allowed on each arc in every moment of time. Moreover, edge $e$ has an associated positive transit time $\tau_e$ which determines the amount of time it takes for flow to travel from the tail to the head of that edge. The objective is to find a dynamic flow that sends in time $T$ as much flow as possible.

We start with the definition of static flows. A static flow $x$ on the static network $N = (V, E, u, V^-, V^+)$ assigns to every arc $e$ a non-negative flow value $x_e$ such that the following flow conservation constraints are obeyed:

$$\sum_{e \in E^+(v)} x_e - \sum_{e \in E^-(v)} x_e = \begin{cases} -y_v, \ v \in V^-, \\ 0, \ v \in V^0, \\ y_v, \ v \in V^+, \end{cases}$$

$$y_v \geq 0, \ \forall\, v \in V,$$

where $E^+(v) = \{(u,v) \,|\, (u,v) \in E\}, \quad E^-(v) = \{(v,u) \,|\, (v,u) \in E\}$.

A static flow $x$ is called feasible if it obeys the capacity constraints

$$0 \leq x_e \leq u_e, \ \forall\, e \in E.$$

A feasible dynamic flow on $N = (V, E, u, \tau, V^-, V^+)$ is a function $x$: $E \times \mathbb{T} \to R_+$ that satisfies the following conditions:

$$\sum_{\substack{e \in E^+(v) \\ t-\tau_e \geq 0}} x_e(t-\tau_e) - \sum_{e \in E^-(v)} x_e(t) = \begin{cases} -y_v(t), \ v \in V^-, \\ 0, \ v \in V^0, \\ y_v(t), \ v \in V^+, \end{cases} \quad \forall\, t \in \mathbb{T}, \, \forall\, v \in V; \tag{1}$$

$$y_v(t) \geq 0, \ \forall\, v \in V, \, t \in \mathbb{T};$$

$$0 \leq x_e(t) \leq u_e(t), \quad \forall\, t \in \mathbb{T}, \ \forall\, e \in E; \tag{2}$$

$$x_e(t) = 0, \ \forall\, e \in E, \ t = \overline{T - \tau_e + 1, T}. \tag{3}$$

Here the function $x$ defines the value $x_e(t)$ of flow entering edge $e$ at time $t$. It is easy to observe that the flow does not enter edge $e$ at time $t$ if it will have to leave the edge after time $T$; this is ensured by condition (3). Capacity constraints (2) mean that in a feasible dynamic flow, at most $u_e$ units of flow can enter the arc $e$ within each integral time step. Conditions (1) represent flow conservation constraints.

We consider the discrete time model, in which all times are integral and bounded by horizon $T$. Time is measured in discrete steps, so that if one unit of flow leaves node $u$ at time $t$ on arc $e = (u, v)$, one unit of flow arrives at node $v$ at time $t + \tau_e$, where $\tau_e$ is the transit time of arc $e$. The time horizon (finite or infinite) is the time until which the flow can travel in the network and defines the makespan $\mathbb{T} = \{0, 1, \ldots, T\}$ of time moments we consider.

The value of the flow $x$ is defined as follows

$$|x| = \sum_{t=0}^{T} \sum_{v \in V^+} y_v(t). \tag{4}$$

Our aim is to find a feasible flow that maximizes the objective function (4).

In the case of many sources and sinks the maximum flow problem can be reduced to the standard one by introducing one additional artificial source and one additional artificial sink and edges leading from the new source to true sources and from true sinks to the new sink. The transit times of these new edges are zero and the capacities of edges connecting the artificial source with all other sources are bounded by the capacities of these sources; the capacities of edges connecting all other sinks with the artificial sink are bounded by the capacities of these sinks.

It is easy to observe that if $\tau_e = 0, \forall e \in E$ and $T = 0$ then the formulated problem becomes the classical maximum flow problem on a static network.

## 3 Main results

In this paper we propose a new approach for solving the formulated problem, which is based on its reduction to a static maximum flow problem which is a well studied problem in operation research and other fields. We show that our problem on network $N = (V, E, u, \tau)$ can be reduced to a static problem on auxiliary static network $N^T = (V^T, E^T, u^T)$; we name it the time-expanded network. The advantage of this approach is that it turns the problem of determining an optimal flow over time into a classical static network flow problem on the time-expanded network.

