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Polynomial time algorithm for solving cyclic games

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Abstract

The problem of finding the value and optimal strategies of players in cyclic games is studied. A polynomial time algorithm for solving cyclic games is proposed.

1 Introduction and Problem Formulation

Let $G = (V, E)$ be a finite directed graph in which every vertex $u \in V$ has at least one leaving edge $e = (u, v) \in E$. A function c: $E \to R$ which assigns a cost $c(e)$ to each edge $e \in E$ is given. In addition the vertex set V is divided into two disjoint subsets V_A and V_B ($V = V_A \cup V_B,\ V_A \cap V_B = \emptyset$) which we will regard as positions sets of two players.

On G we consider the following two-person game from [1,2]. The game starts at position $v_0 \in V$. If $v_0 \in V_A$ then the move is done by the first player, otherwise it is done by the second one. The move means the passage from position v_0 to the neighbor position v_1 through the edge $e_1 = (v_0, v_1) \in E$. After that if $v_1 \in V_A$ then the move is done by the first player, otherwise it is done by the second one and so on indefinitely. The first player has the aim to maximize $\lim\limits_{t\to\infty} \inf \dfrac{1}{t}\sum\limits_{i=1}^{t} c(e_i)$ while the second one has the aim to minimize $\lim\limits_{t\to\infty} \inf \dfrac{1}{t}\sum\limits_{i=1}^{t} c(e_i)$.

In [1,2] it is proved that for this game there exists a value $p(v_0)$ such that the first player has a strategy of moves that insures

$\lim\limits_{t\to\infty} \inf \frac{1}{t}\sum\limits_{i=1}^{t} c(e_i) \geq p(v_0)$ and the second player has a strategy of moves that insures $\lim\limits_{t\to\infty} \inf \frac{1}{t}\sum\limits_{i=1}^{t} c(e_i) \leq p(v_0)$. Furthermore in [1,2]it is shown that the players can achieve the value $p(v_0)$ applying the strategies of moves which do not depend on t. This means that the considered game can be formulated in the terms of stationary strategies. Such statement of the game in [2] is named cyclic game.

The strategies of players in cyclic game are defined as a maps

$$s_A\colon\ u \to v \in V_G(u) \ \text{ for } \ u \in V_A;$$

$$s_B\colon\ u \to v \in V_G(u) \ \text{ for } \ u \in V_B,$$

where $V_G(u)$ represents the set of extremities of edges $e = (u, v) \in E$, i.e. $V_G(u) = \{v \in V \mid e = (u, v) \in E\}$. Since G is a finite graph then the sets of strategies of players

$$S_A = \{s_A\colon\ u \to v \in V_G(u) \ \text{ for } \ u \in V_A\};$$

$$S_B = \{s_B\colon\ u \to v \in V_G(u) \ \text{ for } \ u \in V_B\}$$

are finite sets. The payoff function $F_{v_0}\colon S_A \times S_B \to R$ in cyclic game is defined as follows.

Let $s_A \in S_A$ and $s_B \in S_B$ be the fixed strategies of players. Denote by $G_s = (V, E_s)$ the subgraph of G generated by edges of form $(u, s_A(u))$ for $u \in V_A$ and $(u, s_B(u))$ for $u \in V_B$. Then G_s contains a unique directed cycle C_s which can be reached from v_0 through the edges $e \in E_s$. The value $F_{v_0}(s_A, s_B)$ we consider equal to mean edges cost of cycle C_s, i.e.

$$F_{v_0}(s_A, s_B) = \frac{1}{n(C_s)} \sum_{e \in E(C_s)} c(e),$$

where $E(C_s)$ represents the set of edges of cycle C_s and $n(C_s)$ is a number of the edges of C_s. So, the cyclic game is determined uniquely by the network (G, V_A, V_B, c) and starting position v_0. In [1,2] it is proved that there exist the strategies $s_A^* \in S_A$ and $s_B^* \in S_B$ such that

$$p(v) = F_v(s_A^*, s_B^*) = \max_{s_A \in S_A} \min_{s_B \in S_B} F_v(s_A, s_B) =$$

$$= \min_{s_B \in S_B} \max_{s_A \in S_A} F_v(s_A, s_B), \ \forall\, v \in V.$$

So, the optimal strategies s_A^*, s_B^* of players in cyclic games do not depend on starting position v although for different positions $u, v \in V$ the values $p(u)$ and $p(v)$ may be different. It means that the positions set V can be divided into several classes $V = V^1 \cup V^2 \cup \ldots \cup V^k$ according to values of positions $p^1, p^2, \ldots, p^k$, i.e. $u, v \in V^i$ if and only if $p^i = p(u) = p(v)$. In the case $k = 1$ the network (G, V_A, V_B, c) is named the ergodic network [2]. In [3] it is shown that every cyclic game with arbitrary network (G, V_A, V_B, c) and given starting position v_0 can be reduced to an auxiliary cyclic game on auxiliary ergodic network (G', V_A', V_B', c').

