Computer Science Journal of Moldova, vol.**12**, no.3(36), 2004

On stability of an optimal situation in a finite cooperative game with a parametric concept of equilibrium (from lexicographic optimality to Nash equilibrium)*

Sergey E. Bukhtoyarov, Vladimir A. Emelichev

Abstract

A parametric concept of equilibrium (principle of optimality) in a finite cooperative game in normal form of several players is introduced. This concept is defined by means of the partition of players into coalitions. Lexicographically optimal situation and Nash equilibrium situation correspond to two special cases of this partition. The quantitative analysis of stability of an optimal situation for the independent perturbations of players' payoff functions is performed. The maximum level of such perturbations which save the optimality of a situation is found.

Mathematics Subject Classification 2000: 90C05, 90C10, 90C29, 90C31.

Key words and phrases: finite cooperative game, concept of equilibrium, lexicographically optimal situation, Nash equilibrium situation, stability radius.

1 Introduction

The aim of a game-theoretical model is to find classes of the solutions which are rational (coordinated) for the players (participants) and organizations (coalitions) included into this model. There are cooperative and non-cooperative principles of optimality in a normal form game.

*This work was supported by Fundamental Research Foundation of the Republic of Belarus "Mathematical structures 29" (Grant 913/28).

Usually these principles lead to different optimal situations. There is no unified approach to determining such concepts in the theory of non-antagonistic games. The most famous one is Nash equilibrium concept and its different generalizations. In this work a parametric concept of equilibrium in a normal form game is considered. Such concept leads to generally-optimal situations and its parameter is the way of the partition of players into coalitions. Lexicographically optimal situation and Nash equilibrium situation correspond to two special cases of such a partition (where there are one coalition of all players and the set of one-element coalitions). The analysis of stability of the situation optimal on a given partition for independent perturbations of players' payoff functions is performed. A formula of the stability radius is derived. Thus the maximum level of perturbations which save the optimality of a situation is found.

Observe that a formula of the stability radius of a generally-efficient situation in a normal form game was earlier derived in [1] for another parametric concept of equilibrium. In two special cases that concept lead to Pareto optimal situation and Nash equilibrium situation. Besides that, the questions of stability of vector discrete problems with parametric principles of optimality are discussed in [2–5].

2 Basic definitions, denotations and properties

Let's consider the main subject of investigations in game theory – a finite non-cooperative game in normal form of several players [6,7], where each player $i \in N_n = \{1, 2, \dots, n\},\ n \geq 2$, has the finite set of available strategies $X_i \subset \mathbf{R}, 2 \leq |X_i| < \infty$. Game realization and its outcome (situation) is unambiguously determined by the strategy choice of each player. Let on the game situation set which is the Cartesian product of the sets $X_j,\ j \in N_n$,

$$X = \prod_{j=1}^{n} X_j$$

the linear payoff functions of players

$$f_i(x) = C_i x, \quad i \in N_n,$$

are determined, where C_i is the i-th row of the matrix $C = [c_{ij}] \in \mathbf{R}^{n\times n}$, $x = (x_1, x_2, \dots, x_n)^T$, $x_j \in X_j$, $j \in N_n$. In the result of the game, which further will be called the game with a matrix C, each player i receives the profit $f_i(x)$, which it tries to maximize.

Now we introduce a concept of a generally-optimal situation.

Any nonempty subset $J \subseteq N_n$ is called a coalition of players. For any coalition J and any situation $x^0 = (x_1^0, x_2^0, \dots, x_n^0) \in X$ we determine the set

$$W(x^0, J) = \prod_{j=1}^{n} W_j(x^0, J),$$

where

$$W_j(x^0, J) = \begin{cases} X_j & \text{if } j \in J, \\ \{x_j^0\} & \text{if } j \in N_n \backslash J. \end{cases}$$

Thus $W(x^0, J)$ is the set of the situations reachable from a situation x^0 by the coalition J .

We define the binary relation of the lexicographic order " $\prec$ " in the space $\mathbf{R}^d$ of any dimension $d \in \mathbf{N}$, assuming that for any two vectors $y = (y_1, y_2, \dots, y_d)$ and $y' = (y'_1, y'_2, \dots, y'_d)$

$$y \prec y' \Leftrightarrow y_k < y'_k,$$

where $k = \min\{i \in N_d \ : \ y_i \neq y'_i\}$.

Then the following property is obvious.