The maximum dynamic flow problem can be solved by maximum flow computation on the corresponding time-expanded network, which is a static representation of the dynamic network. Such a time-expanded network contains a copy of the node set of the underlying network for each discrete interval of time, building a time layer. Moreover, for each arc with transit time $\tau_e$ in the given network, there is a copy between each pair of time layers of distance. We define this network as follows:

1. $V^T\colon = \{v(t) \,|\, v \in V,\ t \in \mathbb{T}\}$;
2. $E^T\colon = \{(v(t), w(t+\tau_e)) \,|\, e = (v,w) \in E,\ 0 \le t \le T - \tau_e\}$;
3. $u^T_{e(t)}\colon = u_e(t)$ for $e(t) \in E^T$;

The essence of the time-expanded network is that it contains a copy of the vertices of the dynamic network for each time $t \in \mathbb{T}$, and the transit times and flows are implicit in the edges linking those copies.

Let $e(t) = (v(t), w(t+\tau_e)) \in E^T$ and let $x_e(t)$ be a flow on the dynamic network $N$. The corresponding function $x^T_{e(t)}$ on the time-expanded network $N^T$ is defined as follows:

$$x^T_{e(t)} = x^T_{(v(t),w(t+\tau_e))} = x_e(t),\ \forall\, e(t) \in E^T. \tag{5}$$

We show in the following lemma that dynamic flows $x_e(t)$ with time horizon $T$ are equivalent to static flows $x^T_{e(t)}$ in the time-expanded network.

**Lemma 1.** *The correspondence (5) is a bijection from the set of feasible flows on the dynamic network $N$ onto the set of feasible flows on the time-expanded network $N^T$.*

**Proof.** It is obvious that the correspondence above is a bijection from the set of $T$-horizon functions on the dynamic network $N$ onto the set of functions on the time-expanded network $N^T$. It is also easy to observe that a feasible flow on the dynamic network $N$ is a feasible flow on the time-expanded network $N^T$ and vice-versa. Indeed,

$$0 \le x^T_{e(t)} = x_e(t) \le u_e(t) = u^T_{e(t)},\ \forall\, e \in E,\quad 0 \le t < T.$$

Therefore it is enough to show that each dynamic flow on the dynamic network $N$ is put into the correspondence with a static flow on the time-expanded network $N^T$ and vice-versa.

Henceforward we define $d_v(t) = \begin{cases} -y_v(t), \ v \in V^-, \\ 0, \ v \in V^0, \\ y_v(t), \ v \in V^+, \end{cases} \quad \forall v \in V.$

Let $x_e(t)$ be a dynamic flow on $N$ and let $x^T_{e(t)}$ be a corresponding function on $N^T$. Let's prove that $x^T_{e(t)}$ satisfies the conservation constraints on its static network. Let $v \in V$ be an arbitrary node in $N$ and $t$: $0 \leq t < T$ an arbitrary moment of time:

$$d_v(t) \overset{(i)}{=} \sum_{\substack{e \in E^+(v) \\ t-\tau_e \geq 0}} x_e(t-\tau_e) - \sum_{e \in E^-(v)} x_e(t) =$$

$$= \sum_{e(t-\tau_e) \in E^+(v(t))} x^T_{e(t-\tau(e))} - \sum_{e(t) \in E^-(v(t))} x^T_{e(t)} \overset{(ii)}{=} d^T_{v(t)}. \qquad (6)$$

Note that according to the definition of the time-expanded network the set of edges $\{e(t-\tau_e) | e(t-\tau_e) \in E^+(v(t))\}$ consists of all edges that enter $v(t)$, while the set of edges $\{e(t) | e(t) \in E^-(v(t))\}$ consists of all edges that originate from $v(t)$. Therefore, all necessary conditions are satisfied for each node $v(t) \in V^T$. Hence, $x^T_{e(t)}$ is a flow on the time-expanded network $N^T$.