It is well-known [2,4] that the decision problem associated to cyclic game is in $NP \cap$co-NP but it is not yet known for this problem to be in P. Some exponential and pseudo-exponential algorithms for finding the value and the optimal strategies of players in cyclic are proposed in [1,2,4].

Similar problems on acyclic networks have been considered in [5,6]. New classes of acyclic games have been studied in [5,6] and polynomial time algorithms for its solving have been elaborated. Moreover in [5,6] is made an attempt to reduce the cyclic game to acyclic one. Unfortunately the reduction from [5,6] gives the first player some control over the length of the formed cycles, so that it is not conservative in the general case.

In this paper we propose a polynomial time algorithm for finding the value and optimal strategies of players in cyclic games. The proposed algorithm is based on a new reduction from cyclic games to acyclic one.

2 Some Preliminary Results

First of all we need to remind some preliminary results from [2].

Let (G, V_A, V_B, c) be a network with the properties described in section 1. We denote

$$ext(c,u) = \begin{cases} \max\limits_{v \in V_G(u)} \{c(u,v)\} & \text{for} \quad u \in V_A, \\ \min\limits_{v \in V_G(u)} \{c(u,v)\} & \text{for} \quad u \in V_B, \end{cases}$$

$$VEXT(c,u) = \{v \in V_G(u) \mid c(u,v) = ext(c,u)\}.$$

We shall use the potential transformation $c'(u,v) = c(u,v)+\varepsilon(v)-\varepsilon(u)$ for costs on edges $e = (u,v) \in E$, where $\varepsilon\colon V \to R$ is an arbitrary function on vertex set V. In [2] it is noted that the potential transformation does not change the value and the optimal strategies of players in cyclic games.

Theorem 1 *Let (G, V_A, V_B, c) be an arbitrary network with the properties described in section 1. Then there exist the value $p(v)$, $v \in V$ and the function $\varepsilon\colon V \to R$ which determine a potential transformation $c'(u,v) = c(u,v) + \varepsilon(v) - \varepsilon(u)$ for costs on edges $e = (u,v) \in E$, such that the following properties hold*

a) $p(u) = ext(c', u)$ *for* $v \in V$,

b) $p(u) = p(v)$ *for* $u \in V_A \cup V_B$ *and* $v \in VEXT(c', u)$,

c) $p(u) \geq p(v)$ *for* $u \in V_A$ *and* $v \in V_G(u)$,

d) $p(u) \leq p(v)$ *for* $u \in V_B$ *and* $v \in V_G(u)$,

e) $\max\limits_{e \in E} |c'(e)| \leq 2|V| \max\limits_{e \in E} |c(e)|$.

The values $p(v), v \in V$ on network (G, V_A, V_B, c) are determined unequally and the optimal strategies of players can be found in the following way: fix the arbitrary strategies $s_A^*\colon V_A \to V$ and $s_B^*\colon V_B \to V$ such that $s_A^*(u) \in VEXT(c', u)$ for $u \in V_A$ and $s_B^*(u) \in VEXT(c', u)$ for $u \in V_B$.

The proof of Theorem 1 is given in [2].

Further we shall use the theorem 1 in the case of the ergodic network (G, V_1, V_2, c), i.e. we shall use the following corollary.

Corollary 1 *Let (G, V_A, V_B, c) be an ergodic network. Then there exist the value p and the function $\varepsilon\colon V \to R$ which determine a potential transformation $c'(u,v) = c(u,v) + \varepsilon(v) - \varepsilon(u)$ for costs of edges*

$e = (u, v) \in E$ such that $p = ext(c', u)$ for $u \in V$. The optimal strategies of players can be found as follows: fix arbitrary strategies s_A^*: $V_A \to V$ and s_B^*: $V_B \to V$ such that $s_A^*(u) \in VEXT(c', u)$ for $u \in V_A$ and $s_B^*(u) \in VEXT(c', u)$ for $u \in V_B$.

3 The reduction of cyclic games to ergodic ones

Let us consider an arbitrary network (G, V_A, V_B, c) with given starting position $v_0 \in V$ which determines a cyclic game. In [3] it is shown that this game can be reduced to a cyclic game on auxiliary ergodic network $(G', W_A, W_B, \overline{c})$, $G' = (W, F)$ such that it is preserving the value $p(v_0)$ and $v_0 \in W = V \cup X \cup Y$.