Property 1 *Let $y, y' \in \mathbf{R}^d,\ d \in \mathbf{N}$. If $y_1 < y'_1$, then $y \prec y'$.*

Let $s \in N_n,\ N_n = \bigcup_{r\in N_s} J_r$ is a partition of set N_n into s coalitions, i. e. $J_r \neq \emptyset,\ r \in N_s;\ p \neq q \Rightarrow J_p \cap J_q = \emptyset$. For this partition we define the set $G^n(C, J_1, J_2, \dots, J_s)$ of generally-optimal, or $(J_1, J_2, \dots, J_s)$-optimal, situations in the game with a matrix C as follows

$$G^n(C, J_1, J_2, \dots, J_s) = \{x \in X \ : \ \forall\, r \in N_s \ (\zeta(x, C, J_r) = \emptyset)\},$$

where

$$\zeta(x, C, J_r) = \{x' \in W(x, J_r) \ : \ C_{J_r} x \prec C_{J_r} x'\},$$

C_{J_r} is the submatrix of the matrix C consisting of the rows numbered by J_r.

It is evident that any N_n-optimal situation $x \in G^n(C, N_n)$ (all the players are in one coalition) is lexicographically optimal. Thus all the players are ordered by importance in such way, that any previous player is more important then the next players. This corresponds to the general statement of the optimization problem with sequentially applied criterion [8–10].

Another special case is an $(\{1\}, \{2\}, \ldots, \{n\})$-optimal situation $x \in G^n(C, \{1\}, \{2\}, \ldots, \{n\})$ (individually-optimal), which is called Nash equilibrium situation [11] (see also [6, 7]). In this case the game is non-cooperative and the rationality of an equilibrium situation lies in the fact that any deviations of one player from this situation (while all the others keep to it) gives him no profit.

Thereby the introduction of such coalition characteristics of a situation which allows to generalize such classical concepts as the lexicographic optimality and Nash equilibrium herein is considered as the parametrization of the principle of optimality.

Without loss of generality we suppose that the partition $N_n = \bigcup\limits_{r \in N_s} J_r$ has the following form

$$J_1 = \{1, 2, \ldots, t_1\},\ J_2 = \{t_1 + 1, t_1 + 2, \ldots, t_2\}, \ldots,$$
$$J_s = \{t_{s-1} + 1, t_{s-1} + 2, \ldots, n\}.$$

Then taking into account the separation property of the linear payoff functions $C_i x,\ i \in N_n$, the following property can be directly obtained from the definition of $(J_1, J_2, \ldots, J_s)$-optimal situations set.

Property 2 *For any partition* $(J_1, J_2, \ldots, J_s)$

$$G^n(C, J_1, J_2, \ldots, J_s) = \prod_{r=1}^{s} L^{|J_r|}(C^r, X_{J_r})$$

holds, where each factor $L^{|J_r|}(C^r, X_{J_r})$ *is the set of the lexicographically optimal solutions of the* $|J_r|$*-criterion vector problem*

$$C^r x_{J_r} \to \operatorname{lex} \max_{x_{J_r} \in X_{J_r}},$$

i. e.

$$L^{|J_r|}(C^r, X_{J_r}) = \{x_{J_r} \in X_{J_r} \ : \ \lambda(x_{J_r}, C^r) = \emptyset\},$$

where

$$\lambda(x_{J_r}, C^r) = \{x'_{J_r} \in X_{J_r} \ : \ C^r x_{J_r} \prec C^r x'_{J_r} \ \},$$

C^r is the square matrix of the size $|J_r| \times |J_r|$ formed by the intersection of the rows and columns of a matrix C numbered by J_r; X_{J_r} is the projection of the set X onto J_r, i. e.

$$X_{J_r} = \prod_{j \in J_r} X_j \subset \mathbf{R}^{|J_r|}.$$

It is easy to see that x_{J_r} is the projection of a vector x onto the coordinate axis of the space $\mathbf{R}^n$ numbered by J_r.

It is a well known fact [8–10] that the set $L^{|J_r|}(C^r, X_{J_r})$ is a result of the solution of the sequence of scalar problems

$$L_i = \text{Arg max}\{C_i^r x_{J_r} \ : \ x_{J_r} \in L_{i-1}\}, \quad i \in N_{|J_r|}, \tag{1}$$

where $L_0 = X_{J_r}$; C_i^r is the i-th row of the matrix C^r. Therefore $L^{|J_r|}(C^r, X_{J_r}) = L_{|J_r|} \neq \emptyset$ for any index $r \in N_s$. Then from property 2 it follows that

Property 3 *For any matrix $C \in \mathbf{R}^{n \times n}$ and any partition of the player set into s coalitions $J_1, J_2, \dots, J_s$ the set of $(J_1, J_2, \dots, J_s)$-optimal situations $G^n(C, J_1, J_2, \dots, J_s)$ is non empty.*

