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Linear-time algorithm for the edge-colorability of a graph with prescribed vertex types

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Abstract

We consider the coloring of edges in a graph in which there are vertices of three types. In a feasible edge coloring, each vertex of the first type is incident with at least two edges of the same color, and each vertex of the second type with at least two edges of different colors; while no constraints are required for the vertices of the third type. We present a characterization of colorable graphs, and a linear-time algorithm to decide whether a given graph with prescribed vertex types admits a feasible edge coloring.

1 Introduction

The problems discussed in this paper are motivated by the theory of mixed hypergraphs introduced in [6, 7].

A *mixed hypergraph* is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where X is the vertex set and each of $\mathcal{C}$ and $\mathcal{D}$ is a family of subsets of X, called the $\mathcal{C}$*-edges* and $\mathcal{D}$*-edges*, respectively. A *proper* k*-coloring* of $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is an injective mapping from the vertex set X into a set of k colors so that each $\mathcal{C}$-edge has two vertices with a **c**ommon color and each $\mathcal{D}$-edge has two vertices with **d**ifferent colors. A mixed hypergraph is k*-colorable*

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if it has a proper coloring with at most k colors. A *strict k-coloring* is a proper k-coloring using all of the k colors.

The maximum number of colors in a strict coloring of $\mathcal{H}$ is the *upper chromatic number*, denoted $\bar{\chi}(\mathcal{H})$; and the minimum number is the *lower chromatic number*, $\chi(\mathcal{H})$. Thus, general mixed hypergraphs represent the structures where the problems on both the minimum and maximum number of colors are nontrivial.

In contrast to classic colorings of hypergraphs (see e.g. [1]), mixed hypergraphs may admit no colorings at all. A mixed hypergraph having no colorings is said to be *uncolorable* [6, 7], and otherwise it is called *colorable*. The first paper about uncolorable mixed hypergraphs is [3]. It has been proved in [4] that it is NP-complete to decide whether a given mixed hypergraph is colorable or not.

The following problem arose in considering the colorability problem of mixed hypergraphs $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$. What happens to the complexity of the colorability problem if the "dual hypergraph" of $\mathcal{H}$ is a (multi)graph? We consider duality as it is defined in [1], namely, the *dual* of a hypergraph $\mathcal{H}$ is the hypergraph $\mathcal{H}^*$ whose incidence matrix is the transpose of the incidence matrix of $\mathcal{H}$. In this way, the edges (vertices) of $\mathcal{H}$ are represented with the vertices (edges, respectively) of $\mathcal{H}^*$, while keeping the incidence relation unchanged. Then the dual of $\mathcal{H}$ is a multigraph if and only if $\mathcal{H}$ is regular of degree 2.

Motivated by this correspondence, we introduce the following model. Let $G = (V, E)$ be a multigraph with vertex set V and edge set E, where multiple edges and also loops are allowed. Let, moreover, $V = V^{\mathcal{C}} \cup V^{\mathcal{D}} \cup V^0$ be a given partition of the vertex set, where any of the three subsets may be empty. It will be assumed throughout that every vertex of degree less than 2 belongs to V^0.

Definitions. A *proper edge k-coloring* of G is an injective mapping from the edge set E into a set of k colors so that each $v \in V^{\mathcal{C}}$ has degree at least two in some color — this means either a loop at v or two edges incident with v that have a **c**ommon color — and each $v \in V^{\mathcal{D}}$ is incident with at least two edges with **d**ifferent colors (this latter requirement cannot be satisfied by a loop). A graph G is *edge*

k-colorable if it has a proper edge k-coloring. A *strict edge k-coloring* is a proper edge k-coloring that uses all of the k colors. The maximum number of colors in a strict edge coloring of G is the *upper chromatic index* denoted $\bar{\chi}'(\mathcal{H})$; and the minimum number of colors is the *lower chromatic index*, $\chi'(\mathcal{H})$.

In this way, we obtain colorings of the *edges* of graphs where the problems on both the minimum and maximum number of colors are also nontrivial.

We shall show that, in sharp contrast to the hardness of colorability on unrestricted mixed hypergraphs (and even on 3-uniform hypergraphs), on graphs the problem is solvable in *linear time*. This will be proved in the following section, where also structural characterizations of (un)colorable graphs will be given.

Since in the paper we consider only the colorings of the edges, for the sake of simplicity, in what follows, the word "coloring" will mean *edge coloring*. It is worth mentioning that in this setting the problem solved by Vizing [5] represents a special case, namely when $V^{\mathcal{C}} = V^0 = \emptyset$ and, moreover, all the edges incident with any vertex of $V^{\mathcal{D}}$ are of different colors. The opposite case, namely when $V^{\mathcal{D}} = V^0 = \emptyset$, was investigated in [2].

For more information on mixed hypergraph coloring, see the recent research monograph [8].

2 Colorability

Here the criteria of colorability (uncolorability) will be given. We first design an analogue of the "splitting-contraction algorithm" [7], but with the essential new features that it runs in linear time and either finds a proper coloring or reduces the input graph to a configuration that is trivially uncolorable, namely an isolated loop at a vertex of $V^{\mathcal{D}}$. Then, at the end of the section, we formulate explicit necessary and sufficient conditions for (un)colorability.

Assume that a graph $G = (V, E)$ with vertex partition $V = V^{\mathcal{C}} \cup V^{\mathcal{D}} \cup V^0$ is given. Next, we list some elementary reduction steps, each of them invertible in the natural way, which generate colorable graphs if and only if G itself is colorable. After each step, we indicate in parentheses how a coloring of G can be obtained from that of the reduced graph.

Reductions invariant under colorability

1. Delete an isolated vertex or an isolated edge which is neither a loop nor a multiple edge.

   (Both endpoints of a non-multiple edge have degree 1, belonging to V^0 by assumption, therefore the edge may receive any color.)

