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Supermodular Programming on Lattices

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1 Introduction.

Questions, concerning the optimization of supermodular functions on finite lattices are considered in the paper. The systematic summary of main authors' and other researchers' results known before, new authors' results are given. There should be marked out the following three results among new results.

The first – elaboration of the basic propositions of the theory of maximization of supermodular functions on Boolean lattices (they were worked out only for the problems of minimization before) and establishing of relation between global minimum and maximum of supermodular function for main types of lattices.

The second – elaboration of original combinatorial algorithms of automatized representation of hyper-cubes (booleans) of large dimension on a plane in the form of various diagrams, on which the properties of boolean as a partially ordered set of its vertexes are kept (This provides us with ample opportunities for construction of various schemes of looking through the elements of atomic lattices and for visualization of the optimization process).

The third – carrying out the basic propositions of the theory of optimization of supermodular functions to the main types of lattices: Boolean lattices, lattices with relative supplements (division lattices, lattices of vector subspaces of finite-dimensional vector space, geometrical spaces), lattices equal to Cartesian product of chains, distributive lattices, atomic lattices.

These theoretical results and availability of the great amount of optimization problems for lattices with concrete forms of supermodular functions allow to consider methods and algorithms for solving the problems of optimization of supermodular functions on lattices as a new field of mathematical programming – supermodular programming [19].

2 Basic concepts and definitions, statement of the optimization problems

Let's provide necessary concepts and definitions, following in general, Birkhoff G. and Gratzer G [1, 2].

The partially ordered set $\langle A; \leq \rangle$ is called a lattice, if for all $a, b \in A$ there exist $\sup\{a, b\}$ and $\inf\{a, b\}$. The lattice is called a finite, if the set A is finite. Lower only the finite lattices will be considered.

Let's denote $\sup\{a, b\} = a \vee b$, $\inf\{a, b\} = a \wedge b$ and we shall call $\vee$ a join, and $\wedge$ an intersection. $\vee$ and $\wedge$ are the binary operations on lattices, which map A^2 into A.

Let's remark, that the sense of the operations $\vee$ and $\wedge$ depends on modes of the representation of a set A (and its elements) and on the order relation $\leq$. For example, if A is a set of all subsets of the fixed set I, i.e. the element $a \in A$ is a subset $a \subseteq I$ and vice versa, and the relation $\leq$ means the set-theoretic inclusion $\subseteq$, then the set-theoretic operations $\cup$ and $\cap$ will correspond to operations $\vee$ and $\wedge$. It, in particular, concerns to Boolean lattices. Other examples will be reduced also below.

Let's speak, that in the lattice $\langle A; \leq \rangle$ the element $a \in A$ covers an element $b \in A \quad (a \succ b)$ or b is covered by $a (b \prec a)$, if $a > b$ $(a \geq b \;\; a \neq b)$ and there is no such $x \in A$, that $a > x > b$.

The subset $D(a) \subset A$

$$D(a) = \{x \in A |\, x \succ a \; or \; x \prec a\}$$

is called a neighbourhood of $a \in A$.

The subset A_1 of the lattice $\langle A;\, \vee,\, \wedge\rangle$ is called as convex, if for any $a, b \in A_1$ and $c \in A$ the inequality $a \leq c \leq b$ means that $c \in A_1$. If $a \leq b$, then $[a,\, b] = \{x \in A |\, a \leq x \leq b\}$ is called as an interval. The interval, by it's definition, is a convex sublattice.

Let $a \leq b$, then subset of elements $\{a,\, ...,\, b\} = \{x_0,\, x_1,\, ..., x_{n-1}\}$ is called as a chain, if $x_0 = a$, $x_{n-1} = b$ and $x_i \leq x_{i+1}$ for $0 \leq i < n-1$.

All chains $\{a,\, ...,\, b\} \subseteq [a, b]$. The chain $\{a,\, ...,\, b\} \subseteq [a, b]$ is called as a maximum chain, if $x_i \prec x_{i+1}$, $0 \leq i < n-1$, $x_0 = a$, $x_{n-1} = b$.

Let's consider the function $f(x)$, $x \in A$, given on a lattice $\langle A;\, \vee,\, \wedge\rangle$.

The element $c' \in A$ is called a local minimum of function $f(x)$ on a lattice $\langle A;\, \vee,\, \wedge\rangle$, if

$$f(c') \leq f(x) \text{ for all } \; x \in D(c').$$

The element $c \in A$ is called a global minimum, if

$$f(c) \leq f(x) \text{ for all } \; x \in A.$$

The function $f(x)$ is called supermodular, if for any $a, b \in A$ the following inequality is fulfilled

$$f(a) + f(b) - f(a \vee b) - f(a \wedge b) \leq 0. \tag{1}$$

If the inequality is fulfilled in reverse, then such function is called submodular. A special case of these two functions is the modular function (when the inequality is converted into equality).

The problems of optimization of supermodular functions on arbitrary lattices were formulated and investigated for the first time in [11].

Let $f(x)$ be the supermodular function. It is required to find it's minimum, i.e. to find such $c \in A$, that

$$f(c) = \min_{x \in A} f(x)$$

under the condition (1).

It is known that this problem concerns to a class of NP-difficult problems.

The problem of definition of a maximum of the function $f(x)$ is a dual to this problem, i.e. It is required to find such d, that

$$f(d) = \max_{x \in A} f(x)$$

under the condition (1).

Unlike the problem of minimization, the problem of maximization of the supermodular function can be solved by the polynomial algorithm, this is proved by Lovasz L [9]. But we do not know any effective algorithm developed on the basis of the approach of Lovasz L.

3 Minimization of supermodular functions on lattices

3.1 Minimization of supermodular functions on Boolean lattices

Let's consider a finite set $I = 1, 2, \dots, m$ and the set $B(I)$, the elements of which are all subsets ω of a set I:

$$B(I) = \{\omega \mid \omega \subseteq I\}.$$

Let's designate by $< B(I); \cup, \cap >$ a Boolean lattice, the elements of which are partially ordered by the set-theoretic inclusion $\subseteq$, and let the set-theoretic operations $\cup$ and $\cap$ correspond to operations $\vee$ and $\wedge$.

