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On quasistability of a vector combinatorial problem with Σ-MINMAX and Σ-MINMIN partial criteria *

Vladimir A. Emelichev, Kirill G. Kuzmin, Andrey M. Leonovich

Abstract

We consider one type of stability (quasistability) of a vector combinatorial problem of finding the Pareto set. Under quasistability we understand a discrete analogue of lower semicontinuity by Hausdorff of the many-valued mapping, which defines the Pareto choice function. A vector problem on a system of subsets of a finite set (trajectorial problem) with non-linear partial criteria is in focus. Two necessary and sufficient conditions for stability of this problem are proved.

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1 Introduction

In [1] (see, also, [2 — 4]) it was shown that the coincidence of the Pareto set and the Smale set is the necessary and sufficient condition for quasistability of integer linear programming problems. For a vector combinatorial problem with non-linear partial criteria of the kind MINMAX and MINMIN, the mentioned condition is sufficient but is not necessary. The necessary and sufficient condition of quasistability for criteria MINSUM, MINMAX and MINMIN was provided in [6] (see corollary 2).

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The quasistability analysis was carried out for vector combinatorial problems with lexicographic principle of optimality [7, 8], majority principle of optimality [9] and Pareto principle of optimality (in the case of l_∞-extreme trajectorial problems) [10 — 12], for problems of minimization of linear form on a set permutations [13] and for vector problems of integer linear programming [4, 14].

In this work we focus on generalized partial criteria, which can be linear or bottleneck in partial case. Two necessary and sufficient conditions of quasistability are found. As corollaries, some earlier results from [6, 15] are proved (see below corollaries 1 and 2).

2 Definitions, notation and properties

Let us consider the following model of a vector (n-criterion) trajectorial problem. A set $E = \{e_1,\ e_2, ...,\ e_m\}$, $m \geq 2$, and a system of non-empty subsets (trajectories) $T \subseteq 2^E \backslash \{\emptyset\},\ \ |T| \geq 2$ are given. A vector criterion

$$f(t,\ A) = (f_1(t,\ A_1,\ k_1),\ f_2(t,\ A_2,\ k_2), ...,\ f_n(t,\ A_n,\ k_n)), \quad n \geq 1,$$

is defined on the set of trajectories T. Partial criteria are of the next two kinds:

$$\Sigma\text{-MINMAX}: \ f_i(t,\ A_i,\ k_i) = \max\{g_i(s,\ A_i): \\ s \in S(t,\ k_i)\} \to \min_{t \in T},\ i \in I_1, \tag{1}$$

$$\Sigma\text{-MINMIN}: \ f_i(t,\ A_i,\ k_i) = \min\{g_i(s,\ A_i): \\ s \in S(t,\ k_i)\} \to \min_{t \in T},\ i \in I_2, \tag{2}$$

where

$$g_i(s,\ A_i) = \sum_{j \in N(s)} a_{ij},$$

$$S(t,\ k) = \{s \subseteq t\ :\ |s| = \min\{|t|,\ k\}\},$$

$$N(s) = \{j \in N_m: \ \ e_j \in s\},$$

$$N_m = \{1,\ 2, ...,\ m\},$$

$$I_1 \cup I_2 = N_n, \quad I_1 \cap I_2 = \emptyset,$$

A_i is the $i-$th row of a matrix $A = [a_{ij}] \in \mathbf{R}^{n\times m}$, $k_1,\ k_2, ...,\ k_n$ are natural numbers. We assume that $g_i(\emptyset,\ A_i) = 0$, $S(t,\ 0) = \emptyset$.

For any non-empty set $p \subseteq E$ and number k we will use our notation $f_i(p,\ A_i,\ k)$, assuming that $f_i(p,\ A_i,\ 0) = 0$.

It is easy to see that if $k_i = \max\{|t| \ :\ t \in T\}$, then both (1) and (2) criteria are linear:

$$\text{MINSUM}: \qquad f_i(t,\ A_i,\ k_i) = \sum_{j\in N(t)} a_{ij} \longrightarrow \min_{t\in T}.$$

If $k_i = 1$, then (1) is a bottleneck criterion

$$\text{MINMAX}: \qquad f_i(t,\ A_i,\ k_i) = \max_{j\in N(t)} a_{ij} \longrightarrow \min_{t\in T},$$

and (2) is criterion

$$\text{MINMIN}: \qquad f_i(t,\ A_i,\ k_i) = \min_{j\in N(t)} a_{ij} \longrightarrow \min_{t\in T}.$$

We will denote by I_{SUM} the set of indices from N_n, which correspond to partial criteria of the kind MINSUM.

Under the vector (n-criteria) problem $Z^n(A)$, $n \geq 1$, we understand the problem of finding the Pareto set $P^n(A)$, also mentioned as the set of efficient trajectories:

$$P^n(A) = \{\ t \in T :\ P^n(t,\ A) = \emptyset\ \},$$

where

$$P^n(t,\ A) = \{\ t' \in T :\quad f(t,\ A) \geq f(t',\ A),\quad f(t,\ A) \neq f(t',\ A)\ \}.$$

It is obvious, that $P^1(A)$ (where A is an m-dimensional vector) is the set of all optimal solutions of the scalar trajectorial problem $Z^1(A)$. The most of well-known problems of graph theory, boolean programming problems and many problems of the scheduling theory [2, 14 — 17] are instances of trajectorial problems.

Observe, that $\sum$-minimax and $\sum$-minimin criteria are used in formulation of appointment problems [17].