Moreover, one can state obvious

Property 4 *A situation $x \notin G^n(C, J_1, J_2, \dots, J_s)$ if and only if there exists such an index $r \in N_s$, that $x_{J_r} \notin L^{|J_r|}(C^r, X_{J_r})$.*

We define the norm l_∞ for any natural number q in the q-dimensional space $\mathbf{R}^q$ as follows

$$||z||_\infty = \max\{|z_i| \ : \ i \in N_q\}, \ z = (z_1, z_2, \dots, z_q) \in \mathbf{R}^q.$$

Under the norm $||C||_\infty$ of a matrix $C = [c_{ij}] \in \mathbf{R}^{n\times n}$ we understand the norm of the vector $(c_{11}, c_{12}, \dots, c_{n\,n-1}, c_{nn}) \in \mathbf{R}^{nn}$. We also will use the norm l_1 :

$$||z||_1 = \sum_{i\in N_q} |z_i|.$$

For arbitrary number $\varepsilon > 0$ we define the set of perturbation matrices

$$\Omega(\varepsilon) = \{B \in \mathbf{R}^{n\times n} \ : \ ||B||_\infty < \varepsilon\}.$$

3 The formula of stability radius

By analogy with [1, 2, 5, 12] under the stability radius of a $(J_1, J_2, \dots, J_s)$-optimal situation $x \in G^n(C, J_1, J_2, \dots, J_s)$ we will understand the number

$$\rho^n(x, C, J_1, J_2, \dots, J_s) = \begin{cases} \sup\ \Xi & \text{if } \Xi \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\Xi = \{\varepsilon > 0 \ : \ \forall\, B \in \Omega(\varepsilon) \quad (x \in G^n(C + B, J_1, J_2, \dots, J_s))\}.$$

Such a situation always exists by virtue of property 3.

Theorem 1 *Let $C \in \mathbf{R}^{n\times n}$, $s \in N_n$, $n \geq 2$. For the stability radius $\rho^n(x, C, J_1, J_2, \dots, J_s)$ of a generally-optimal situation $x \in G^n(C, J_1, J_2, \dots, J_s)$ in the game with a matrix C the following formula holds*

$$\rho^n(x, C, J_1, J_2, \dots, J_s) = \min_{r\in N_s} \ \min_{x'_{J_r}\in X_{J_r}\setminus\{x_{J_r}\}} \frac{C_1^r(x_{J_r} - x'_{J_r})}{||x_{J_r} - x'_{J_r}||_1}. \qquad (2)$$

Proof. Let us denote the right part of formula (2) as φ and also let $\rho := \rho^n(x, C, J_1, J_2, \dots, J_s)$ for shortness of the subsequent text.

First we prove the inequality $\rho \geq \varphi$. If $\varphi = 0$, then this inequality is evident.

Let $\varphi > 0,\ B \in \Omega(\varphi)$. Directly from the definition of the number φ it follows that for any index $r \in N_s$ and any vector $x'_{J_r} \in X_{J_r} \setminus \{x_{J_r}\}$ the inequalities

$$||B^r||_\infty \ \leq \ ||B||_\infty \ < \ \varphi \ \leq \frac{C_1^r(x_{J_r} - x'_{J_r})}{||x_{J_r} - x'_{J_r}||_1}$$

take place. Then we derive

$$(C_1^r + B_1^r)(x_{J_r} - x'_{J_r}) = C_1^r(x_{J_r} - x'_{J_r}) + B_1^r(x_{J_r} - x'_{J_r}) \geq$$
$$\geq C_1^r(x_{J_r} - x'_{J_r}) - ||B_1^r||_\infty ||x_{J_r} - x'_{J_r}||_1 > 0.$$

Therefore, according to property 1, we get $(C^r+B^r)x'_{J_r} \prec (C^r+B^r)x_{J_r}$, i. e. $\lambda(x_{J_r}, C^r + B^r) = \emptyset$. Thus

$$\forall r \in N_s \quad \left(x_{J_r} \in L^{|J_r|}(C^r + B^r, X_{J_r})\right).$$

Taking into account property 4 we conclude

$$\forall B \in \Omega(\varphi) \quad (x \in G(C + B, J_1, J_2, \ldots, J_s)),$$

what proves the inequality $\rho \geq \varphi$.