For the first time the problem of minimization of supermodular functions on a Boolean lattice has been considered by V.P.Cherenin, and he has offered his method "of successive calculations" for it's solution [3].

Let's reduce without the proofs the basic theorem "of successive calculations" and two rejection rules received by V.P.Cherenin. (These results are reformulated with the use of the above-stated definitions and designations).

Theorem 1 (the basic theorem of the method "of successive calculations").

The supermodular function $f(\omega)$, defined on a Boolean lattice $< B(I); \cup, \cap >$, on any maximum chain $\{\Theta, ..., I\}$, containing a local minimum c', monotonically decreases down to c' and monotonically increases after c'.

As the corollaries from this theorem two rejection rules are deduced.

The first rejection rule. If for any two elements $\omega_1,\ \omega_2$, such, that $\omega_1 \subset \omega_2$, it will appear, that $f(\omega_1) < f(\omega_2)$, then it is possible to exclude from consideration all $2^{m-|\omega_2|}$ elements $\omega \supseteq \omega_2$. (As among them there will be no local minima of function $f(\omega)$).

Let's remark, that the first rejection rule excludes from consideration an interval $[\omega_2, I]$.

The second rejection rule. If for any two elements $\omega_1,\ \omega_2$, such, that $\omega_1 \subset \omega_2$, it will appear, that $f(\omega_1) > f(\omega_2)$, then it is possible to exclude from consideration all $2^{|\omega_1|}$ elements $\omega \subseteq \omega_1$. (As among them there will be no local minima of function $f(\omega)$).

Let's remark, that the second rejection rule excludes from consideration an interval $[\Theta, \omega_1]$.

The development of the method "of successive calculations" further was performed by studying of properties of local minima, by deriving of new rejection rules, by perfecting and modification of algorithms [5-8].

Let's consider a chain $\{a, ..., b\} = \{x_0, x_1, ..., x_{n-1}\}$.

Let

$$f(x_k) = \min_{0 \leq i \leq n-1} f(x_i).$$

Chain $\{x_0, x_1, ..., x_{n-1}\}$ we shall call unimodal, if the function $f(x_i)$ monotonically decreases with index i running from 0 up to k and monotonically increases with index i running from k up to $n-1$. Special cases of a unimodal chain will be: a chain, in which $x_k = x_{n-1}$ and chain, in which $x_k = x_0$.

It follows from the basic theorem, that any chain containing a local minimum, is unimodal. However it is shown, that not any unimodal chain contains a local minimum, and that there exist maximal not unimodal chains.

A series of the theorems about properties and structures of local minima of supermodular functions is proved.

Let's consider an interval $[\omega_1, \omega_2]$. Let's designate:

$$\Delta_1(i) = \begin{cases} f(\omega_1) - f(\omega_1 \cup \{i\}), & \text{if } f(\omega_1) - f(\omega_1 \cup \{i\}) \geq 0, \\ 0 & , \text{ if } f(\omega_1) - f(\omega_1 \cup \{i\}) < 0; \end{cases}$$

$$\Delta_2(i) = \begin{cases} f(\omega_2) - f(\omega_2 \backslash \{i\}), & \text{if } f(\omega_2) - f(\omega_2 \backslash \{i\}) \geq 0, \\ 0 & , \text{ if } f(\omega_2) - f(\omega_2 \backslash \{i\}) < 0. \end{cases}$$

$$f(c') = \min_{\omega_1 \subseteq \omega \subseteq \omega_2} f(\omega)$$

$$f_1(c') = f(\omega_1) - \sum_{i \in \omega_2 \backslash \omega_1} \Delta_1(i)$$

$$f_2(c') = f(\omega_2) - \sum_{i \in \omega_2 \backslash \omega_1} \Delta_2(i).$$

Theorem 2 $f(c') \geq \max\{f_1(c'),\ f_2(c')\}$.

The next rejection rule follows from this theorem:

The third rejection rule. If for any two elements $\omega_1 \subset \omega_2$ it will appear, that either $f_1(c') \geq f(\tilde{c})$, or $f_2(c') \geq f(\tilde{c})$, where $f(\tilde{c})$-is a known value of function $f(\omega)$, it is possible to exclude from consideration all $2^{|\omega_2 \backslash \omega_1|}$ elements $\omega \in [\omega_1, \omega_2]$.

The theorems which permit to determine a two-sided estimation for number of elements in an optimal subset $c \in I$ are proved.

3.2 Minimization of supermodular functions on lattices with relative supplements

Let $\langle A; \vee, \wedge \rangle$ be a lattice, $a, b \in A$, $a \leq b$ and $x \in [a, b]$. The element $x^* \in [a, b]$, such, that $x \wedge x^* = a$ and $x \vee x^* = b$ is called a relative supplement of an element x in an interval $[a, b]$. The lattice $\langle A; \vee, \wedge \rangle$ is called a *lattice with relative supplements* (*RS-lattice*), if there exists a relative supplement x^* for any $x \in A$ in any interval containing x.

Let's consider a problem of minimization of supermodular function $f(x)$ on a finite lattice with relative supplements. It's required to find $c \in A$, such, that

$$f(c) = \min_{x \in A} f(x).$$

Unlike the Boolean lattice, x^* can be not unique in the case of RS-lattice. The Boolean lattice is a special case of a RS-lattice, however, the **theorem 1** and rejection rules are transferred on RS-lattices [11].

Theorem 3 *On any maximal chain $\{a, \ \ldots, \ b\}$, containing a local minimum x', the function $f(x)$ monotonically decreases down to x' and monotonically increases after x'.*

Three rejection rules, which are used in algorithms for searching of a minimum of supermodular function on a finite RS-lattice are proved.

The class of finite RS-lattices is rather wide, and contains many known lattices. The examples of RS-lattices are: Boolean lattice, lattice of a finite set division, lattice of vectorial subspaces of a finite-dimensional vector space, geometrical lattices. Let's note, that the finite lattices being the product of chains, having length more than 1, do not belong to a class of RS-lattices.