The graph $G' = (W, F)$ is obtained from G if each edge $e = (u, v)$ is changed by a triple of edges $e^1 = (u, x), e^2 = (x, y), e^3 = (y, v)$ with the costs $\overline{c}(e^1) = \overline{c}(e^2) = \overline{c}(e^3) = c(e)$. Here $x \in X, y \in Y$ and $u, v \in V; W = V \cup X \cup Y$. In addition in G' each vertex x is connected with v_0 by edge (x, v_0) with the cost $\overline{c}(x, v_0) = M$ (M is a great value) and each edge (y, v_0) is connected with v_0 by edge (y, v_0) with the cost $\overline{c} = (y, v_0) = -M$. In $(G', W_A, W_B, \overline{c})$ the sets W_A and W_B are defined as follows: $W_A = V_A \cup Y$; $W_B = V_B \cup X$.

It is easy to observe that this reduction can be done in linear time.

4 Acyclic games on networks and polynomial algorithms for its solving

We consider the following auxiliary games from [5,6].

4.1 Acyclic c-game on acyclic networks

Let $G = (V, E)$ be a finite directed graph without directed cycles. Assume that in G there exists a vertex v_f which is attainable from each vertex $v \in V$, i.e. v_f is a sink in G. In addition we consider that a function c: $E \to R$ is given on edge set E and the vertex set V is divided into two disjoint subsets V_A and V_B ($V = V_A \cup V_B$, $V_A \cap V_B = \emptyset$) which we regard as positions sets of two players.

On G we consider the two-person game from [5]. The game starts at given position $v_0 \in V \setminus \{v_f\}$. If $v_0 \in V_A$ then the move is done by the first player otherwise it is done by the second one. Like in cyclic game here the move means the passage from position v_0 to the neighbor position v_1 through the edge $e_1 = (v_0, v_1)$. Again if $v_A \in V_A$ then the move is done by the first player, otherwise it is done by the second one and so on while the final position v_f is not reached. As soon as the final position v_f is reached the game is over. The final step of the game we denote by t. In this game the first player has the aim to maximize the integral cost $\sum\limits_{i=1}^{t} c(e_i)$ and the second player has the aim to minimize $\sum\limits_{i=1}^{t} c(e_i)$.

In [5] it is shown that for such game there exists a value $\hat{p}(v_0)$ such that the first player has a strategy of moves that insures $\sum\limits_{i=1}^{t} c(e_i) \geq \hat{p}(v_0)$ and the second player has a strategy of moves that insures $\sum\limits_{i=1}^{t} c(e_i) \leq \hat{p}(v_0)$. Moreover the players can achieve this value using the stationary strategies of moves. This game in [5] is named acyclic c-game. The strategies of players in acyclic c-game can be defined as the maps

$$s_A\colon\ u \to v \in V_G(u) \ \text{ for } \ u \in V_A \setminus \{v_0\};$$

$$s_B\colon\ u \to v \in V_G(u) \ \text{ for } \ u \in V_B \setminus \{v_0\}.$$

Since G is a finite graph then the set of strategies of players

$$S_A = \{s_A\colon\ u \to v \in V_G(u) \ \text{ for } \ u \in V_A \setminus \{v_0\}\};$$

$$S_B = \{s_B\colon\ u \to v \in V_G(u) \ \text{ for } \ u \in V_B \setminus \{v_0\}\}$$

are finite sets. The payoff function $\hat{F}_{v_0}\colon\ S_A \times S_B \to R$ in acyclic c-game is defined as follows.

Let $s_A \in S_A$ and $s_B \in S_B$ be the fixed strategies of players. Denote by $T_s = (V, E_s)$ the tree with root vertex v_f in G generated by edges of form $(u, s_A(u))$ for $u \in V_A \setminus \{v_0\}$ and $(u, s_B(u))$ for $u \in V_B \setminus \{v_0\}$. Then in T_s there exists a unique directed path $P_{T_s}(v_0, v_f)$ from v_0 to v_f. We put

$$\hat{F}_{v_0}(s_A, s_B) = \sum_{e \in E\left(P_{T_s}(v_0, v_f)\right)} c(e),$$

where $E(P_{T_s}(v_0, v_f))$ represents the set of edges of the directed path $P_{T_s}(v_0, v_f)$.

In [5,6] it is proved that for acyclic c-game on network (G, V_1, V_2, c) there exist the strategies of players s_A^*, s_B^* such that

$$\hat{p}(v) = \hat{F}_v(s_A^*, s_B^*) = \max_{s_A \in S_A} \min_{s_B \in S_B} \hat{F}_v(s_A, s_B) = \min_{s_B \in S_B} \max_{s_A \in S_A} \hat{F}_v(s_A, s_B) \tag{1}$$

and s_A^*, s_B^* do not depend on starting position $v \in V$, i.e. (1) holds for every $v \in V$. Note that here for different positions u and v the values $p(u)$ and $p(v)$ may be different.

The equality (1) is evident in the case when $ext(c, u) = 0, \forall\, u \in V$. In this case $p(u) = 0, \forall\, u \in V$ and the optimal strategies of players can be obtained by fixing the maps s_A^*: $V_A \setminus \{v_f\} \to V$ and s_B^*: $V_B \setminus \{v_f\} \to V$ such that $s_A^*(u) \in VEXT(c, u)$ for $u \in V_A \setminus \{v_f\}$ and $s_B^*(u) \in VEXT(c, u)$ for $u \in V_B \setminus \{v_f\}$.

