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The splitting method and Poincaré's theorem: (II) – matrix, polynomial and language

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Abstract

This paper is the continuation of the paper which appeared in the previous issue: we revisited Poincaré's theorem in the light of the *splitting method* which was introduced by the author in [5], especially in the geometric aspect of the question.

This new part is also based on the definition of a *combinatoric* tiling which was detailed in the previous issue. Indeed, this definition has a natural algebraic continuation as long as it involves a matrix, hence a polynomial. We discuss here the connection of these objects which we provide the reader in full extent, with the notion of languages which are attached in such a case which we call the *language of the splitting*

We show that in all the cases under study, the language of the splitting is not regular.

*Key-words:*hyperbolic tessellations, algorithmic approach

1 Introduction

For the convenience of the reader, here we sketchily remind the definitions and the results of the first part.

Poincaré's theorem is a famous result about tessellations in the hyperbolic plane by triangles.

A tessellation of a polygon is a tiling which is obtained by recursively reflecting it in its sides and the images in their sides: this defines the tiles. The tiling property requires that the interior of the tiles are pairwise disjoint and that any point of the plane belongs to the closure of at least one tile. We say that the considered polygon generates a tiling by tessellation.

Poincaré's theorem, being established in the late 19^{th} century, says that a triangle T generates a tiling by tessellation if the interior angles of T are of the form $\frac{\pi}{p}$, $\frac{\pi}{q}$ and $\frac{\pi}{p}$ and if p, q and r satisfy the inequality: $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$. Notice that the latter inequality simply says that T is indeed a triangle in the hyperbolic plane.

Several proofs of this result where given, among them elementary ones, see, for instance, [1].

In this paper, we revisit the proof of this theorem in the light of the new method which was introduced by the author in [5] and which we call the **splitting method**.

1.1 The splitting method

It lies on the following notion which is a generalisation of [5]:

Definition 1 – *Consider finitely many sets $S_0, \ldots, S_k$ of some geometric metric space X which are supposed to be closed with non-empty interior, unbounded and simply connected. Consider also finitely many closed simply connected bounded set $P_1, \ldots, P_h$ with $h \leq k$. Say that the S_i's and P_ℓ's constitute a* **basis of splitting** *if and only if:*

(i) X splits into finitely many copies of S_0,

(ii) any S_i splits into one copy of some P_ℓ and finitely many copies of S_j's,

where **copy** *means an* **isometric image***, and where, in the condition (ii), the copies may be of different S_j's, S_i being possibly included.*

As usual, it is assumed that the interiors of the copies of P_ℓ and the copies of the S_j's are pairwise disjoint.

The set S_0 is called the **head** *of the basis and the P_ℓ's are called the* **generating tiles***.*

Consider a basis of splitting of X, if any. We recursively define a tree A which is associated with the basis as follows. First, we split S_0 according to the condition (ii) of Definition 1. This gives us a copy of say P_0 which we call the *root* of A and which we call also the *leading tile* of S_0. In the same way, by the condition (ii) of Definition 1, the splitting of each S_i provides us with a copy of some P_ℓ which we call the *leading tile* of S_i. The splitting provides us also with k_i *regions*, S_{i_1}, …, $S_{i_{k_i}}$ which enter the splitting of S_i. The regions which enter the splitting of S_0 according to the condition (ii) of Definition 1 are called the *regions* of the first generation. Assume that we have all the regions of the n^{th} generation: S_{n_1}, …, $S_{n_{m_n}}$. By definition, their leading tiles constitute the nodes of the n^{th} generation. We split again these S_j's according to the condition (ii). We obtain m_n tiles which are called the tiles of the $n+1^{\text{th}}$ generation and, for each S_{n_h} which is some S_i, we have a splitting which is the isometric image of the splitting of S_i as it is above indicated. We say that the leading tiles of these copies of the splitting of S_i are called the *sons* of the leading tile of S_{n_h}. By definition, the sons of the leading tile of S_{n_h} belong to the $n+1^{\text{th}}$ generation. The union of all the sons of the nodes of the n^{th} generation constitutes the nodes of the $n+1^{\text{th}}$ generation.

This recursive process generates an infinite tree with finite branching. This tree, A, is called the **spanning tree of the splitting**, where the *splitting* refers to the basis of splitting $S_0, \ldots, S_k$.

