Computer Science Journal of Moldova, vol.**12**, no.3(36), 2004

On stability in game problems of finding Nash set

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Abstract

A finite game of several players in the case of linear payoff functions is considered. Stability of the problem of finding the set of Nash equilibrium situations is investigated. The formula of the stability radius of this problem is derived.

AMS Mathematics Subject Classification: 90C27, 90C29.

Key words: non-cooperative finite game, Nash equilibrium situation, stability radius.

Consider the main subject of investigations in game theory – a finite non-cooperative game of some players in normal form [1], [2], when each player $i \in N_n = \{1, 2, \dots, n\}$, $n \geq 2$, has finite number of available strategies $X_i \subseteq \mathbf{R}$, $2 \leq |X_i| < \infty$. Game realization and its outcome (situation) is unambiguously determined by the strategy choice of each player. They make this choice independently from each other.

Let on the set of all situations $X = \prod_{i \in N_n} X_i$ the linear payoff functions of players

$$f_i(x) = C_i x, \qquad i \in N_n,$$

is defined. Here C_i is the i-th row of matrix $C = (c_{ij})_{n \times n} \in \mathbf{R}^{nn}$, $x = (x_1, x_2, \dots, x_n)^T$, $x_i \in X_i$, $i \in N_n$. In the result of the game which is called the game with matrix C, each player i gets the profit $f_i(x)$.

Consider the problem $Z^n(C)$ of finding the set of Nash equilibrium situations $NE^n(C)$ in the game with matrix C [3]:

$$NE^n(C) = \{x \in X : \ \forall\, i \in N_n \ \ (\pi(x, C_i) = \emptyset)\},$$

where

$$\pi(x, C_i) = \{x' \in W(x,i) : \ C_i x' > C_i x\},$$

$$W(x,i) = \prod_{j \in N_n} W_j(x,i),$$

$$W_j(x,i) = \begin{cases} X_j & \text{if } j = i, \\ \{x_j\} & \text{if } j \neq i. \end{cases}$$

Thus $W(x,i)$ is the set of situations reachable by player i from situation x.

As it is known, rationality of equilibrium situation lies in the fact, that any deviations of one player from this situation give him no profit.

It is easy to show (see, for example, [4]), that for any matrix $C \in \mathbf{R}^{nn}$ the set of Nash equilibrium situations $NE^n(C)$ is not empty.

By analogy with [5] — [8], under the stability radius of the problem $Z^n(C)$ we mean the number

$$\rho^n(C) = \begin{cases} \sup \Xi & \text{if } \Xi \neq \emptyset, \\ 0 & \text{if } \Xi = \emptyset, \end{cases}$$

where

$$\Xi = \{\varepsilon > 0 : \ \forall \ B \in \mathcal{B}(\varepsilon) \ \ (NE^n(C+B) \subseteq NE^n(C))\},$$

$$\mathcal{B}(\varepsilon) = \{B \in \mathbf{R}^{nn} : \ ||B||_\infty < \varepsilon\},$$

$$||B||_\infty = \max\{|b_{ij}| : \ (i,j) \in N_n \times N_n\}.$$

In other words, the stability radius is the maximum level of perturbations of elements of matrix C, which does not lead to appearance of new equilibrium situations. It is reasonable to define the stability radius to be equal to infinity, if $NE^n(C+B) \subseteq NE^n(C)$ for any matrix $B \in \mathbf{R}^{nn}$. Therefore we have

Property 1. *If $\rho^n(C) = \infty$, then $\overline{NE^n}(C) := X \setminus NE^n(C) = \emptyset$.*

Denote

$$I(x,C) = \{i \in N_n : \ \pi(x, C_i) \neq \emptyset\},$$

$$K(C) = \{i \in N_n : \ c_{ii} \neq 0\}.$$

Property 2. *For any index $k \in N_n$, such that $c_{kk} \neq 0$, there exists such situation $x^0 \in \overline{NE^n}(C)$, that $I(x^0, C) = \{k\}$.*

Since by virtue of the inequality $c_{kk} \neq 0$ for any situation $x \in NE^n(C)$ there exists situation $x^0 \in W(x,k)$, that $C_k(x^0 - x) < 0$, i. e. $\pi(x^0, C_k) \neq \emptyset$, then for any index $i \in N_n \setminus \{k\}$ and any situation $x' \in W(x^0, i)$ one easily derives

$$C_i(x' - x^0) = c_{ii}(x'_i - x^0_i) = c_{ii}(x'_i - x_i) = C_i(x' - x) \leq 0.$$

It follows that $\pi(x^0, C_i) = \emptyset$, $i \in N_n \setminus \{k\}$. Summing up, one obtains property 2.