If the network (G, V_A, V_B, c) has the property $ext(c, u) = 0, \forall\, u \in V \setminus \{v_f\}$ then it is named the network in canonic form [5]. So, for the acyclic c-game on network in canonic form the equality (1) holds and $p(v) = 0, \forall\, v \in V$.

In general case the equality (1) can be proved using the properties of the potential transformation $c'(u, v) = c(u, v) + \varepsilon(v) - \varepsilon(u)$ on edges $e = (u, v)$ of the network, where ε: $V \to R$ is an arbitrary real function on V. The fact is that such transformation of the costs on edges of the network in acyclic c-game do not change the optimal strategies of players, although the values $\hat{p}(v)$ of the positions $v \in V$ are changed by $\hat{p}(v) + \varepsilon(v_0) - \varepsilon(v)$. It means that for arbitrary function ε: $V \to R$ the optimal strategies of the players in acyclic c-games on networks

(G, V_A, V_B, c) and on (G, V_A, V_B, c') are the same. Using such property in [5] the following theorem is proved.

Theorem 2 *For arbitrary acyclic network* (G, V_A, V_B, c) *there exists a function* $\varepsilon\colon R \to R$ *which determines a potential transformation* $c'(u,v) = c(u,v) + \varepsilon(v) - \varepsilon(u)$ *on edges* $e = (u,v) \in E$ *such that* (G, V_A, V_B, c') *has the canonic form. The value* $\varepsilon(v), v \in V$*, which determines* $\varepsilon\colon V \to R$*, can be found by using the following recursive formula*

$$\begin{cases} \varepsilon(v_f) = 0, \\ \varepsilon(u) = \begin{cases} \max\limits_{v \in V_G(u)} \{c(u,v) + \varepsilon(v)\}, & \textit{for} \;\; u \in V_A \setminus \{v_f\}, \\ \min\limits_{v \in V_G(u)} \{c(u,v) + \varepsilon(v)\}, & \textit{for} \;\; u \in V_B \setminus \{v_f\}. \end{cases} \end{cases} \tag{2}$$

According to this theorem the optimal strategies of players in acyclic c-game can be found using the following algorithm.

Algorithm 1.

1. Find the values $\varepsilon(v), v \in V$, according to recursive formula (2) and the corresponding potential transformation $c'(u,v) = c(u,v) + \varepsilon(v) - \varepsilon(u)$ for edges $(u,v) \in E$.

2. Fix the arbitrary maps $s_A^*\colon V_A \setminus \{v_f\} \to V$ and $s_B^*\colon V_B \setminus \{v_f\} \to V$ for which $s_A^*(u) \in VEXT(c', u)$ for $u \in V_A \setminus \{v_f\}$ and $s_B^*(u) \in VEXT(c', u)$ for $u \in V_B \setminus \{v_f\}$.

Remark 1 *The values* $\varepsilon(u), u \in V$*, represent the values of the acyclic c-game on* (G, V_A, V_B, c) *with starting position u, i.e.* $\varepsilon(u) = \hat{p}(u), \forall\, u \in V$.

Remark 2 *Algorithm 1 needs* $O(n^2)$ *elementary operations because such number of operations need the tabulation of values* $\varepsilon(u), u \in V$*, by using formula (2).*

4.2 Acyclic c-game on arbitrary networks

Let us consider the game from section 4.2 when G is an arbitrary graph, i.e. G may contain directed cycles. In this case the subgraph $T_s = (V, F_s)$, generated by given strategies of players

$$s_A\colon\ u \to v \in V_G(u) \ \text{ for } \ u \in V_A \setminus \{v_f\};$$
$$s_B\colon\ u \to v \in V_G(u) \ \text{ for } \ u \in V_B \setminus \{v_f\},$$

may contain directed cycles. So, for given s_A and s_B either exists a unique directed path $P_{T_s}(v_0, v_f)$ from v_0 to v_f in T_s or such a path does not exist in T_s. In the second case there exists a unique directed cycle C_s which can be reached from v_0 through the edges of T_s. If in T_s there exists a unique directed path $P_{T_s}(v_0, v_f)$ from v_0 to v_f then $\hat{F}_{v_0}(s_A, s_B)$ we define as

$$\hat{F}_{v_0}(s_A, s_B) = \sum_{e \in E\left(P_{T_s}(v_0, v_f)\right)} c(e).$$

If in T_s there are no directed paths from v_0 to v_f then $\hat{F}_{v_0}(s_A, s_B)$ we define in the following way.