Definition 2 – *Say that a tiling of X is* **combinatoric** *if it has a basis of splitting and if the spanning tree of the splitting yields exactly the restriction of the tiling to S_0, where S_0 is the head of the basis.*

As in [6], in this paper also, we consider only the case when we have a single generating tile, *i.e.* when $h = 1$.

In previous works by the author and some of its co-authors, a lot of partial corollaries of that result were already proved as well as the extension of this method to other cases, all in the case when X is the hyperbolic plane or the hyperbolic $3D$ space.

Here, we remind the results which were established for $I\!H^2$ and $I\!H^3$:

Theorem 1 – (Margenstern-Morita, [8, 9]) *The tiling $\{5,4\}$ of the hyperbolic plane is combinatoric.*

Theorem 2 – (Margenstern-Skordev, [10]) *The tilings* $\{s, 4\}$ *of the hyperbolic plane are combinatoric, with* $s \geq 5$.

Theorem 3 – (Margenstern-Skordev, [11, 12, 13]) *The tiling* $\{5, 3, 4\}$ *of the hyperbolic* $3D$ *space is combinatoric.*

Recall that the theorem of Poincaré considers that the angles of the triangle which generates the tessellation are of the form $\frac{\pi}{p}$, $\frac{\pi}{q}$ and $\frac{\pi}{p}$. In this situation, in [6] we proved the following theorem:

Theorem 4 – (Margenstern, [6]) *When* p, q, $r \geq 3$, *the tiling which is generated by the recursive reflection of a triangle with angles* $\frac{\pi}{p}$, $\frac{\pi}{q}$ *and* $\frac{\pi}{r}$ *with* $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ *is combinatoric. It is also the case for all* q, $r \geq 4$ *when* $p = 2$.

Now, notice that for the tiling property by tessellation, it is only needed that the angles of the triangle are of the form $\frac{2\pi}{h}$, $\frac{2\pi}{k}$ and $\frac{2\pi}{\ell}$ with the condition $\frac{1}{h} + \frac{1}{k} + \frac{1}{\ell} < \frac{1}{2}$. If h, k and ℓ are all even, we find again the condition of Poincaré's theorem. As announced before, in most cases, the tiling which is generated by the triangle by tessellation is combinatoric. But it is not always the case and we need a weaker notion:

Definition 3 – *Say that a tiling is* **quasi-combinatoric** *if it has a sub-tiling which is combinatoric.*

Recall that a *sub-tiling* of a tiling is a partition of the same set where the members of the partition are unions of tiles of the initial tiling. We also can view a sub-tiling as a partition over the partition which is defined by the tiling.

From the definition of a combinatoric tiling, it is not difficult to see that a sub-tiling of a tiling $\mathcal{T}$ is generated by *super-tiles* which split into finitely many tiles of $\mathcal{T}$.

In the following, we shall see that in the cases when we are not able to prove whether the tiling is combinatoric, it turns out that the tiling is always quasi-combinatoric.

We have the following result:

Theorem 5 – (Margenstern, [6]) *If we consider a triangle with angles* $\frac{2\pi}{h}$, $\frac{2\pi}{k}$ *and* $\frac{2\pi}{\ell}$ *with* $\frac{1}{h}+\frac{1}{k}+\frac{1}{\ell}<\frac{1}{2}$, *the tiling which is generated by the recursive reflections of this triangle is always quasi-combinatoric.*

It is combinatoric when $h = 2p$, $k = 2q$ *and* $\ell = 2r$ *for the values of* p, q, *and* r *which are indicated in theorem* 4.

When $h = 2p+1$ *and* $k = \ell = 2q$, *and then* $h \geq 3$ *and* $\frac{1}{p}+\frac{1}{q}<\frac{1}{2}$, *the tiling is combinatoric when* $h \geq 5$.

When $h = k = \ell = 2p+1$ *and then* $h \geq 7$, *the tiling is always combinatoric.*

Notice that in the first part of the paper, we proved that if h, k and ℓ are not all even, then either they are all odd, or one of these numbers is odd and the two others are even and equal to each other.

2 Matrices and polynomials

From [5], we know that when a tiling is combinatoric, there is a polynomial which is attached to the spanning tree of the splitting.