3.3 Minimization of supermodular functions on lattices being a Cartesian product of chains

Let A_i, $i \in I = \{1, 2, ..., m\}$ be arbitrary finite sets, such, that $|A_i| = s(i) + 1 < +\infty$ and $A_i = \{x_i^0, x_i^1, ..., x_i^{s(i)}\}$. A_i are linearly ordered, and $x_i^0 = 0$ is a minimum element for all A_i:

$$0 = x_i^0 < x_i^1 < \dots < x_i^{s(i)}.$$

Let's consider the Cartesian product of these sets

$$A = \prod_{i \in I} A_i.$$

Let's designate an arbitrary element of a set A through $X = (x_1, x_2, ..., x_m)$, where $x_i \in A_i$, $i \in I = \{1, 2, ..., m\}$. Let's define operations $\vee$ and $\wedge$ on a set A as follows: if $X, Y \in A$ and $X = (x_1, x_2, ..., x_m)$, $Y = (y_1, y_2, ..., y_m)$ then $X \vee Y = (\sup(x_1, y_1), ..., \sup(x_m, y_m))$ and $X \wedge Y = (\inf(x_1, y_1), ..., \inf(x_m, y_m))$. It is obvious, that for any $X, Y \in A$, $X \vee Y$ and $X \wedge Y$ are always defined and belong to set A. The obtained lattice (we shall designate it $\langle A; \vee, \wedge \rangle$) is called a Cartesian product of chains. The element $X = (0, 0, ..., 0)$ is a zero and the element $X = (x_1^{s(1)}, x_1^{s(2)}, ..., x_m^{s(m)})$ is a unity of a lattice.

The property of a unimodality of maximal chains containing a local minimum, is broken here, that is visible from the following example (see figure). The function $f(X)$ with $A_i = \{0, 1, 2\}$, $i \in I = \{1, 2\}$ is represented in the figure. $f(X)$ is supermodular and has two global minima – $f(2, 0) = 0$ and $f(0, 2) = 0$ and local minimum – $f(2, 2) = 3$. But there is no maximal unimodal chain containing a local minimum.

By introduction of the new partial order for elements of a set A it becomes possible to reduce a problem on a lattice $\langle A; \vee, \wedge \rangle$ to problems on the lower semilattice $\langle A; \cap \rangle$ and upper semilattice $\langle A; \cup \rangle$, in which the set A is represented as the join of specially constructed booleans.

Theorem 4 *The supermodular function $f(X)$, given on a lattice $\langle A; \vee, \wedge \rangle$, on any maximal chain of the lower semilattice $\langle A; \cap \rangle$ connecting zero element with a local minimum, monotonically decreases, and on any maximum chain of the upper semilattice $\langle A; \cup \rangle$ connecting a local minimum to unity, monotonically increases.*

The first, second and third rejection rules are also proved for these semilattices [6,7,12].

During the last time these results were carried to the distributive lattices [17].

4 Maximization of supermodular functions on lattices

Let $f(x)$ be a supermodular function given on a lattice $\langle A; \vee, \wedge \rangle$. It is required to find it's maximum, i.e. to find such $d \in A$, that

$$f(d) = \max_{x \in A} f(x)$$

4.1 4.1. Correlation between global maximum and global minimum of supermodular function.

The theorems establishing correlation between global maximum and global minimum of supermodular function on various types of lattices were proved [16,18].

4.1.1 Case of a Boolean lattice

Let's consider a finite set $I = 1, 2, \ldots, m$ and set $B(I)$, the elements of which are all subsets ω of a set I:

$$B(I) = \{\omega \mid \omega \subseteq I\}.$$

Let's designate a Boolean lattice by $< B(I); \cup, \cap >$, the elements of which are partially ordered by set-theoretic inclusion $\subseteq$, and set-theoretic operations $\cup$ and $\cap$ correspond to operations $\vee$ and $\wedge$.

Let function $f(\omega)$, given on each subset $\omega \in B(I)$, be supermodular, i.e.:

$$f(\delta) + f(\gamma) - f(\delta \cup \gamma) - f(\delta \cap \gamma) \;\; \forall \delta, \gamma \in B(I)$$

Theorem 5 *Let $d \in B(I)$ be a global maximum of function $f(\omega)$, and $c \in B(I)$ be a global minimum of function $f(\omega)$, i.e.*

$$f(d) = \max_{\omega \in B(I)} f(\omega) f(c) = \min_{\omega \in B(I)} f(\omega).$$

Then

$$f(d) \leq f(I) + f(\Theta) - f(c).$$

4.1.2 Case of lattice with relative supplements

Let $\langle A; \vee, \wedge \rangle$ be a lattice with relative supplements.

Let $f(x)$ be supermodular function on $\langle A; \vee, \wedge \rangle$, i.e.

$$f(x) + f(y) - f(x \vee y) - f(x \wedge y) \ \ \forall x, y \in A$$

Let's consider an interval $[a, b]$, containing a lattice $\langle A; \vee, \wedge \rangle$.

Theorem 6 *Let $d \in [a, b]$ be a global maximum of function $f(x)$, and $c \in [a, b]$ be a global minimum of function $f(x)$, i.e.*

$$f(d) = \max_{x \in [a, b]} f(x) f(c) = \min_{x \in [a, b]} f(x).$$

Then

$$f(d) \leq f(b) + f(a) - f(c).$$

4.1.3 Case of lattice being a Cartesian product of chains

Let $\langle A; \vee, \wedge \rangle$ be a lattice being a Cartesian product of chains, i.e.

$$A = \prod_{i-1}^{m} A_i$$

where

$$A_i = \{x_i^0 = 0, x_i^1, ..., x_i^{s(i)}\}.$$

Let $f(x)$ be a supermodular function on a lattice $\langle A; \vee, \wedge \rangle$, i.e.

$$f(x) + f(y) - f(x \vee y) - f(x \wedge y) \ \ \forall x, y \in A$$

Theorem 7 *Let $d \in A$ be a global maximum of function $f(x)$, and $c \in A$ be a global minimum of function $f(x)$ i.e.*

$$f(d) = \max_{x \in A} f(x) \; f(c) = \min_{x \in A} f(x).$$

Then

$$f(d) \le f(x_1^{s(1)}, \ldots, x_m^{s(m)}) + f(0, \ldots, 0) - f(c).$$

4.2 Rejection rules of not optimum solutions for a case of a Boolean lattice

Definition:

The function $f(\omega)$ has a local maximum on a subset $d_u \in B(I)$, if the following conditions are fulfilled:

$$f(d_u) \ge f(d_u \cup \{i\}) \;\; \forall i \in I \backslash d_u \; f(d_u) \ge f(d_u \backslash \{i\}) \;\; \forall i \in d_u$$

Let's consider two subsets ω_1 and ω_2 such, that $\Theta \subset \omega_1 \subset \omega_2 \subset I$. Let's designate $\omega_1 \subset \tilde{d} \subset \omega_2$ such, that

$$f(\tilde{d}) = \max_{\omega_1 \subset \omega \subset \omega_2} f(\omega).$$

Let's designate local maxima of function $f(\omega)$ through $\tilde{d}_u$ for $\omega_1 \subset \omega \subset \omega_2$.