Turn to the proof of the inequality $\rho \leq \varphi$. According to the definition of the number φ there exists such an index $k \in N_s$ and such a vector $x^0_{J_k} \in X_{J_k} \setminus \{x_{J_k}\}$, that

$$C_1^k(x_{J_k} - x^0_{J_k}) = \varphi ||x_{J_k} - x^0_{J_k}||_1. \tag{3}$$

For any number $\varepsilon > \varphi$ construct the perturbation matrix $\widehat{B} = \left[\widehat{b}_{ij}\right] \in \Omega(\varepsilon)$ with the elements

$$\widehat{b}_{ij} = \begin{cases} \alpha & \text{if } (i,j) \in \{t_{k-1}+1\} \times J_k,\ x_j \geq x^0_j, \\ -\alpha & \text{if } (i,j) \in \{t_{k-1}+1\} \times J_k,\ x_j < x^0_j, \\ 0 & \text{if } (i,j) \in N_n \times N_n \setminus \{t_{k-1}+1\} \times J_k, \end{cases}$$

where $\varepsilon > \alpha > \varphi$. Then in view of (3) we get

$$(C_1^k + \widehat{B}_1^k)(x_{J_k} - x^0_{J_k}) = C_1^k(x_{J_k} - x^0_{J_k}) + \widehat{B}_1^k(x_{J_k} - x^0_{J_k}) =$$

$= C_1^k(x_{J_k} - x^0_{J_k}) - \alpha||x_{J_k} - x^0_{J_k}||_1 = \varphi||x_{J_k} - x^0_{J_k}||_1 - \alpha||x_{J_k} - x^0_{J_k}||_1 < 0.$

From the proved inequality by virtue of property 1 we derive

$$(C_1^k + \widehat{B}_1^k)x_{J_k} \prec (C_1^k + \widehat{B}_1^k)x^0_{J_k},$$

i. e.

$$x_{J_k} \notin L^{|J_k|}(C^k + \widehat{B}^k, X_{J_k}).$$

In view of property 4 it follows that the formula

$$\forall \varepsilon > \varphi \quad \exists \widehat{B} \in \Omega(\varepsilon) \quad (x \notin G(C + \widehat{B}, J_1, J_2, \dots, J_s))$$

is valid and, hence, the inequality $\rho \leq \varphi$ holds.

Theorem is proved.

4 Corollaries

Let's introduce the set of strict situations for a given partition $(J_1, J_2, \dots, J_s)$:

$$S^n(C, J_1, J_2, \dots, J_s) = \{x \in X \ : \ \forall r \in N_s \quad \forall x'_{J_r} \in X_{J_r} \setminus \{x_{J_r}\} \\ (C_1^r x_{J_r} > C_1^r x'_{J_r})\}.$$

It is easy to see that only two cases are possible

$$S^n(C, J_1, J_2, \dots, J_s) = \emptyset \quad \text{or}$$

$$S^n(C, J_1, J_2, \dots, J_s) = G^n(C, J_1, J_2, \dots, J_s).$$

A situation $x \in G^n(C, J_1, J_2, \dots, J_s)$ is called stable if $\rho^n(C, J_1, J_2, \dots, J_s) > 0$.

Corollary 1 *A $(J_1, J_2, \dots, J_s)$-optimal situation is stable if and only if it is strict.*

Clearly, $G^n(C, J_1, J_2, \dots, J_s) = \{x^0\}$ if $x^0 \in S^n(C, J_1, J_2, \dots, J_s)$. Therefore the following corollary takes place.

Corollary 2 *If a $(J_1, J_2, \ldots, J_s)$-optimal situation x^0 is stable, then*

$$G^n(C, J_1, J_2, \ldots, J_s) = \{x^0\}.$$

Taking into account formulas (1) and (2) it is easy to see that the reverse statement is not valid in the general case, i. e. if a $(J_1, J_2, \ldots, J_s)$-optimal situation x^0 is a unique, then it can be either stable or unstable.

Corollary 2 implies

Corollary 3 *If $|G^n(C, J_1, J_2, \ldots, J_s)| > 1$, then no one situation $x \in G^n(C, J_1, J_2, \ldots, J_s)$ is stable.*

Corollary 4 *For the stability radius of a lexicographically optimal situation $x \in G^n(C, N_n)$ of the game with a matrix C the formula*

$$\rho^n(x, C, N_n) = \min_{x' \in X \setminus \{x\}} \frac{C_1(x - x')}{||x - x'||_1}$$

holds.

Corollary 5 *For the stability radius of a Nash equilibrium situation $x \in G^n(C, \{1\}, \{2\}, \ldots, \{n\})$ of the game with a matrix C the formula*

$$\rho^n(x, C, \{1\}, \{2\}, \ldots, \{n\}) = \min\{|c_{ii}| \ : \ i \in N_n\}$$

holds.

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Sergey E. Bukhtoyarov, Vladimir A. Emelichev, Received December 20, 2004

Belarusian State University
ave. Fr. Skoriny, 4,
Minsk, 220050, Belarus.
E–mail : *emelichev@bsu.by*