More precisely, we have the following result:

Theorem 6 – (Margenstern, [5]) *Let* $\mathcal{T}$ *be a combinatoric tiling, and denote a basis of splitting for* $\mathcal{T}$ *by* S_0, ..., S_k *with* $P_0, \ldots, P_h$ *as its generating tiles. Let* $\mathcal{A}$ *be the spanning tree of the splitting. Let* M *be the square matrix with coefficients* m_{ij} *such that* m_{ij} *is the number of copies of* S_{j-1} *which enter the splitting of* S_{i-1} *in the condition* (ii) *of the definition of a basis of splitting. Then the number of nodes of* $\mathcal{A}$ *of the* n^{th} *generation are given by the sum of the coefficients of the first row of* M^n. *More generally, the number of nodes of the* n^{th} *generation in the tree which is constructed as* $\mathcal{A}$ *but which is rooted in a node being associated to* S_i *is the sum of the coefficients of the* $i+1^{\text{th}}$ *row of* M^n.

This matrix is called the **matrix of the splitting** and we call **polynomial of the splitting** the characteristic polynomial of this matrix, being possibly divided by the greatest power of X which it contains as a factor. Denote the polynomial by P. From P, we easily infer the induction equation which allows us to compute very easily the number u_n of nodes of the n^{th} level of $\mathcal{A}$. This gives us also the number of nodes of each kind at this level by the coefficients of M^n on the first row: we use the same equation with different initial values. The sequence $\{u_n\}_{n\in I\!N}$ is called the **recurrent sequence of the splitting**.

2.1 The matrices of the splittings

In this sub-section, we give the matrix of the splitting in all the cases which are listed by the theorems 4 and 5.

First, we remind that we have two kinds of regions from which we constituted the basis of the splitting as it is indicated in the first part of the paper, see [6].

Below, the figure 1 reminds the definition of the basic regions which we use:

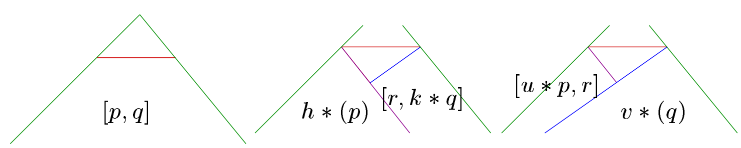


Figure 1. The basic regions:
on the left hand: the splitting of an angular sector,
in the middle, and on the right hand,
two different splittings of a truncated sector

For the general case of the theorem of Poincaré, when $p,q,r \geq 3$, we take the following sets, in this order, as a basis of splitting:

$$(p),\ (q),\ (r),\ [p,q],\ [q,r],\ [r,p],\ [p,2{*}r],\ [2{*}p,q],\ [r,2{*}q].$$

where (p), (q) and (r) are angular sectors of respective angles $\dfrac{\pi}{p}$, $\dfrac{\pi}{q}$ and $\dfrac{\pi}{r}$. Also, in order to simplify the writings, we set $a = p-3$, $b = q-3$ and $c = r-3$. We obtain the 9×9 matrix which is displayed by the table 1.

0	0	0	0	1	0	0	0	0
0	0	0	0	0	1	0	0	0
0	0	0	1	0	0	0	0	0
a+1	0	0	0	0	0	0	0	1
0	b+1	0	0	0	0	1	0	0
0	0	c+1	0	0	0	0	1	0
0	0	c	0	0	0	0	1	0
a	0	0	0	0	0	0	0	1
0	b	0	0	0	0	1	0	0

Table 1. The splitting matrix of the general case, when $p, q, r \geq 3$.

We remind the reader that a line in the table indicates how is split the corresponding region. Take as an example the fourth line, which corresponds to $[p, q]$. We obtain that this truncated sector is split into a copy of T, then $p-2$ copies of (p) and a copy of $[r, 2*q]$. We denoted such a splitting by $[p, q] \Rightarrow (p-2)(p) + [r, 2*q]$ and the presence of the copies of (p) on the left hand is due to the fact that the angle of the truncated on the left side is defined by (p).

In the particular case when $p = 2$ and $q, r \geq 4$ in which the tiling is also combinatoric, as a basis of splitting we have the following sets:

$(\overline{2})$, (q), (r), $[\overline{2}, r]$, $[\overline{2}, q]$,
$[q, r]$, $[2*q, \overline{2}]$, $[\overline{2}, 2*r]$, $[q, 2*r]$, $[q, 3*r]$, $[2*q, r]$, $[r, 3*q]$.