The first rejection rule.

If for any $i_0 \in \omega_2 \backslash \omega_1$ it has appeared, that $f(\omega_1) < f(\omega_1 \cup i_0)$, then $i_0 \in \tilde{d}_u$, i.e. i_0 belongs to all local maxima $\tilde{d}_u$, so it belongs to global maxima $\tilde{d}$ among all $\omega_1 \subset \omega \subset \omega_2$.

The second rejection rule.

If for any $i_0 \in \omega_2 \backslash \omega_1$ it has appeared, that$f(\omega_2) < f(\omega_2 \backslash i_0)$, then $i_0 \notin \tilde{d}_u$ and $i_0 \notin \tilde{d}$.

The third rejection rule.

Let $\bar{f}_1(\tilde{d})$ and $\bar{f}_2(\tilde{d})$ be upper estimations for a value of a local maximums for subsets $\omega_1 \subset \omega \subset \omega_2$, which are calculated under the formulas:

$$\bar{f}_1(\tilde{d}) = f(\omega_2) + \sum_{i\in\omega_2\backslash\omega_1} [f(\omega_1) - f(\omega_1 \cup \{i\})]$$

$$\bar{f}_2(\tilde{d}) = f(\omega_1) + \sum_{i\in\omega_2\backslash\omega_1} [f(\omega_2) - f(\omega_2\backslash\{i\})].$$

Let some temporarily-optimum solution d' with its value $f(d')$ be fixed during searching a global maximum d and its value $f(d)$. Then the third rejection rule is formulated as follows: if for any subsets $\omega_1 \subset \omega_2$ it will appear either $\bar{f}_1(\tilde{d}) \leq f(d')$, or $\bar{f}_2(\tilde{d}) \leq f(d')$, then it is not necessary to consider all $2^{|\omega_2\backslash\omega_1|}$ subsets $\omega_1 \subset \omega \subset \omega_2$, as obviously $f(\tilde{d}) \leq f(d')$ and, consequently $f(\tilde{d}) \leq f(d)$.

5 General approach to optimization on lattices with the use of atomic lattices

5.1 Optimization on atomic lattices

The element covering zero, is called atom of partially ordered set with zero. The element covered by unity is called coatom of partially ordered set with unity.

Let's call a finite lattice and semilattice *atomic (coatomic)*, if any their element can be presented as the join of atoms (intersection of coatoms). Such lattices are called, as well, dot lattices.

Let $\langle B; \vee, \wedge\rangle$ be a finite atomic lattice (or lower semilattice $\langle B; \cap\rangle$). Let's designate a set of its atoms by $A(B)$.

If $\langle B; \vee, \wedge\rangle$ is atomic lattice, then for any subset $Y \subseteq A(B)$ exists the join $\sup Y = \bigvee_{x\in Y} x$. Hence, there exists a map φ of a set of all subsets $2^{A(B)}$ of a set of atoms into a set B, keeping the partial order. It ensures applicability of algorithms of looking through the elements of a set $2^{A(B)}$ in a combination with rejection rules, like it is carried out for Boolean lattices $\langle 2^{A(B)}; \cup, \cap\rangle$. But the map φ can be not one-to-one, as to various subsets $A(B)$ there can correspond the same element from B. Therefore it is necessary to formulate rules excluding recurrings during construction of elements of a lattice $\langle B; \vee, \wedge\rangle$. These rules depend on a mode of representation of any element of a lattice $\langle B; \vee, \wedge\rangle$ as a join of its atoms [11,13].

If $\langle B; \cap\rangle$ is the lower atomic semilattice, then the join $\sup Y = \bigvee\limits_{x\in Y} x$ exists not for every subset $Y \subseteq A(B)$. It is possible to show, that there exists a map φ of some subset $C \subseteq 2^{A(B)}$ into a set B, keeping the partial order. During the construction of algorithm of looking through the elements of a semilattice $\langle B; \cap\rangle$ it is necessary to formulate, except the rules excluding recurring during the construction of elements of a semilattice $\langle B; \cap\rangle$, the rules excluding from consideration such subsets $Y \subseteq 2^{A(B)}$, for which Y does not belong C. Similarly it is possible to act during research of a finite coatomic lattice (or upper semilattice $\langle B; \cup\rangle$).

If a considered lattice is not atomic (not coatomic), then it is necessary to try to transform a given lattice into atomic (coatomic) lattice or a semilattice by introduction of the new order, so that the desirable properties of function being minimized (for example, a property of a supermodularity) will be kept.

The given approach has been applied for a lattice "Cartesian product" and division lattices. The rules for elimination of elements, not belonging to these lattices, and excluding recurrings have been obtained [11,12,13]. The properties of the supermodular functions on the distributive lattices have been also studied [17]. The rejection rules were carried out for finding the functions minimum values.

5.2 Representation of N-dimensional hyper-cube on a plane in the form of diagrams

As follows from the part 2, it is important to study the Boolean lattice properties, as it can help for the solution of the optimization problems on lattices of general type. Therefore the visualization of hyper-cubes (Booleans) receives a great importance. One of the ways to resolve this problem is to represent the hyper-cubes (HC) in the form of diagrams.

Different methods of automatization of these diagrams construction [19] are described below.

N-dimensional hyper-cube can be represented on the plane by the different methods. Let us stop here on two of them, which seem to be the most interesting from the point of view of obtaining the various representations of HC on a plane. First of them is the orthogonal projection of HC in the Euclidean Space R^N on a plane of observation. The second method allows to construct the diagrams of Boolean lattices of various shapes, which may not always correspond to real HC projections, but conserve all Boolean lattice properties.