Let C_s be the cycle which can be reached from v_0 and $P'_{T_s}(v_0, u_0)$ represent a unique directed path in T_s which connects v_0 with the cycle C_s ($P'_{T_s}(v_0, u_0)$ and C_s have no common edges). Then we put

$$\hat{F}_{v_0}(s_A, s_A) = \begin{cases} +\infty & \text{if} \quad \displaystyle\sum_{e \in E(C_s)} c(e) > 0, \\ \displaystyle\sum_{e \in E\left(P'_{T_s}(v_0, u_0)\right)} c(e) & \text{if} \quad \displaystyle\sum_{e \in E(C_s)} c(e) = 0, \\ -\infty & \text{if} \quad \displaystyle\sum_{e \in E(C_s)} c(e) < 0. \end{cases}$$

We consider the problem of finding the strategies s_A^*, s_B^* for which

$$\hat{F}_{v_0}(s_A^*, s_B^*) = \max_{s_A \in S_A} \min_{s_B \in S_B} \hat{F}_{v_0}(s_A, s_B).$$

This problem [7] is named the max-min path problem in c-game.

It is easy to observe that for this problem may be the case when the players couldn't reach the final position v_f. We shall formulate the condition when for the player in this game there exists a settle value $\hat{p}(v)$ for each $v \in V$.

Theorem 3 *Let (G, V_A, V_B, c) be a network with given final position, where G is an arbitrary directed graph. Moreover let us consider that $\sum\limits_{e \in E(C_s)} c(e) \neq 0$ for every cycle C_s from G. Then for c-game on (G, V_A, V_B, c) the condition*

$$\hat{p}(v) = \hat{F}_v(s_A^*, s_B^*) = \max_{s_A \in S_A} \min_{s_B \in S_B} \hat{F}_v(s_A, s_B) =$$

$$= \min_{s_B \in S_B} \max_{s_A \in S_A} \hat{F}_v(s_A, s_B), \; \forall\, v \in V$$

holds if and only if there exists a function ε: $V \rightarrow R$ which determines a potential transformation $c'(u,v) = c(u,v) + \varepsilon(v) - \varepsilon(u)$ on edges $(u,v) \in E$ such that $ext(c', u) = 0$, $\forall\, v \in V$. The optimal strategies of players in c-game on (G, V_A, V_B, c) can be found by fixing the arbitrary maps s_A^*: $V_A \setminus \{v_f\} \rightarrow V$ and s_B^*: $V_B \setminus \{v_f\} \rightarrow V$ such that $s_A^*(u) \in VEXT(c', u)$ for $u \in V_A$ and $s_B^*(v) \in VEXT(c', u)$ for $u \in V_B$.

Proof. $\Rightarrow$ Let us consider that the condition

$$\hat{p}(v) = \hat{F}_v(s_A^*, s_B^*) = \max_{s_A \in S_A} \min_{s_B \in S_B} \hat{F}_v(s_A, s_B) = \min_{s_B \in S_B} \max_{s_A \in S_A} \hat{F}_v(s_A, s_B)$$

holds for every $v \in V$ and $\hat{p}(v) < \infty$, $\forall\, v \in V$. It is easy to observe that if we put $\varepsilon(v) = \hat{p}(v)$ for $v \in V$ then the function ε: $V \rightarrow R$ determines a potential transformation $c'(u,v) = c(u,v) + \varepsilon(v) - \varepsilon(u)$ on edges $(u,v) \in E$ such that holds $ext(c', u) = 0$, $\forall\, u \in V$.

$\Leftarrow$ Let us consider that there exists a potential transformation $c'(u,v) = c(u,v) + \varepsilon(v) - \varepsilon(u)$ of costs of edges $(u,v) \in E$ such that $ext(c', u) = 0$, $\forall\, u \in V$. Then for a c'-game on network (G, V_A, V_B, c') the following condition

$$\max_{s_A \in S_A} \min_{s_B \in S_B} \hat{F}'_v(s_A, s_B) = \min_{s_B \in S_B} \max_{s_A \in S_A} \hat{F}'_v(s_A, s_B), \; \forall\, v \in V$$

holds. Since the potential transformation do not change the optimal strategies of players then

$$\max_{s_A \in S_A} \min_{s_B \in S_B} \hat{F}_v(s_A, s_B) = \min_{s_B \in S_B} \max_{s_A \in S_A} \hat{F}_v(s_A, s_B),\ \forall\, v \in V.\ \Box$$

Corollary 2 *The difference $\varepsilon(v) - \varepsilon(v_0), v \in V$, in Theorem 3 determines the cost of max-min path from v to v_f, i.e. $\varepsilon(v) = \hat{p}(v)$, $\forall\, v \in V$.*

Algorithm 2.