We will perturb the parameters of the vector criterion $f(t, A)$ by the addition of the matrix $A \in \mathbf{R}^{nm}$ with matrices from the set

$$\mathcal{B}(\varepsilon) = \{B \in \mathbf{R}^{n\times m} : \ ||B|| < \varepsilon\},$$

where $\varepsilon > 0$, $||\ .\ ||$ is Chebyshev norm in the space $\mathbf{R}^{n\times m}$, i. e.

$$||B|| = \max\{\ |b_{ij}| : \ (i,\ j) \in N_n \times N_m\ \}, \quad B = [b_{ij}] \in \mathbf{R}^{n\times m}.$$

Problem $Z^n(A+B)$ is called perturbed, matrix B is called perturbing.

As usual [3 — 6, 10 — 12], under quasistability of the problem $Z^n(A)$ we will understand a discrete analogue of lower semicontinuity (by Hauzdorf) of the many-valued maping, that assigns the Pareto set $P^n(A)$ to any matrix $A \in \mathbf{R}^{nm}$. Thus, quasistablility is the property of preservation of all the efficient trajectories under "small" independent perturbations of the matrix A elements. By that, the problem $Z^n(A)$ is quasistable if and only if

$$\exists\ \varepsilon > 0 \quad \forall\ B \in \mathcal{B}(\varepsilon) \quad (P^n(A) \subseteq P^n(A+B)).$$

This implies the next two properties.

Property 1. *The problem $Z^n(A)$ is not quasistable if*

$$\exists\ t \in P^n(A) \qquad \forall\ \varepsilon > 0 \qquad \exists\ B \in \mathcal{B}(\varepsilon) \qquad (t \notin P^n(A+B)).$$

Property 2. *The problem $Z^n(A)$ is quasistable if*

$$\forall\ t \in P^n(A) \qquad \exists\ \varepsilon > 0 \qquad \forall\ B \in \mathcal{B}(\varepsilon) \qquad (t \in P^n(A+B)).$$

It was shown in [14] that the coincidence of the sets of efficient and strictly efficient trajectories is necessary and sufficient for quasistability of the problem $Z^n(A)$ only if at least one partial criterion is linear (see property 1). In this work we derive two necessary and at the same time sufficient conditions of quasistability of the problem $Z^n(A)$ with arbitrary combination of criteria (1) and (2).

For any $i \in N_n$, $t \in T$ and $t' \in T\backslash\{t\}$, when $t \cap t' \neq \emptyset$, put

$$\xi_i(t,\ t',\ A_i) = \begin{cases} f_i(t \cap t', A_i, k_i) - f_i(t \cap t', A_i, k_i - 1) - \\ \qquad -f_i(\Delta(t,t'), A_i, 1) & \text{if } i \in I_1, \\ -(f_i(t \cap t', A_i, k_i) - f_i(t \cap t', A_i, k_i - 1) - \\ \qquad -f_i(\Delta(t,t'), A_i, 1)) & \text{if } i \in I_2, \end{cases}$$

where

$$\Delta(t,\ t') = (t \cup t') \setminus (t \cap t').$$

Evidently, that the inequality $t \neq t'$ implies $\Delta(t,\ t') \neq \emptyset$.

For a non-empty set $p \subseteq E$, an index $i \in N_n$ and a number $k \in N_m$ we denote

$$U_i(p,\ A_i,\ k) = \begin{cases} \operatorname{Arg\,max}\{g_i(s,\ A_i)\ :\ s \in S(p,\ k)\} & \text{if } i \in I_1, \\ \operatorname{Arg\,min}\{g_i(s,\ A_i)\ :\ s \in S(p,\ k)\} & \text{if } i \in I_2. \end{cases}$$

It is natural to assume $U_i(p,\ A_i,\ 0) = \emptyset$.

The next properties are evident.

Property 3. *Let $i \in I_{SUM}$. The equality $U_i(t,\ A_i,\ k_i) = U_i(t',\ A_i,\ k_i)$ is valid if and only if the trajectories t and t' are equal.*

Property 4. *If $s,\ s' \subseteq E$, $s \cap s' = \emptyset$, then for any index $i \in N_n$ the equality*

$$g_i(s,\ A_i) + g_i(s',\ A_i) = g_i(s \cup s',\ A_i)$$

is true.

Property 5. *If for some index $i \in N_n$ and some trajectories t and t' the equalities*

$$U_i(t,\ A_i,\ k_i) = U_i(t',\ A_i,\ k_i) = U$$

hold, then

$$t \cap t' \neq \emptyset,$$

$$U_i(t \cup t',\ A_i,\ k_i) = U_i(t \cap t',\ A_i,\ k_i) = U,$$

$$f_i(t \cup t',\ A_i,\ k_i) = f_i(t,\ A_i,\ k_i) = f_i(t',\ A_i,\ k_i) = f_i(t \cap t',\ A_i,\ k_i).$$

Property 6. *Let $\beta > 0$, $i \in N_n$, $s \in U_i(t,\ A_i,\ k_i)$, $t \in T$, $q \in N(s)$, b be an m-dimensional perturbing vector with coordinates*

$$b_j = \begin{cases} \beta, & \text{where } j = q, \\ 0, & \text{where } j \in N_m \setminus \{q\}. \end{cases}$$

Then

$$f_i(t,\ A_i + b,\ k_i) = f_i(t,\ A_i,\ k_i) + \beta, \qquad \text{where } i \in I_1,$$

$$f_i(t,\ A_i - b,\ k_i) = f_i(t,\ A_i,\ k_i) - \beta, \qquad \text{where } i \in I_2.$$