Let's describe the first method. In the aim to obtain various projections of HC (to be exact, of its tops and ribs) it is necessary:

1. To construct the N-dimensional HC in space R^N;

2. To fix one of the coordinate planes in space R^M (here, in general, $M \geq N$);

3. To turn the HC around one of the coordinate subspaces in space R^M on the given angle $\alpha = (\alpha_1, ..., \alpha_k)$;

4. To receive a projection of the HC in the chosen coordinate plane.

The operator of turn in space R^M looks as follows:

$$A = \begin{bmatrix} U_1 & & & & 0 \\ & \ddots & & & \\ & & U_k & & \\ & & & E_p & \\ 0 & & & & -E_q \end{bmatrix},$$

where

$$U_i = \begin{bmatrix} \cos\alpha_i & -\sin\alpha_i \\ \sin\alpha_i & \cos\alpha_i \end{bmatrix},$$

E_S – a single matrix of the $s\,(s = p, q)$ order, $2k + p + q = M$.

The following cases are possible:

1. $p = M$ – identical transformation;

2. $q = M$ – the rotation is the central symmetry;

3. $p + q = M$ – the rotation is a symmetry concerning a p-plane (reflection from a p-plane);

4. Operator A does not contain E_p and $-E_q$ submatrixes – rotation is a turn around a unique stationary point;

5. Operator A contains submatrixes U_i and E_p, but does not contain submatrix $-E_q$ – rotation is as a turn around a p-plane;

6. Operator A contains submatrixes U_i and $-E_q$, but does not contain a submatrix E_p – rotation is a rotary reflection from a $(M-q)$ plane.

Thus, the turning of N-dimensional HC in M-dimensional Euclidean Space on different angles around various coordinate subspaces permit to obtain the different projections of HC on plane (as well as on any coordinate subspace of dimension less than N). In some of these projections the tops and the ribs will coincide and superimpose each other. However, it is always possible to construct the projections, in which all 2^N tops will be visible, like in the HC itself. For example, in the case of 4D HC we can write the operator A in the following form:

$$A = \begin{bmatrix} \cos\alpha_1 & -\sin\alpha_1 & 0 & 0 \\ \sin\alpha_1 & \cos\alpha_1 & 0 & 0 \\ 0 & 0 & \cos\alpha_2 & -\sin\alpha_2 \\ 0 & 0 & \sin\alpha_2 & \cos\alpha_2 \end{bmatrix}.$$

This corresponds to the case 4, when the rotation is a turn around a unique stationary point. If we set here the angles $\alpha_1 = \alpha_2 = \pi/4$ radians, then we'll receive the centrally symmetrical projection of 4D HC, where all the tops will be separated and visible (Fig.1).

The second method of N-dimensional HC representation on a plane is of special interest. First of all, we have to set the initial frame of N ribs of HC, which come from the top with coordinates $(0, ..., 0)$. These N two-dimensional vectors can be considered as a projection of corresponding ribs of HC, omitting the question if this projection is really possible in the Euclidean space. Such an approach permits to set the initial N ribs of given lengths and directions on a plane, independent from each other, and then to construct the "projection" of HC on this plane according to the given frame. The described procedure allows to realize different deformations with the HC image on a plane, conserving, thus, all of its properties as a Boolean lattice.

A specific case is of the particular interest, when initial N frame vectors are of the same length and the angle between every two neighbouring vectors is equal to π/N radians. In this case the exterior contour of the HC image on a plane represents the regular $2N$-gon, and the whole figure is centrally symmetrical (Fig.1).

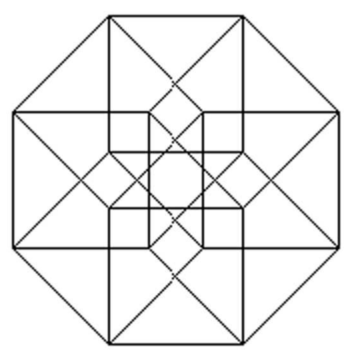

Figure 1.

For HC of any dimension, except the case of $N = 2^K$ ($K = 0, 1, 2, \ldots$ – integer), this representation of HC corresponds to the orthogonal projection from the space R^N towards one of its main diagonals direction to the plane of observation. It means, in general, the coincidence of some HC tops in this projection. In the special case, when $N = 2^K$ and the number of tops is $P = 2^N = 2^{2^K}$, there exists another direction, except the direction of the HC main diagonal, the projection towards which gives equal lengths of the all ribs projections, and none of tops projections superimposes another. The simple example of this is the two-dimensional cube (quadrate), for which $N = 2 = 2^1$ (Fig.2).

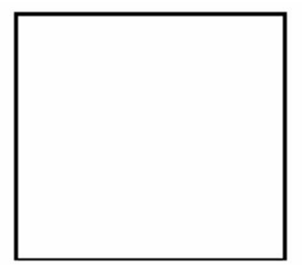

Figure 2.

Fig.3 represents the 4D HC diagram constructed in this manner. You can see that this diagram is identical to the one in the Fig.1, while it was constructed with the first method and represents a real projection of 4D HC to a plane.

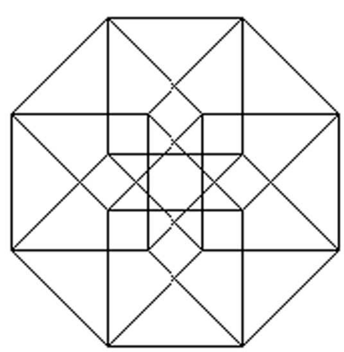

Figure 3.

The algorithms have been developed and the computer programs elaborated for both methods of HC representation on a plane, described above. They permit either to turn N-dimensional HC in space R^M and to obtain its various projections to planes, or to set initial frame vectors and, next, to construct the pseudo-projections (or diagrams) of HC with the preservation of its properties as Boolean lattice.

By introduction of metric the worked out algorithms will allow to distribute non-uniformly the vertexes of HC in the space, deforming it and keeping the partial order of it's vertexes. This allows, in case of keeping the property of supermodularity, to use also values and properties of some other function, which are known in the area of disposition of "deformed" hypercube, while solving the problem of optimization of supermodular function.