1. We construct an auxiliary acyclic graph $H = (W, \overline{E})$, where

$$W = \{w_0^0\} \cup W^1 \cup W^2 \cup \ldots \cup W^n,\ W^i \cap W^j = \emptyset,\, i \neq j;$$

$$W^i = \{w_0^i, w_1^i, \ldots, w_{n-1}^i\},\ i = \overline{1, n};$$
$$\overline{E} = E^0 \cup E^1 \cup E^2 \cup \ldots \cup E^{n-1};$$
$$E^i = \{(w_k^{i+1}, w_l^i) |\, (v_k, v_l) \in E\},\ i = \overline{1, n-1};$$
$$E^0 = \{(w_k^i, w_0^0) |\, (v_k, v_0) \in E,\ i = \overline{1, n}\}.$$

The vertex set W of H is obtained from V if it is doubled n times and then a sink vertex w_0^0 is added. The edge subset $E^i \subseteq \overline{E}$ in H connects the vertices of the set W^{i+1} and the vertices of the set W^i in the following way: if in G there exists an edge $(v_k, v_l) \in E$ then in H we add the edge (w_k^{i+1}, w_l^i). The edge subset $E^0 \subseteq \overline{E}$ in H connects the vertices $w_k^i \in W^1 \cup W^2 \cup \ldots \cup W^n$ with the sink vertex w_0^0, i.e. if there exists an edge $(v_k, v_0) \in E$ then in H we add the edges $(w_k^i, w_0^0) \in E^0$, $i = \overline{1, n}$.

After that we define the acyclic network (H_0, W_A, W_B, c_0), $H_0 = (W_0, E_0)$ where H_0 is obtained from H by deleting the vertices $w_k^i \in W$ from which the vertex w_0^0 couldn't be attainable. The sets W_A, W_B and the cost function c_0: $E_0 \to R$ are defined as follows:

$$W_A = \{w_k^i \in W_0 |\, v_k \in V_A\},\ W_B = \{w_k^i \in W_0 |\, v_k \in V_B\};$$

$$c_0(w_k^{i+1}, w_k^i) = c(v_k, v_l) \ \text{ if } \ (v_k, v_l) \in E \ \text{ and } \ (w_k^{i+1}, w_k^i) \in E_0;$$

$$c_0(w_k^i, w_0^0) = c(v_k, v_0) \ \text{ if } \ (v_k, v_0) \in E \ \text{ and } \ (w_k^i, w_0^0) \in E_0.$$

2. On auxiliary acyclic network (H_0, W_A, W_B, c_0) with sink vertex w_0^0 we consider the acyclic c-game. According to recursive formula (2) we find $\varepsilon(w_k^i)$ for every $w_k^i \in H_0$, i.e. $\varepsilon(w_k^i) = \hat{p}(w_k^i)$, $\forall\, w_k^i \in H_0$.

3. Fix $U = \{v_0\}$; $i = 0$.
4. $i = i + 1$.
5. Find the edge set $E(U) = \{(v_k, v_l) \in E |\, v_l \in U, v_k \in V \setminus U\}$ and the vertex set $V(U) = \{v_k \in V |\, e = (v_k, v_l) \in E(U)\}$.
6. Fix $\varepsilon(v_k) = \varepsilon(w_k^i)$ for $v_k \in V(U)$.
7. Select the vertex set $V^0(U)$ from $V(U)$ where $v_k \in V^0(U)$ if the following conditions are satisfied:
   a) $c(v_k, v_q) + \varepsilon(v_q) - \varepsilon(v_k) \leq 0$, $\forall\, v_q \in U \cup V(U), (v_k, v_q) \in E$ if $v_k \in V_A \cap V(U)$;
   b) $c(v_k, v_q) + \varepsilon(v_q) - \varepsilon(v_k) \geq 0$, $\forall\, v_q \in U \cup V(U), (v_k, v_q) \in E$ if $v_k \in V_B \cap V(U)$;
   c) there exists an edge $(v_k, v_l) \in E$ such that
   $$c(v_k, v_l) + \varepsilon(v_l) - \varepsilon(v_k) = p(v_0).$$
8. If $V^0(U) = \emptyset$ go to step 10.
9. Change U by $U \cup V^0(U)$.
10. If $U \neq V$ then go to step 4, otherwise go to next step 11.
11. End.

The correctness of this algorithm follows from the algorithm from [8].

Remark 3 *Algorithm 4 finds the values $p(v_0)$ and optimal strategies of players in time $O(n^2|E|^2)$.*

4.3 Acyclic l-game on networks

Let (G, V_A, V_B, c) be the network with the properties described in section 1. So, $G = (V, E)$ represents a directed graph without directed cycles and G contains a sink vertex $v_0 \in V$. On E a function c: $E \to R$ is defined and on V a partition $V = V_A \cup V_B$ $(V_A \cap V_B = \emptyset)$ is given, where V_A and V_B are considered as positions sets of two players.