Property 7. *Let $i \in N_n$, $s \subset p \subseteq E$, $|s| = k_i$. The set s belongs to $U_i(p,\ A_i,\ k_i)$ if and only if*

$$\forall j \in N(s) \ \ \forall l \in N(p \backslash s) \ \ (a_{ij} \geq a_{il}), \quad \text{where } i \in I_1,$$

$$\forall j \in N(s) \ \ \forall l \in N(p \backslash s) \ \ (a_{ij} \leq a_{il}), \quad \text{where } i \in I_2.$$

From property 7 we easily derive

Property 8. *Let $i \in N_n$, $s \subset p \subseteq E$, $|s| = k_i$. The set s belongs to $U_i(p,\ A_i,\ k_i)$ if and only if*

$$\min\{a_{ij} :\ j \in N(s)\} \geq \max\{a_{ij} :\ j \in N(p \backslash s)\}, \quad \text{where } i \in I_1,$$

$$\max\{a_{ij} :\ j \in N(s)\} \leq \min\{a_{ij} :\ j \in N(p \backslash s)\}, \quad \text{where } i \in I_2.$$

Property 9. *For any index $i \in N_n$, the inclusion $s \in U_i(p,\ A_i,\ k_i)$, where $|s| = k_i$, implies*

$$f_i(p,\ A_i,\ k_i - 1) = f_i(s,\ A_i,\ k_i - 1).$$

Proof. Obviously, it is sufficient to prove this property assuming that $s \neq p$ and $k_i > 1$.

Case 1: $i \in I_1$. Let

$$\widehat{s} \in U_i(s,\ A_i,\ k_i - 1).$$

Then $|\widehat{s}| = k_i - 1$. Thus $\widehat{s} \subset s \subset p$. Therefore, using property 8 twice, we derive

$$\min_{j \in N(\widehat{s})} a_{ij} \geq \max_{j \in N(s \backslash \widehat{s})} a_{ij} \geq \min_{j \in N(s)} a_{ij} \geq \max_{j \in N(p \backslash s)} a_{ij} \geq \min_{j \in N(p \backslash s)} a_{ij} \geq$$

$$\geq \min_{j \in N(p \backslash \widehat{s})} a_{ij}.$$

Hence, applying property 8 again, we have

$$\widehat{s} \in U_i(p,\ A_i,\ k_i - 1).$$

This relation with the condition $\widehat{s} \in U_i(s,\ A_i,\ k_i - 1)$ imply

$$f_i(p,\ A_i,\ k_i - 1) = g_i(\widehat{s},\ A_i) = f_i(s,\ A_i,\ k_i - 1).$$

This equality was required.

Case 2: $i \in I_2$. The proof is analogous to case 1. □

Remark 1. *It is easy to see, that the index i, appearing in properties 7 and 8, can not belong to the set I_{SUM}.*

3 Lemmas

Lemma 1. *For any index $i \in N_n$ and any trajectories $t \neq t'$, if*

$$U_i(t,\ A_i,\ k_i) \neq U_i(t',\ A_i,\ k_i), \tag{3}$$

where $k_i \in N_m$, then there exists a set $s \in U_i(t,\ A_i,\ k_i) \cup U_i(t',\ A_i,\ k_i)$, such that $s \not\subseteq t \cap t'$.

Proof. Suppose the opposite. Let

$$\forall s \in U_i(t,\ A_i,\ k_i) \cup U_i(t',\ A_i,\ k_i) \ \ (s \subseteq t \cap t'). \tag{4}$$

Then for any set $s \in U_i(t,\ A_i,\ k_i)$ the relations

$$|t \cap t'| \geq |s| = \min\{|t|,\ k_i\}$$

are true. This implies

$$\min\{|t\cap t'|,\ k_i\} = \min\{|t|,\ k_i\}.$$

Therefore we get

$$S(t\cap t',\ k_i) \subseteq S(t,\ k_i).$$

Consequently,

$$U_i(t\cap t',\ A_i,\ k_i) \subseteq U_i(t,\ A_i,\ k_i).$$

From this, taking into account

$$U_i(t,\ A_i,\ k_i) \subseteq U_i(t\cap t',\ A_i,\ k_i),$$

by virtue of (4), we have

$$U_i(t\cap t',\ A_i,\ k_i) = U_i(t,\ A_i,\ k_i).$$

Analogously, as $s \subseteq t\cap t'$ is true for any $s \in U_i(t',\ A_i,\ k_i)$, we easily derive

$$U_i(t\cap t',\ A_i,\ k_i) = U_i(t',\ A_i,\ k_i).$$

Finally, we have the equality

$$U_i(t,\ A_i,\ k_i) = U_i(t',\ A_i,\ k_i)$$

that contradicts with condition (3). □

Lemma 2. *For some index $i \in N_n$ and some trajectories $t \neq t'$, if*

$$U_i(t,\ A_i,\ k_i) = U_i(t',\ A_i,\ k_i), \qquad (5)$$

where $k_i \in N_m$, then the inequalities

$$|t\cap t'| \geq k_i, \qquad (6)$$

$$\xi_i(t,\ t',\ A_i) > 0 \qquad (7)$$

are true.