5.3 Some examples of the hyper-cube representation on a plane

Below you can see the examples of the HC representations constructed in two different ways for each HC dimension:

Way A – the HC diagram is constructed by the first method from 5.2 (as an orthogonal symmetric projection to a plane from N-dimensional space).

Way B – the HC diagram is constructed by the second method from 5.2 (with a symmetric deformation of the initial N ribs).

Representation of 3D HC diagrams.

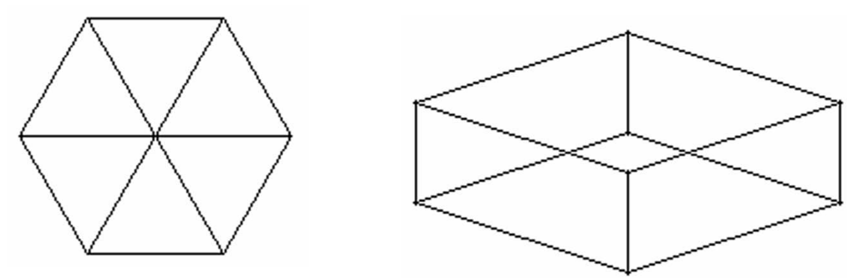

Figure 4. Way A Way B

Representation of 4D HC diagrams.

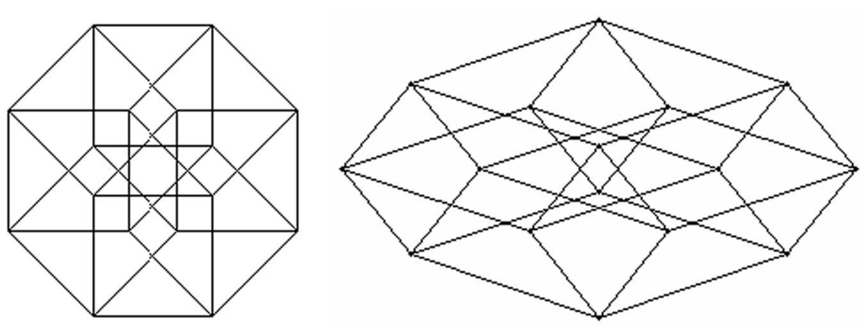

Figure 5. Way A Way B

Representation of 5D HC diagrams.

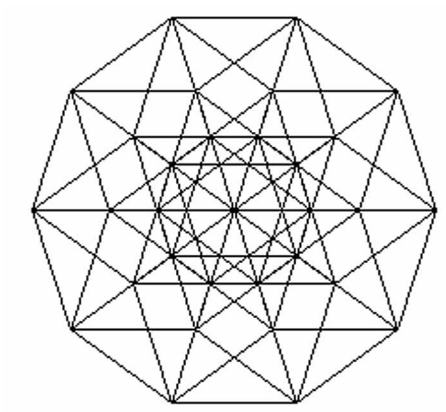

Figure 6. Way A

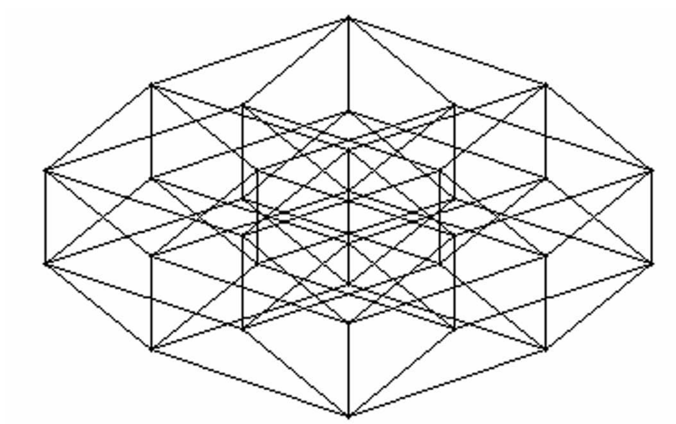

Figure 7. Way B

Representation of 6D HC diagrams.

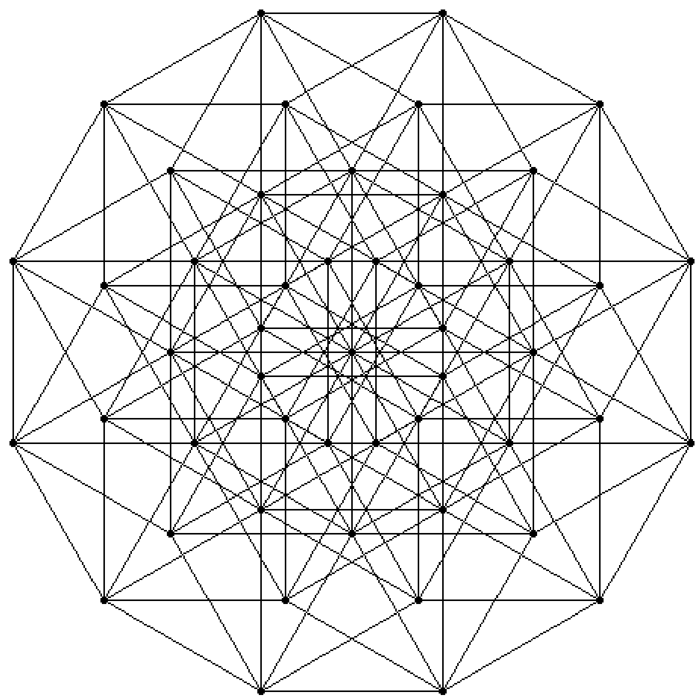

Figure 8. Way A

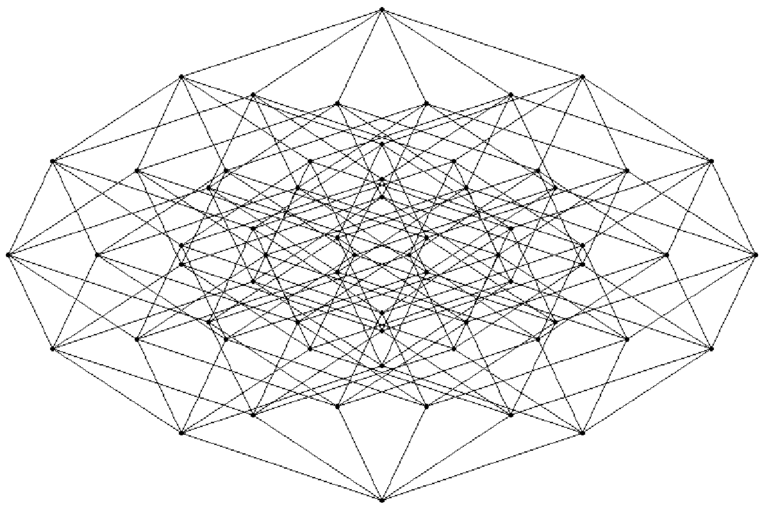

Figure 9. Way B

Representation of 8D HC diagrams.