We study the following acyclic l-game from [5]. Again the strategies of players we define as the maps

$$s_A\colon\ u \to v \in V_G(u) \ \text{ for } \ u \in V_A \setminus \{v_f\};$$
$$s_B\colon\ u \to v \in V_G(u) \ \text{ for } \ u \in V_B \setminus \{v_f\}.$$

The payoff function $\overline{F}_{v_0}\colon\ S_A \times S_B \to R$ in this game we define as follows.

Let $s_A \in S_A$ and $s_B \in S_B$ be the fixed strategies of players. Then the tree $T_s = (V, E_s)$ contains a unique directed path $P_{T_s}(v_0, v)$ from v_0 to v_f. We put

$$\overline{F}_{v_0}(s_A, s_B) = \frac{1}{n\left(P_{T_s}(v_0, v_f)\right)} \sum_{e \in E\left(P_{T_s}(v_0, v_f)\right)} c(e),$$

where $E\left(P_{T_s}(v_0, v_f)\right)$ is the set of edges of the path $P_{T_s}(v_0, v_f)$ and $n\left(P_{T_s}(v_0, v_f)\right)$ is the number of its edges.

In [5,6] it is proved that the function $\overline{F}_{v_0}(s_A, s_B)$ satisfies the following condition:

$$\begin{aligned} \overline{p}(v_0) = \overline{F}_{v_0}(s_A^*, s_B^*) = \max\nolimits_{s_A \in S_A} \min\nolimits_{s_B \in S_B} \overline{F}_{v_0}(s_A, s_A) = \\ = \min\nolimits_{s_B \in S_B} \max\nolimits_{s_A \in S_A} \overline{F}_{v_0}(s_A, s_B). \end{aligned} \tag{3}$$

In this game the optimal strategies s_A^* and s_B^* of players depend on starting position v_0. But the players can achieve the settle value $\overline{p}(v_0)$ starting from v_0 by using the successive choosing of optimal strategies of moves from one position to another while the final position v_f is not reached.

The equality (3) can be proved by using the Theorem 1 (corollary 1). It is easy to observe that if for given acyclic l-game we identify the positions v_0 and v_f we obtain the cyclic game with an auxiliary ergodic network (G', V_A', V_B', c') and given starting position v_0. In this ergodic network the graph G' is obtained from G when the vertices v_0 and v_f are identified. In the graph G' all directed cycles pass through the vertex v_0. Applying the corollary 1 of Theorem 1 to the obtained cyclic games on ergodic network (G', V_A', V_B', c') with starting position v_0, we obtain the proof of the following theorem [4].

Theorem 4 *Let be given an acyclic l-game on network Let (G, V_A, V_B, c) with starting position v_0. Then there exist a value $\overline{p}(v_0)$ and a function ε: $V \rightarrow R$ which determine a potential transformation $c'(u,v) = c(u,v) + \varepsilon(v) - \varepsilon(u)$ of costs on edges $e = (u,v) \in E$ such that the following conditions hold*

a) $\overline{p}(v_0) = ext(c', u), \forall\, u \in V \setminus \{v_f\}$;

b) $\varepsilon(v_0) = \varepsilon(v_f)$

the optimal strategies of players in acyclic l-game can be found as follows: fix the arbitrary maps s_A^*: $V_A \setminus \{v_f\} \rightarrow V$ and s_B^*: $V_B \setminus \{v_f\} \rightarrow V$ such that $s_A^*(u) \in VEXT(c', u)$ for $u \in V \setminus \{v_f\}$ and $s_B^*(u) \in VEXT(c', u)$.

This theorem follows from corollary 1 of Theorem 1 because the acyclic l-game on network Let (G, V_A, V_B, c) with starting position v_0 can be regarded as a cyclic game on network Let (G', V_A', V_B', c') with starting position v_0. This network is obtained from the network (G, V_A, V_B, c) when the vertices v_0 and v_f are identified.

Taking into consideration remark 2 and corollary 2 of Theorem 3 we obtain the following result.

Corollary 3 *The difference $\varepsilon(v) - \varepsilon(v_0), v \in V$, in Theorem 4 represents the costs of max-min path from v to v_f in acyclic c-game on network $(G, V_A, V_B, \overline{c})$ where $\overline{c}(u,v) = c(u,v) - \overline{p}(v_0)$, $\forall\,(u,v) \in E$.*

So, if $\overline{p}(v_0)$ is known then the optimal strategies s_A^* and s_B^* of players in acyclic l-games can be found in time $O(n^2)$ by reducing acyclic l-game to acyclic c-game on $(G, V_A, V_B, \overline{c})$.

On the basis of this property in [5] the polynomial algorithm for solving l-games is elaborated. In [6] an attempt to elaborate strongly polynomial time algorithm for l-games is made. But the algorithm from [6] contains the same flow as the algorithm of reducing cyclic games to acyclic one.

5 Polynomial time algorithm for solving cyclic games

On the basis of the obtained results we can propose polynomial time algorithm for solving cyclic games.