Proof. Let us prove inequality (6) at first. Suppose (on the contrary) that the inequality $|t \cap t'| < k_i$ is true. Then, by virtue of property 5, the relations

$$|s| = |t \cap t'| < k_i$$

hold for any set $s \in U_i(t \cap t', A_i, k_i)$. Therefore, we have

$$\min\{|t'|, k_i\} = \min\{|t|, k_i\} = |s| < k_i.$$

Consequently,

$$|t \cap t'| = |s| = |t| = |t'|,$$

i. e. $t = t'$. This contradiction proves inequality (6).

Now let us prove inequality (7). Suppose the opposite, i.e. that the inequality

$$\xi_i(t, t', A_i) \le 0 \tag{8}$$

holds. Let

$$s \in U_i(t \cap t', A_i, k_i - 1), \; s' \in U_i(\Delta(t, t'), A_i, 1). \tag{9}$$

Without loss of generality, let us put

$$s' \subseteq t \backslash t'.$$

Then, taking into account (9), we have

$$s \cap s' = \emptyset, \tag{10}$$

$$s \cup s' \subseteq t, \tag{11}$$

$$s \cup s' \not\subseteq t'. \tag{12}$$

From (6), (9) and (10), by virtue of property 4 and the definition of $\xi_i(t, t', A_i)$, we derive

$$\xi_i(t, t', A_i) = \begin{cases} f_i(t \cap t', A_i, k_i) - g_i(s \cup s', A_i) & \text{if} \quad i \in I_1, \\ g_i(s \cup s', A_i) - f_i(t \cap t', A_i, k_i) & \text{if} \quad i \in I_2. \end{cases}$$

Hence, by property 5 (on account of (5)) and inequality (8), we conclude

$$f_i(t,\ A_i,\ k_i) \le g_i(s \cup s',\ A_i) \quad \text{if} \quad i \in I_1, \tag{13}$$

$$f_i(t,\ A_i,\ k_i) \ge g_i(s \cup s',\ A_i) \quad \text{if} \quad i \in I_2. \tag{14}$$

Further, relations (6) and (9) (on account of $|\Delta(t,\ t')| \ge 1$) imply

$$|s| = k_i - 1 \qquad \text{and} \qquad |s'| = 1,$$

Then, by virtue of (10), we derive

$$|s \cup s'| = k_i.$$

Using (11), we conclude that $s \cup s' \in S(t,\ k_i)$. On account of (13) and (14), this implies

$$s \cup s' \in U_i(t,\ A_i,\ k_i).$$

By (12), we have

$$U_i(t,\ A_i,\ k_i) \ne U_i(t',\ A_i,\ k_i).$$

This is a contradiction with (5). □

Lemma 3. *Suppose* $i \in N_n$, t, $t' \in T$, $t \ne t'$,

$$|t \cap t'| \ge k_i \in N_m,$$

$$s \in U_i(t \cap t',\ A_i,\ k_i).$$

Then

$$\xi_i(t,\ t',\ A_i) =$$

$$= \begin{cases} \min\{a_{ij} : j \in N(s)\} - \max\{a_{ij} : j \in N(\Delta(t,t'))\} & \text{if } i \in I_1, \\ \min\{a_{ij} : j \in N(\Delta(t,t'))\} - \max\{a_{ij} : j \in N(s)\} & \text{if } i \in I_2. \end{cases} \tag{15}$$

Proof. By virtue of property 9 and the definition of the set $U_i(t \cap t',\ A_i,\ k_i)$, we have

$$f_i(t\cap t',\ A_i,\ k_i) - f_i(t\cap t',\ A_i,\ k_i - 1) = g_i(s,\ A_i) - f_i(s,\ A_i,\ k_i - 1). \tag{16}$$

Case 1: $i \in I_1$. Then, on account of (16), we derive

$$f_i(t \cap t',\ A_i,\ k_i) - f_i(t \cap t',\ A_i,\ k_i - 1) = g_i(s,\ A_i) - \max_{j \in N(s)} (g_i(s,\ A_i) - a_{ij}) =$$

$$= \min_{j \in N(s)} a_{ij}.$$

Hence, by the definition of $\xi_i(t,\ t',\ A_i)$ (when $i \in I_1$) and the evident equality

$$f_i(\Delta(t,\ t'),\ A_i,\ 1) = \max\{a_{ij} :\ j \in N(\Delta(t,\ t'))\},$$

we get inequality (15) for $i \in I_1$.

Case 2: $i \in I_2$. Then, by virtue of (16), we have

$$f_i(t \cap t',\ A_i,\ k_i) - f_i(t \cap t',\ A_i,\ k_i - 1) = g_i(s,\ A_i) - \min_{j \in N(s)} (g_i(s,\ A_i) - a_{ij}) =$$

$$= \max_{j \in N(s)} a_{ij}.$$

Hence, by the definition of $\xi_i(t,\ t',\ A_i)$ (when $i \in I_2$) and the evident inequality

$$f_i(\Delta(t,\ t'),\ A_i,\ 1) = \min\{a_{ij} :\ j \in N(\Delta(t,\ t'))\},$$

we get equality (15) for $i \in I_2$. $\square$

Lemma 4. *Suppose* $i \in N_n,\ t,\ t' \in T,\ t \neq t'$,

$$|t \cap t'| \geq k_i \in N_m, \tag{17}$$

$$\xi_i(t,\ t',\ A_i) > 0. \tag{18}$$

Then the equality

$$f_i(t,\ A_i,\ k_i) = f_i(t',\ A_i,\ k_i)$$

is true.