Property 3. *Set $K(C)$ is empty if and only if set $\overline{NE^n}(C)$ is empty.*

Necessity of this statement follows directly from the definitions of sets $K(C)$ and $\pi(x, C_i)$, $i \in N_n$, and sufficiency can be easily shown, if we assume contrary and apply property 2.

Theorem. *For stability radius $\rho^n(C)$ of the problem $Z^n(C)$, $n \geq 2$, the following formula is valid*

$$\rho^n(C) = \begin{cases} \min\{|c_{ii}| : \ i \in K(C)\} & \text{if } K(C) \neq \emptyset, \\ \infty & \text{if } K(C) = \emptyset. \end{cases}$$

Proof. If $K(C) = \emptyset$, then by virtue of property 3 the equality $\overline{NE^n}(C) = \emptyset$ holds and, hence, according to property 1 we have $\rho^n(C) = \infty$.

Let further $K(C) \neq \emptyset$. Then

$$\psi := \min\{|c_{ii}| : \ i \in K(C)\} > 0$$

and by virtue of property 3 we obtain $\overline{NE^n}(C) \neq \emptyset$.

At first, let us prove inequality $\rho^n(C) \geq \psi$. For any situation $x \in \overline{NE^n}(C)$ there exist index $l \in N_n$ and situation $x' \in W(x, l)$ such that inequality

$$C_l(x' - x) > 0$$

holds. From this for any matrix $B \in \mathcal{B}(\psi)$ we easily derive

$$(C_l+B_l)(x'-x) = |C_l(x'-x)|+B_l(x'-x) = |c_{ll}||x'_l-x_l|+b_{ll}(x'_l-x_l) \geq$$

$$\geq |c_{ll}||x'_l - x_l| + |b_{ll}||x'_l - x_l| \geq (|c_{ll}| - ||B||_\infty)|x'_l - x_l| > 0,$$

i. e. $x' \in \pi(x, C_l + B_l)$. Therefore $x \in \overline{NE^n}(C + B)$. Thus, we obtain

$$\forall\ B \in \mathcal{B}(\psi) \qquad (NE^n(C + B) \subseteq NE^n(C)).$$

Hence, estimation $\rho^n(C) \geq \psi$ is valid.

Let us now prove inequality $\rho^n(C) \leq \psi$. We choose arbitrary number $\varepsilon > \psi$ and index $k \in N_n$, for which equality $c_{kk} = \psi$ holds. We define matrix $B^* = (b^*_{ij})_{n\times n} \in \mathcal{B}(\varepsilon)$ with elements

$$b^*_{ij} = \begin{cases} -c_{kk} & \text{if } i = j = k, \\ 0 & \text{otherwise.} \end{cases}$$

Property 2 ensures (by virtue of $c_{kk} \neq 0$) that there exists such situation $x^0 \in \overline{NE^n}(C)$, for which $I(x^0, C) = \{k\}$, i. e. for any index $i \in N_n \setminus \{k\}$ set $\pi(x^0, C_i)$ is empty. Therefore, taking into account the structure of matrix B^*, we have

$$\pi(x^0, C_i + B^*_i) = \pi(x^0, C_i) = \emptyset, \qquad i \in N_n \setminus \{k\},$$

$$\pi(x^0, C_k + B^*_k) = \pi(x^0, (c_{k1}, c_{k2}, \dots, c_{kk-1}, 0, c_{kk+1}, \dots, c_{kn})) = \emptyset.$$

It follows that $x^0 \in NE^n(C + B^*)$, i. e.

$$NE^n(C + B^*) \not\subseteq NE^n(C).$$

Further, since $B^* \in \mathcal{B}(\varepsilon)$ the inequality $\rho^n(C) < \varepsilon$ becomes obvious for any number $\varepsilon > \psi$. Hence, $\rho^n(C) \leq \psi$.

The theorem is proved.

The problem $Z^n(C)$ is called stable if $\rho^n(C) > 0$.

Corollary 1. *The problem $Z^n(C)$, $n \geq 2$, is stable for any matrix $C \in \mathbf{R}^{nn}$.*

Following [5], by rigidity radius of the problem $Z^n(C)$ we understand the number

$$\widetilde{\rho^n}(C) = \begin{cases} \sup \Omega & \text{if } \Omega \neq \emptyset, \\ 0 & \text{if } \Omega = \emptyset, \end{cases}$$

where

$$\Omega = \{\varepsilon > 0 : \ \forall\, B \in \mathcal{B}(\varepsilon) \ \ (NE^n(C+B) = NE^n(C))\}.$$

Taking into account the result of work [4], where the formula of stability radius of Nash equlibrium situation in the game with matrix C is obtained, we easily state

Corollary 2. *For rigidity raduis $\widetilde{\rho^n}(C)$ of the problem $Z^n(C)$, $n \geq 2$, the followng formula is valid*

$$\widetilde{\rho^n}(C) = \min\{|c_{ii}| : \ i \in N_n\}.$$

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