2. Delete an isolated loop incident with a vertex of $V \setminus V^{\mathcal{D}} = V^{\mathcal{C}} \cup V^0$.

   (The loop may receive any color.)

3. Delete a non-isolated loop and put its vertex into V^0.

   (The vertex, say v, of the deleted loop remains non-isolated after the reduction, i.e. it will be incident with a remaining edge of some color. We assign the loop to the same color if $v \in V^{\mathcal{C}}$ in the original vertex partition of G, and to a new color otherwise.)

4. Delete an edge uv that has both endpoints in $V^{\mathcal{D}}$, and put both u and v into V^0.

   (A new color may be used for the deleted edge to color G.)

5. If one endpoint u of an edge uv has degree 1, then delete vertex u, delete edge uv, and put the vertex v into V^0.

   (By the degree-1 assumption, we have $u \in V^0$. Hence, the color of uv can be chosen properly with respect to v, similarly to Step 3.)

6. If $v \in V^0$, delete v and all its incident edges, and put all its neighbours into V^0.

   (The selection of colors on the edges incident with v is analogous to the previous case.)

7. If $v \in V^{\mathcal{C}}$ has degree 2, say uv, vw are its two incident edges, then delete v and replace uv, vw by a new edge uw. If u and w are adjacent already in G, this yields a new parallel edge.

   (The original edges uv and vw may receive the color of the new edge uw; this will be proper for v. Note that in every proper coloring of the original G, the two edges incident with v must have the same color.)

8. If $v \in V^{\mathcal{C}}$ has two neighbors $u, w \in V^{\mathcal{D}}$, then delete the edges uv, vw and put u, v, w into V^0.

   (We may assign *the same new color* to the two deleted edges, hence making all the three vertices u, v, w properly colored in G.)

9. If $uv \in E$ has multiplicity at least 3, then delete two of these parallel edges and put u, v into V^0.

   (Each of the deleted edges can make one of u and v properly colored in G.)

Starting from G, we repeatedly apply the above steps, as long as at least one of them is possible. Observe that if the degree of a vertex decreases in some step, then the vertex in question is put into V^0, i.e. the degree constraint remains valid and hence the transformations generate structures within the same class. When the procedure eventually stops, we denote by G^0 the graph obtained, and call it the *reduced graph of* G. (It may also be the case that the reduced graph is the null-graph, without any vertices.)

Lemma 1 *In the reduced graph G^0, the set V^0 is empty and all vertices have degree at least 2.*

Proof. If some vertex does not satisfy these properties, then at least one of the operations above can be applied, i.e. the graph is not reduced yet. $\square$

Lemma 2 *In the reduced graph G^0, every $\mathcal{C}$-vertex is adjacent to at least two other $\mathcal{C}$-vertices, while the $\mathcal{D}$-vertices are mutually nonadjacent.*

Proof. All vertices of degree at most 1, as well as all $\mathcal{C}$-vertices of degree 2 are eliminated, so that every $\mathcal{C}$-vertex of G^0 has degree at least 3. Since such a vertex can have at most one $\mathcal{D}$-neighbor, it must be adjacent with at least two $\mathcal{C}$-vertices. Also, the edges with both endpoints in $V^{\mathcal{D}}$ are eliminated during the process. □

Now we are in a position to characterize (un)colorable graphs.

Theorem 1 *A graph G is colorable if and only if the operations above do not produce any isolated loop on a $v \in V^{\mathcal{D}}$.*

Proof. We define a coloring as follows.

- If both endpoints of an edge are $\mathcal{C}$-vertices, then assign color 1 to it.
- If at least one endpoint of an edge is a $\mathcal{D}$-vertex, assign a *distinct* color to it.

By the preceding lemmas, if G^0 is non-empty, then it has minimum degree at least 2, and every $\mathcal{C}$-vertex is adjacent to at least two $\mathcal{C}$-vertices. Thus, the coloring just defined is proper on every $\mathcal{C}$-vertex, and also on every $\mathcal{D}$-vertex incident with more than one edge. Hence, the coloring is not proper only if G^0 has a $\mathcal{D}$-vertex of degree 2 incident with just one edge, which is then necessarily a loop.

Conversely, it is trivial that an isolated loop at a $V^{\mathcal{D}}$-vertex is uncolorable. □

Observe that Step 7 is the only operation where a modified vertex may remain in $V^{\mathcal{D}}$. Hence, an isolated $\mathcal{D}$-loop can be obtained from an isolated cycle only. In this way, we obtain the following alternative characterization.

Theorem 2 *A graph G is uncolorable if and only if it has a connected component which is a cycle with precisely one $\mathcal{D}$-vertex and all the other vertices of this cycle are in $V^{\mathcal{C}}$.* □

The cycle component involved in this theorem can be a loop, or a cycle of length two consisting of precisely two parallel edges (with one $\mathcal{C}$-vertex and one $\mathcal{D}$-vertex), or a simple cycle of length at least 3 (without loops and multiple edges). If just the colorability of G has to be checked — a *decision problem*, as opposed to the *search* problem of finding a proper coloring if it exists — then the above characterization admits an even faster algorithm.

Theorem 3 *The (un)colorability of G can be tested by the following algorithm:*

- *Find the set V_2 of degree-2 vertices in $V^{\mathcal{C}} \cup V^{\mathcal{D}}$.*
- *Find the cycles induced by V_2.*
- *Check if (at least) one of those cycles has one vertex in $V^{\mathcal{D}}$ and all the other vertices in $V^{\mathcal{C}}$.*

If a cycle with the property described in the last step is found, then G is uncolorable; and otherwise it is colorable. □

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