Proof. Let $s \in U_i(t \cap t',\ A_i,\ k_i)$. Then, on account of (17), we have

$$|s| = k_i. \tag{19}$$

To prove the lemma, it is sufficient to prove the equalities

$$f_i(t,\ A_i,\ k_i) = g(s,\ A_i) = f_i(t',\ A_i,\ k_i).$$

In order to prove the above mentioned equalities, it is sufficient to show that the equality

$$f_i(t,\ A_i,\ k_i) = g(s,\ A_i) \tag{20}$$

is valid (trajectories t and t' appear in conditions of lemma 4 symmetrically).

On account of (19), the equality (20) is equivalent to the inclusion

$$s \in U_i(t,\ A_i,\ k_i).$$

We will suppose that s is a proper subset of the trajectory t. If not ($s = t$, i.e. $|t| = k_i$), then equality (20) is evident.

Case 1: $i \in I_1$. Then, on account of (18) and lemma 3, we have

$$\min\{a_{ij}:\ j \in N(s)\} \geq \max\{a_{ij}:\ j \in N(\Delta(t,\ t'))\} \geq$$
$$\geq \max\{a_{ij}:\ j \in N(t \backslash t')\}. \tag{21}$$

The definition of the set s implies $s \subseteq t \cap t'$. If $s \subset t \cap t'$, then, by property 8 (taking into account (19)), we derive

$$\min\{a_{ij}:\ j \in N(s)\} \geq \max\{a_{ij}:\ j \in N((t \cap t') \backslash s)\}.$$

From here, by (21), we get

$$\min\{a_{ij}:\ j \in N(s)\} \geq \max\{a_{ij}:\ j \in N(t \backslash s)\}. \tag{22}$$

If $s = t \cap t'$, then the inequality (22) is true again on account of (21).

By virtue of (19), using the inequality (22) and property 8, we conclude that $s \in U_i(t,\ A_i,\ k_i)$.

Case 2: $i \in I_2$. By (18) and lemma 3, we derive

$$\max\{a_{ij} : j \in N(s)\} \leq \min\{a_{ij} : j \in N(\Delta(t, t'))\} \leq \min\{a_{ij} : j \in N(t \backslash t')\}. \quad (23)$$

As it has been shown, $s \subseteq t \cap t'$.

If $s \subset t \cap t'$, then, by virtue of property 8, we have

$$\max\{a_{ij} : j \in N(s)\} \leq \min\{a_{ij} : j \in N((t \cap t') \backslash s)\}.$$

On account of (23), this implies the inequality

$$\max\{a_{ij} : j \in N(s)\} \leq \min\{a_{ij} : j \in N(t \backslash s)\}. \quad (24)$$

If $s = t \cap t'$, then inequality (24) is true again by virtue of (23).

Taking into account (19) and using inequality (24) and property 8, we conclude $s \in U_i(t, A_i, k_i)$. $\square$

Remark 2. *Evidently, the index i, that appears in lemmas 2 – 4, can not belong to the set I_{SUM}.*

4 Theorems

Let us assign to an arbitrary trajectory $t \in P^n(A)$ the set of equivalent efficient trajectories

$$Q^n(t, A) = \{ t' \in T \setminus \{t\} : f(t', A) = f(t, A)\}.$$

Theorem 1. *For a vector trajectorial problem $Z^n(A)$, $n \geq 1$, with any combination of partial criteria* (1) *and* (2), *the next statements are equivalent:*

$\mathbf{1}^0$ *the problem $Z^n(A)$ is quasistable,*

$\mathbf{2}^0$ $\forall t \in P^n(A)$
$(Q^n(t, A) \neq \emptyset \Rightarrow \forall t' \in Q^n(t, A) \ \forall i \in N_n \ (U_i(t, A_i, k_i) = U_i(t', A_i, k_i))$,

$\mathbf{3}^0$ $\forall t \in P^n(A)$
$(Q^n(t, A) \neq \emptyset \Rightarrow \forall t' \in Q^n(t, A) \ \forall i \in N_n \ (|t \cap t'| \geq k_i \ \& \ \xi_i(t, t', A_i) > 0))$.

Proof. $\mathbf{1}^0 \Rightarrow 2^0$. Let the problem $Z^n(A)$ be quasistable. Suppose, contrary to the statement $\mathbf{2}^0$, that there exist trajectories $t \in P^n(A)$ and $t' \in Q^n(t, A)$ such that the sets $U_q(t,\ A_q,\ k_q)$ and $U_q(t',\ A_q,\ k_q)$ do not coincide for some number $q \in N_n$. Then lemma 1 implies that a set $s \in U_q(t,\ A_q,\ k_q) \cup U_q(t',\ A_q,\ k_q)$, $s \not\subseteq t \cap t'$ can be found. Therefore, there exists an index $l \in N(s)$, such that $l \in N(t \setminus t')$ or $l \in N(t' \setminus t)$. Let us consider these two cases.

Case 1: $l \in N(t \backslash t')$. Let us construct a perturbing matrix $B = [b_{ij}]_{n\times m}$ depending on the membership of the index q in the set I_1 or I_2. Put $0 < \beta < \varepsilon$.

If $q \in I_1$, then the elements of the perturbing matrix B are defined by

$$b_{ij} = \begin{cases} \beta & \text{if } \ i = q,\ j = l, \\ 0 & \text{otherwise.} \end{cases}$$

Then, taking into account property 8 and equivalence of the trajectories t and t', we have

$$f_i(t,\ A_i + B_i,\ k_i) - f_i(t',\ A_i + B_i,\ k_i) = \begin{cases} \beta & \text{if } \ i = q, \\ 0 & \text{if } \ i \in N_n \setminus \{q\}. \end{cases}$$

Thus $P^n(t,\ A + B) \neq \emptyset$. As a result,

$$\exists\ t \in P^n(A) \ \ \forall\ \varepsilon > 0 \ \ \exists\ B \in \mathcal{B}(\varepsilon) \ \ (t \notin P^n(A + B)).$$

Consequently, by virtue of property 2, the problem $Z^n(A)$ is not quasistable. This contradicts to $\mathbf{1}^0$.