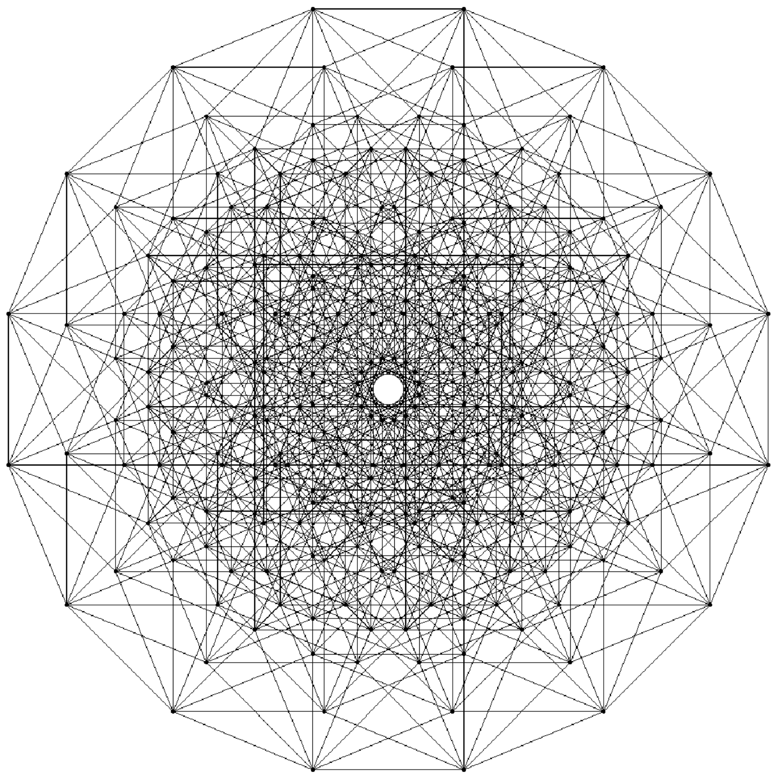

Figure 10. Way A

16D HC orthogonal symmetric projection (way A, only apexes). It can be seen on this image and on the image below that,

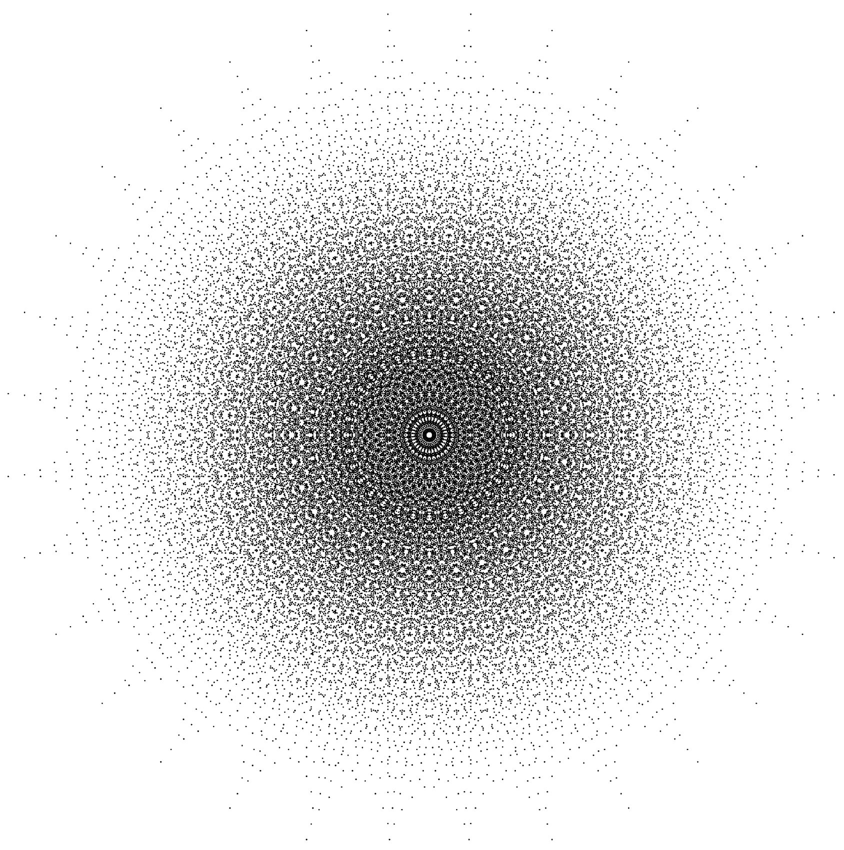

Figure 11.

as it was noticed in 5.2, in the case of 16D HC orthogonal symmetric projection none of the apexes projections superimposes another.

Central part of the 16D HC orthogonal symmetric projection (scale 10:1).