We consider an acyclic game on ergodic network (G, V_A, V_B, c) with given starting position v_0. The graph $G = (V, E)$ is considered to be strongly connected and $V = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$. Assume that v_0 belongs to the cycle C_{s^*} determined by the optimal strategies of players s_A^* and s_B^*.

We construct an auxiliary acyclic graph $H_r = (\overline{W}_r, \overline{E}_r)$, where

$$\overline{W}_r = \{w_0^0\} \cup W^1 \cup W^2 \cup \ldots \cup W^r,\ W^i \cap W^j = \emptyset,\ i \neq j;$$

$$W^i = \{w_0^i, w_1^i, \ldots, w_{n-1}^i\},\ i = \overline{1, r};$$
$$\overline{E} = E^0 \cup E^1 \cup E^2 \cup \ldots \cup E^{r-1};$$
$$E^i = \{(w_k^{i+1}, w_l^i) |\, (v_k, v_l) \in E\},\ i = \overline{1, r-1};$$
$$E^0 = \{(w_k^i, w_0^0) |\, (v_k, v_0) \in E,\ i = \overline{1, r}\}.$$

The vertex set W_r of H_r is obtained from V if it is doubled r times and then a sink vertex w_0^0 is added. The edge subset $E^i \subseteq \overline{E}$ in H_r connects the vertices of the set W^{i+1} and the vertices of the set W^i in the following way: if in G there exists an edge $(v_k, v_l) \in E$ then in H_r we add the edge (w_k^{i+1}, w_l^i). The edge subset $E^0 \subseteq \overline{E}$ in H_r connects the vertices $w_k^i \in W^1 \cup W^2 \cup \ldots \cup W^r$ with the sink vertex w_0^0, i.e. if there exists an edge $(v_k, v_0) \in E$ then in H_r we add the edges $(w_k^i, w_0^0) \in E^0$, $i = \overline{1, r}$.

After that we define the acyclic network (H_r', W_A, W_B, c_r'), $H_r' = (W_r, E_r)$ where H_r' is obtained from H_r by deleting the vertices $w_k^i \in \overline{W}_r$ from which the vertex w_0^0 couldn't be attainable. The sets W_A, W_B and the cost function c_r': $E_r \to R$ are defined as follows:

$$W_A = \{w_k^i \in W_0 |\, v_k \in V_A\},\ W_B = \{w_k^i \in W_0 |\, v_k \in V_B\};$$

$$c_r'(w_k^{i+1}, w_k^i) = c(v_k, v_l) \ \text{ if } \ (v_k, v_l) \in E \ \text{ and } \ (w_k^{i+1}, w_k^i) \in E_r;$$

$$c'_r(w_k^i, w_0^0) = c(v_k, v_0) \text{ if } (v_k, v_0) \in E \text{ and } (w_k^i, w_0^0) \in E_r.$$

Now we consider the acyclic c-game on acyclic network (H'_r, W_A, W_B, c'_r) with sink vertex w_0^0 and starting position w_0^r.

Lemma 1 *Let $p = p(v_0)$ be the value of the ergodic game on G and the number of edges of the maxim cycle in G is equal to r. Moreover, let $p_r(W_0^r)$ be the value of the l-game on (H'_r, W_A, W_B, c'_r) with starting position w_0^r. Then $p(v_0) = p_r(w_0^r)$.*

Proof. It is evident that there exist a bijective mapping between the set of cycles with exactly r edges (which contains the vertex v_0) in G and the set of directed paths with r edges from w_0^r to w_0^0 in H'_r. Therefore $p(v_0) = p_r(w_0^r)$. $\square$

On the basis of this lemma we can propose the following algorithm for finding the optimal strategies of players in cyclic games.

Algorithm 3.

We construct the acyclic networks (H'_r, W_A, W_B, c'_r), $r = 2, 3, \dots, n$ and for each of them solve l-game. In such a way we find the values $p_2(w_0^2), p_3(w_0^3), \dots, p_n(w_0^n)$ for these l-games. Then we consequtively fix $p = p_2(w_0^2), p_3(w_0^3), \dots, p_n(w_0^n)$ and each time solve the c-game on network (G, V_A, V_B, c'), where $c' = c - p$. Fixing each time the values $\varepsilon'(v_k) = \widehat{p}'(v_k)$ for $v_k \in V$ we check if the following condition

$$ext(\overline{c}', v_k) = 0. \ \forall v_k \in V$$

is satisfied, where $\overline{c}'(v_k, v_l) = c'(v_k, v_l) + \varepsilon(v_l) - \varepsilon(v_k)$. We find r for which this condition holds and fix the respective maps s_A^* and s_B^* such that $s_A^*(v_k) \in VEXT(c', v_k)$ for $u \in V_A$ and $s_B^*(v_k) \in VEXT(c', v_k)$ for $u \in V_B$. So, s_A^* and s_B^* represent the optimal strategies of player in cyclic games on G.

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