If $q \in I_2$, then we construct the perturbing matrix B accordingly to the formula

$$b_{ij} = \begin{cases} -\beta & \text{if } i = q,\ j = l, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, taking into account property 8 and equivalence of the trajectories t and t' we see that the equalities

$$f_i(t,\ A_i + B_i,\ k_i) - f_i(t',\ A_i + B_i,\ k_i) = \begin{cases} -\beta & \text{if } \ i = q, \\ 0 & \text{if } \ i \in N_n \setminus \{q\} \end{cases}$$

are valid. Therefore, we have $P^n(t',\ A+B) \neq \emptyset$, i. e. t' is not an efficient trajectory in the perturbed problem $Z^n(A+B)$. Hence, as $t' \in P^n(A)$, on account of property 1, we conclude that the problem $Z^n(A)$ is not quasistable. Consequently, we have a contradiction with $\mathbf{1}^0$ again.

Case 2: $l \in N(t'\backslash t)$. Since $t' \in P^n(A)$ and $t \in Q^n(t',\ A)$, the proof in this case is analogous to case 1 (it is sufficient to interchange t and t').

$\mathbf{2}^0 \Rightarrow \mathbf{3}^0$. This implication follows from lemma 2.

$\mathbf{3}^0 \Rightarrow \mathbf{1}^0$. Suppose that $t \in P^n(A)$. Let us show that, when $\mathbf{3}^0$ is true, no one trajectory $t' \neq t$ can belong to the set $P^n(t,\ A)$ under "small" perturbations of the matrix A.

Case 1: $t' \in T \setminus Q^n(t,\ A)$. Then $f(t,\ A) \neq f(t',\ A)$. This implies that there is an index $r \in N_n$, such that inequality $f_r(t',\ A_r,\ k_r) > f_r(t,\ A_r,\ k_r)$ is true. Therefore, by virtue of continuity in $\mathbf{R}^m$ of the function $f_i(t,\ A_i,\ k_i)$, there exists a number $\varepsilon = \varepsilon(t') > 0$, such that for any matrix $B \in \Omega(\varepsilon)$ the inequality $f_r(t',\ A_r + B_r,\ k_r) > f_r(t,\ A_r + B_r,\ k_r)$ holds. Consequently,

$$\forall\ t' \in T\backslash Q^n(t,\ A) \qquad \forall\ B \in \Omega(\varepsilon^*) \qquad (t' \notin P^n(t,\ A+B)), \tag{25}$$

where $\varepsilon^* = \min\{\ \varepsilon(t') : \ \ t' \in T \setminus Q^n(t,\ A)\ \ \&\ \ t' \neq t\}$.

Case 2: $t' \in Q^n(t,\ A)$. Then, by virtue of $\mathbf{3}^0$, for any index $i \in N_n$ the inequalities $|t \cap t'| \geq k_i$ and $\xi_i(t,\ t',\ A_i) > 0$ are valid. Hence, on account of continuity in $\mathbf{R}^m$ of the function $\xi_i(t,\ t',\ A_i)$, we have

$$\forall\ i \in N_n \qquad \exists\ \varepsilon_i(t') > 0 \qquad \forall\ B \in \Omega(\varepsilon_i(t')) \qquad (\xi_i(t,\ t',\ A_i + B_i) > 0).$$

Therefore, by lemma 4, the equality $f_i(t,\ A_i + B_i) = f_i(t',\ A_i + B_i)$ is true for any index $i \in N_n$, where $B \in \Omega(\varepsilon_i(t'))$. As a result we get

$$\forall\ t' \in Q^n(t,\ A) \qquad \forall\ B \in \Omega(\varepsilon_*) \qquad (t' \notin P^n(t,\ A+B)), \tag{26}$$

where

$$\varepsilon_* = \min_{t' \in Q^n(t,\ A)}\ \min_{i \in N_n} \varepsilon_i(t').$$

Formulas (25) and (26) mean, that t is an efficient trajectory in the perturbed problem $Z^n(A+B)$ for any matrix $B \in \Omega(\varepsilon)$, if $\varepsilon =$

$\min\{\varepsilon^*, \varepsilon_*\}$. Consequently, taking into account property 2, we see that the problem $Z^n(A)$ is quasistable. $\square$

Evidently, theorem 1 can be formulated in the following way.

Theorem 2. *For a vector trajectorial problem* $Z^n(A)$, $n \geq 1$ *with any combination of the partial criteria* (1) *and* (2), *the next statements are equivalent:*

1^0 *the problem* $Z^n(A)$ *is quasistable,*

2^0 *one of the two conditions*

$$P^n(A) = R^n(A) \qquad or \qquad \emptyset \neq P^n(A) \backslash R^n(A) = \hat{P}^n(A) \tag{27}$$

is true,

3^0 *one of the conditions*

$$P^n(A) = R^n(A) \qquad or \qquad \emptyset \neq P^n(A) \backslash R^n(A) = \check{P}^n(A)$$

holds, where

$$\hat{P}^n(A) = \big\{t \in P^n(A) \backslash R^n(A) : \ \forall t' \in Q^n(t, \ A) \ \ \forall i \in N_n \\ (U_i(t, \ A_i, k_i) = U_i(t', \ A_i, k_i))\big\},$$

$$\check{P}^n(A) = \big\{t \in P^n(A) \backslash R^n(A) : \ \forall t' \in Q^n(t, \ A) \ \ \forall i \in N_n \\ (|t \cap t'| \geq k_i \ \ \& \ \ \zeta_i(t, \ t', \ A_i) > 0)\big\},$$

$$R^n(A) = \{t \in P^n(A) : \quad Q^n(t, \ A) = \emptyset\}$$

is a set of strictly efficient trajectories, i. e. the Smale set [18].