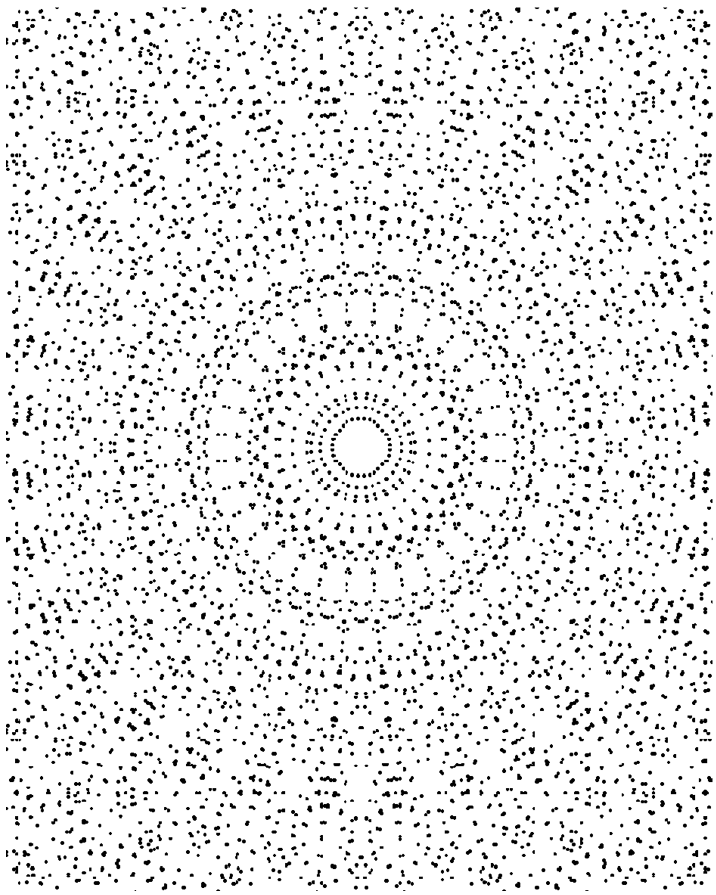

Figure 12.

5.4 About the basic algorithms of optimization

The wide experience of a solution of problems of optimization for lattices with concrete types of supermodular functions was accumulated. The following problems, for example, were concerned this type of problems: static and dynamic optimal enterprise allocation problems with a step-wise cost function, the vertex graph coloring problem, the optimal work grouping problem, optimal partitioning of a set of production points into duplicating systems, problem of optimal distribution of the investments between areas of a region, and many others [4-8,10-18].

The various algorithms of the exact and approximate solution of the problems are developed. These algorithms essentially use atomic structure of lattices.

In algorithms of finding an exact solution there is organized the directed looking through the vertexes of lattices, during which, the appropriate rejection rules are being used, these rules exclude from consideration great amount of not optimal vertexes. The considered problems, as a rule, concern to a class of NP-difficult problems. However experience of a solution of the large number of practical problems has allowed to deduce the following statistical estimation of number vertexes of lattices, which have been looked through, for which the calculation of values of optimized function $f(x)$ was carried out:

$$N \approx km^3, \quad \frac{1}{m^2} < k < m^2,$$

where m is a number of atoms in the appropriate Boolean lattice;

k – coefficient depending on concrete values of a problem parameters,

N – is a number of calculations of values of function $f(x)$. Consequently

$$(m+1) \leq N < m^5$$

The approached algorithms allow to find solutions with their error estimation. Thus the number of the calculations of function $f(x)$ does not exceed magnitude [18]

$$N = (m^2 + 3m - 2)/2.$$

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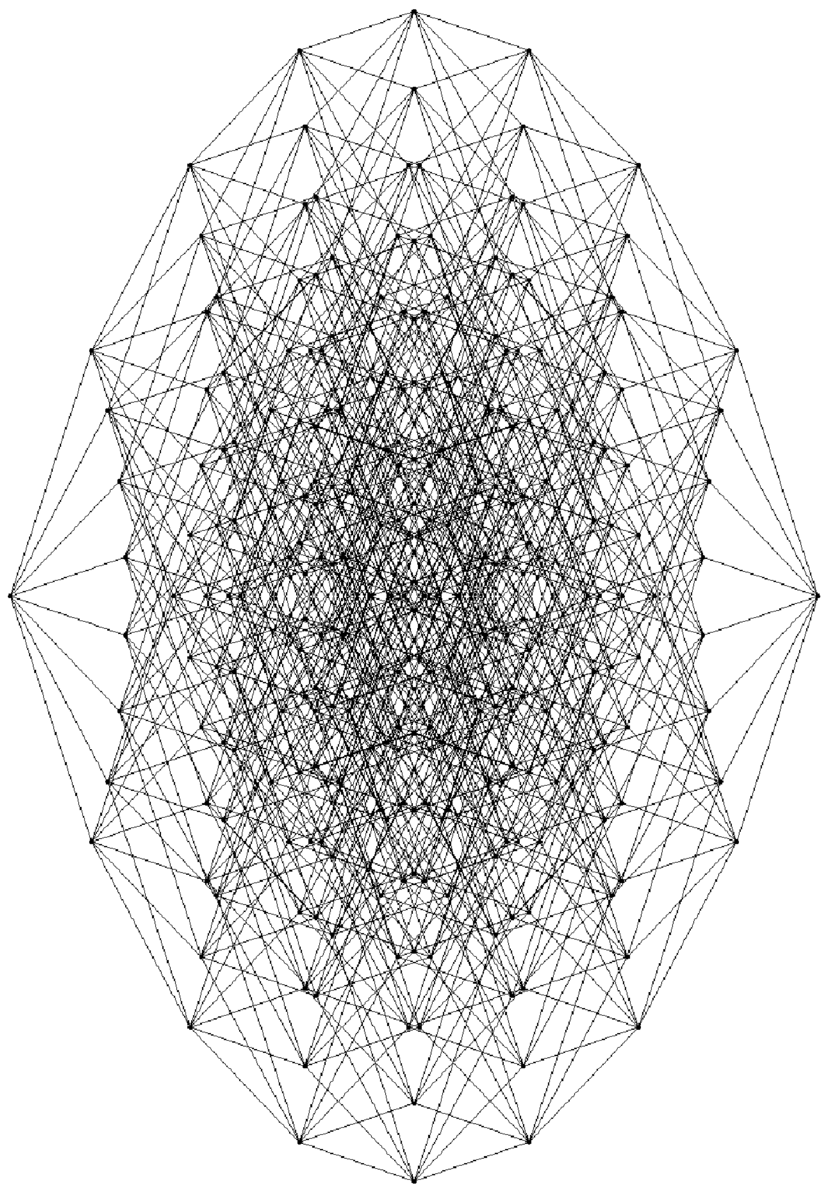

Figure 13. Way B