Let $x^T_{e(t)}$ be a static flow on the time-expanded network $N^T$ and let $x_e(t)$ be a corresponding function on the dynamic network $N$. Let $v(t) \in V^T$ be an arbitrary node in $N^T$. The conservation constraints for this node in the static network are expressed by equality (ii) from (6), which holds for all $v(t) \in V^T$ at all times $t$: $0 \leq t < T$. Therefore, equality (i) holds for all $v \in V$ at all times $t$: $0 \leq t < T$ and $x_e(t)$ is a flow on the dynamic network $N$. $\square$

**Corollary 1.** *The following condition is true:*

$$\sum_{t \in \mathbb{T}} \sum_{v \in V^-} y_v(t) = \sum_{t \in \mathbb{T}} \sum_{v \in V^+} y_v(t).$$

**Proof.** The proof of this corollary can be obtained by examining three sets of vertices $V^-, V^+$ and $V^0$ and summing equations (1) in all moments of time. In such a way, we obtain the considered condition. □

If we define a flow correspondence to be $x^T_{e(t)}: = x_e(t)$, the maximum flow problem on dynamic networks can be solved by applying network flow optimization techniques for static flows directly to the expanded network. Thus to obtain the solution, we construct the time-expanded network, after that we solve the classical maximum flow problem on the static network and then we reconstruct the solution of the static problem to the dynamic problem. Therefore, we can solve the dynamic maximum flow problem by reducing it to the maximum flow problem on static networks.

## 4 Algorithm

Let the dynamic network $N$ be given. Our object is to solve the maximum flow problem on $N$. Proceedings are following:

1. Building the time-expanded network $N^T$ for the given dynamic network $N$.

2. Solving the classical maximum flow problem on the static network $N^T$, using one of the known algorithms (see, for example, [1, 10, 11, 12]).

3. Reconstructing the solution of the static problem on $N^T$ to the dynamic problem on $N$. □

Now let us examine the complexity of this algorithm including the time necessary to solve the resulting problem on the static time-expanded network. The process of building the time-expanded network and of reconstruction the solution of the static maximum flow problem to the dynamic one has the complexity $O(nT + mT)$, where $n$ is the number of nodes in the dynamic network and $m$ is the number of edges in this network. The complexity of step (2) depends on the complexity of the algorithm used for maximum flow problem in static networks.

If such an algorithm has complexity $O(f(n', m'))$, where $n'$ and $m'$ are number of nodes and edges in the network, then the complexity of solving the maximum flow problem on the time-expanded network employing the same algorithm is $O(f(nT, mT))$.

## 5 The dynamic problem on the given time interval

In the above sections we have discussed the problem of maximum dynamic flow from the zero time moment to the fixed time horizon $T$. In such problems we find the maximum amount of flow until the time $T$. In many practical cases it is necessary to know the maximum flow in the time period from $t_1$ to $t_2$, where $t_1 < t_2$. The solution of this problem is based on the Ford-Fulkerson theorem about the maximum flow and minimal cut [1]. We have to construct a time-expanded network, the discrete time moments of which form the following makespan $\mathbb{T} = \{t_1, t_1 + 1, \ldots, t_2 - 1, t_2\}$. In that way, constructing such a time-expanding network and finding the maximum flow in this network we can obtain the maximum flow in the dynamic network for the time period from $t_1$ to $t_2$.

## 6 Generalization

We would like to note that the same argumentation can be held to solve the maximum flow network problem on the dynamic networks in the case when, instead of the condition (2) in the definition of the feasible dynamic flow, the following condition takes place:

$$r_e^1(t) \leq x_e(t) \leq r_e^2(t), \quad \forall\, t \in \mathbb{T},\ \forall\, e \in E,$$

where $r_e^1(t)$ and $r_e^2(t)$ are lower and upper boundaries of the capacity of the edge $e$ correspondingly. In this case we introduce one additional artificial source $b_1$ and one additional artificial sink $b_2$. For every arc $e = (u, v)$, where $r_e^1 \neq 0$ we introduce arcs $(b_1, v)$ and $(u, b_2)$ with $r^1$ and 0 as the upper and lower boundaries of the capacity of the edges.

We reduce $r^2$ to $r^2 - r^1$, but $r^1$ to 0. We also introduce the arc $(b_2, b_1)$ with $r^2_{(b_2,b_1)} = \infty$ and $r^1_{(b_2,b_1)} = 0$. The transit times of all introduced arcs are zero. In such a mode we obtain a new network, on which we can solve the standard maximum flow problem.

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