5 Corollaries

As corollaries, we obtain some results known before.

Corollary 1 [14]. *The equality*

$$P^n(A) = R^n(A)$$

is sufficient for the quasistability of a problem $Z^n(A)$, $n \geq 1$, *with partial criteria of the kind* (1) *and* (2). *This equality is necessary, when* $I_{SUM} \neq \emptyset$.

Proof. The sufficiency follows from theorem 2.

Necessity. Let the problem $Z^n(A)$ be quasistable and $I_{SUM} \neq \emptyset$. Quasistability of the problem $Z^n(A)$, by virtue of theorem 2, implies that one of conditions (27) holds. Evidently, the second of mentioned conditions can not be valid, because the inequality

$$U_i(t,\ A_i,\ k_i) \neq U_i(t',\ A_i,\ k_i)$$

holds by property 3 for any $i \in I_{SUM}$, $t \in P^n(A) \backslash R^n(A)$ and $t' \in Q^n(t,\ A)$ (on account of $t \neq t'$). Therefore, $P^n(A) = R^n(A)$. $\square$

Further, let $N_n = I_{SUM} \cup I_{MAX} \cup I_{MIN}$, where I_{SUM}, I_{MAX} and I_{MIN} are the sets of indexes from N_n, which are assigned to partial criteria MINSUM, MINMAX and MINMIN respectively.

For any pair of different trajectories $t,\ t' \in T$ and any index $i \in N_n$, put

$$\zeta_i(t,\ t',\ A_i) = \begin{cases} f_i(t',\ A_i,\ k_i) - f_i(t,\ A_i,\ k_i) & \text{if } i \in I_{SUM}, \\ f_i(t',\ A_i,\ k_i) - f_i(t \setminus t',\ A_i,\ k_i) & \text{if } i \in I_{MAX}, \\ f_i(t' \setminus t,\ A_i,\ k_i) - f_i(t,\ A_i,\ k_i) & \text{if } i \in I_{MIN}. \end{cases}$$

For any vector $A_i \in \mathbf{R}^m$, suppose that

$$f_i(\emptyset,\ A_i,\ k_i) = \begin{cases} 0 & \text{if } i \in I_{SUM}, \\ -\infty & \text{if } i \in I_{MAX}, \\ +\infty & \text{if } i \in I_{MIN}. \end{cases}$$

Corollary 2 [6]. *An n-criterial* ($n \geq 1$) *trajectorial problem with arbitrary combination of partial criteria of the kind MINSUM, MINMAX and MINMIN* ($I_{SUM} \cup I_{MAX} \cup I_{MIN} = N_n$) *is quasistable if and only if the formula*

$$\forall\ t \in P^n(A)\ \ (\ Q^n(t,\ A) \neq \emptyset\ \Rightarrow\ \forall\ t' \in Q^n(t,\ A)\ \ \forall\ i \in N_n \quad (\zeta_i(t,\ t',\ A_i) > 0)\ ) \qquad (28)$$

holds.

Proof. Sufficiency. It is easy to see, that (28) implies satisfiability of one of the conditions

$$P^n(A) = R^n(A) \quad \text{or} \quad \emptyset \neq P^n(A)\backslash R^n(A) = \widetilde{P}^n(A),$$

where

$$\widetilde{P}^n(A) = \big\{t \in P^n(A)\backslash R^n(A): \ \ \forall\, t' \in Q^n(t,\ A) \ \ \forall\, i \in N_n$$

$$(\zeta_i(t,\ t',\ A_i) > 0)\big\}$$

If $P^n(A) = R^n(A)$, then the problem $Z^n(A)$ is quasistable by property 1.

Let $P^n(A) \neq R^n(A)$. Then $\widetilde{P}^n(A) \neq \emptyset$. Therefore, for any trajectories $t \in P^n(A)\backslash R^n(A)$ and $t' \in Q^n(t,\ A)$ the inequalities

$$\zeta_i(t,\ t',\ A_i) > 0, \quad i \in N_n \tag{29}$$

are valid. Hence, by definition of the quantity $\zeta_i(t,\ t',\ A_i)$, the equality $I_{MAX} \cup I_{MIN} = N_n$ is true, i. e. $i \notin I_{SUM}$. Evidently (due to the theorem 2), to prove quasistability of the problem $Z^n(A)$ in this case, it is sufficient to show that the inequalities $|t \cap t'| \geq 1$ and $\xi_i(t,\ t',\ A_i) > 0$ hold for any $t \in P^n(A)\backslash R^n(A)$ and $t' \in Q^n(t,\ A)$, $i \in I_{MAX} \cup I_{MIN}$.

First, the inequality $|t \cap t'| \geq 1$ is evident. Indeed, if $|t \cap t'| = 0$, then the equalities

$$f_i(t \setminus t',\ A_i,\ 1) = f_i(t,\ A_i,\ 1) = f_i(t',\ A_i,\ 1) = f_i(t' \setminus t,\ A_i,\ 1)$$

hold because of equivalence of the trajectories t and t'. Thus $\zeta_i(t,\ t',\ A_i) = 0$, i.e. there is a contradiction with (29). Hence, $|t \cap t'| \geq 1$.

Further, we shall show that $\xi_i(t,\ t',\ A_i) > 0$. For that, let us consider the following cases.

Case 1: $i \in I_{MAX}$. Then, taking into account (29) and equivalence of the trajectories t and t', we have

$$0 < \zeta_i(t,\ t',\ A_i) = f_i(t',\ A_i,\ 1) - f_i(t \setminus t',\ A_i,\ 1),$$

$$0 < \zeta_i(t',\ t,\ A_i) = f_i(t,\ A_i,\ 1) - f_i(t' \setminus t,\ A_i,\ 1) =$$
$$= f_i(t',\ A_,\ 1) - f_i(t' \setminus t,\ A_i,\ 1).$$

This implies
$$f_i(\Delta(t,\ t'),\ A_i,\ 1) < f_i(t',\ A_i,\ 1).$$

Therefore the inequality

$$f_i(\Delta(t,\ t'),\ A_i,\ 1) < f_i(t \cap t',\ A_i,\ 1)$$

is true. Consequently, $\xi_i(t,\ t',\ A_i) > 0$.

Case 2: $i \in I_{MIN}$. Then, taking into account (29) and equivalence of the trajectories t and t', we derive

$$0 < \zeta_i(t,\ t',\ A_i) = f_i(t' \setminus t,\ A_i,\ 1) - f_i(t,\ A_i,\ 1),$$

$$0 < \zeta_i(t',\ t,\ A_i) = f_i(t \setminus t',\ A_i,\ 1) - f_i(t',\ A_i,\ 1) =$$
$$= f_i(t \setminus t',\ A_i,\ 1) - f_i(t,\ A_i,\ 1).$$

Therefore, we conclude

$$f_i(\Delta(t,\ t'),\ A_i,\ 1) > f_i(t,\ A_i,\ 1).$$

Thus, we have the inequality

$$f_i(\Delta(t,\ t'),\ A_i,\ 1) > f_i(t \cap t',\ A_i,\ 1).$$

Consequently, $\xi_i(t,\ t',\ A_i) > 0$.

From the above proof (when $P^n(A) \neq R^n(A)$), we conclude, that statement $\mathbf{3}^0$ of theorem 2 holds. This implies that the problem $Z^n(A)$ is quasistable.

To prove the necessity, suppose the opposite. Let the problem $Z^n(A)$ be quasistable, but condition (28) is not true, i.e. there exist different trajectories $t \in P^n(A)\backslash R^n(A)$ and $t' \in Q^n(t,\ A)$, such that the inequality

$$\zeta_i(t,\ t',\ A_i) \leq 0 \tag{30}$$

holds for some index $i \in N_n$.

We shall consider three possible cases.

Case 1: $i \in I_{SUM}$. Then, by virtue of statement $\mathbf{2}^0$ of theorem 1, $U_i(t,\ A_i,\ k_i) = U_i(t',\ A_i,\ k_i)$. Consequently, by property 3, $t = t'$. It leads to contradiction.

Case 2: $i \in I_{MAX}$. Then $k_i = 1,\ i \in I_1$ and, theorem 1, $|t \cap t'| \geq 1$. Therefore, on account of inequality (30) and the definition of $\zeta_i(t,\ t',\ A_i)$ and $\xi_i(t,\ t',\ A_i)$, we derive

$$0 \geq \zeta_i(t,\ t',\ A_i) = f_i(t',\ A_i,\ 1) - f_i(t \setminus t',\ A_i,\ 1) \geq$$

$$\geq f_i(t \cap t',\ A_i,\ 1) - f_i(\Delta(t,\ t'),\ A_i,\ 1) = \xi_i(t,\ t',\ A_i).$$

This contradicts to the condition $\mathbf{3}^0$ of theorem 1.

Case 3: $i \in I_{MIN}$. Then $k_i = 1,\ i \in I_2$. Therefore, using (30) and $|t \cap t'| \geq 1$ (on account of theorem 1), we have

$$0 \geq \zeta_i(t,\ t',\ A_i) = f_i(t' \setminus t,\ A_i,\ 1) - f_i(t,\ A_i,\ 1) \geq$$

$$\geq f_i(\Delta(t,\ t'),\ A_i,\ 1) - f_i(t \cap t',\ A_i,\ 1) = \xi_i(t,\ t',\ A_i).$$

This contradicts to the condition $\mathbf{3}^0$ of theorem 1. □

From property 2 for $n = 1$ we get

Corollary 3 [19]. *A singlecriterial linear (with criterion of the kind MINSUM) problem $Z^1(A)$, where $A \in \mathbf{R}^m$, is quasistable if and only if $|P^1(A)| = 1$.*

Remark 3. *On account of equivalence of any two norms in a finite-dimensional linear space* [20]*, all results of the present work are true not only for Chebyshev, but for any norm in the space of perturbing matrices $\mathbf{R}^{n\times m}$.*

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V. A. Emelichev, K. G. Kuzmin, A. M. Leonovich, Received January 29, 2004

Belorussian State University,
Mechanics-Mathematics Department,
ave. Fr. Skoriny, 4, Minsk, 220050, Belarus.
E–mail: *emelichev@bsu.by*