We introduce again a, b and c with the same meaning as previously. We obtain the matrix which is displayed by the table 2.

In the equilateral case, when p is even, taking as a basis of splitting the sets $(\overset{\bullet}{p})$, $[\overset{\bullet}{p}, \overset{\bullet}{p}]$ and $[\overset{\bullet}{p}, 2*\overset{\bullet}{p}]$. When p is odd, we take as a basis of splitting the sets $(\overset{\bullet}{p})$, $[p, p+\overset{\bullet}{p}]$, $[p+\overset{\bullet}{p}, p+\overset{\bullet}{p}]$ and $[p, 2*\overset{\bullet}{p}+p]$. In order to simplify the writings, we set $a = \lfloor \frac{p}{2} \rfloor - 3$. The matrices which we obtain for these two cases are displayed by the tables 3.

0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0
0	0	$c{+}1$	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	1	0	0
0	0	c	0	0	0	1	0	0	0	0	0
0	0	$c{-}1$	0	0	0	1	0	0	0	0	0
0	b	0	0	0	0	0	1	0	0	0	0
0	$b{-}1$	0	0	0	0	0	1	0	0	0	0

Table 2. The splitting matrix of the general case,
when $p = 2$ and $q, r \geq 4$.

0	1	0
$a{+}1$	0	1
a	0	1

0	1	0	0
$a{+}1$	0	1	0
$a{+}1$	0	0	1
a	0	1	0

Tables 3. The splitting matrices of the equilateral case,
to left: p even; to right: p odd.

At last, for the isosceles case, when $p \geq 7$, a basis of splitting is given by the sets:

$$(\overset{\bullet}{p}),\ (q)^-,\ (q)^+,\ [p,q],\ [q,q],\ [p,2*q],\ [\overset{\bullet}{p}{+}p,q],$$
$$[q,\overset{\bullet}{p}{+}p],\ [q,2\overset{\bullet}{p}{+}p],\ [q,2*q],\ [2*q,\overset{\bullet}{p}{+}p],\ [2*q,q].$$

We obtain the matrix which is displayed by the table 4.

In the table 4, we use again the convention that $a = \lfloor\frac{p}{2}\rfloor - 3$, and $b0 = \lfloor\frac{}{2}\rfloor$, $b\epsilon = b0 + (q-3)\ mod\ 2$, and $b1 = b0 + 1$. It is not difficult to see that $\lfloor\frac{}{2}\rfloor = b\epsilon$ and that $b1 = b\epsilon + (q-2)\ mod\ 2$.

Indeed, this comes from the fact that in the splittings $(5.a)$ up to $(5.\ell)$, when a term of the form $(q-a)(q)$ occurs, we have to take into account that each occurrence of (q) is either $(q)^-$ or $(q)^+$. More precisely, $(q-a)(q)$ splits into something of the form $(q_1)\Big((q)^-(q)^+\Big)^{\left\lfloor \frac{(q-a)}{2} \right\rfloor}(q_2)$ where (q_1) is either $\emptyset$ or $(q)^+$ and (q_2) is either $\emptyset$ or $(q)^-$. This explains the rule which is given for $b0$, $b\epsilon$ and $b1$ in the table 4. The justification of the grouping with $(q)^-$ and $(q)^+$ is given by the figures 9 which illustrates the four possible cases of the position of the angle $(\overset{\bullet}{p})$ in T and the side which is chosen for splitting with (q). The corresponding expressions are given by the table 5, below.

0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0
0	$b1$	$b\epsilon$	0	0	0	1	0	0	0	0	0
0	$b1$	$b\epsilon$	0	0	1	0	0	0	0	0	0
0	$b\epsilon$	$b0$	0	0	0	1	0	0	0	0	0
$a+1$	0	0	0	0	0	0	0	0	1	0	0
$a+1$	0	0	0	0	0	0	0	0	0	0	1
a	0	0	0	0	0	0	0	0	0	0	1
0	$b0$	$b\epsilon$	0	0	0	0	0	0	0	1	0
0	$b0$	$b\epsilon$	0	0	0	0	0	1	0	0	0
0	$b\epsilon$	$b0$	0	0	1	0	0	0	0	0	0

Table 4. The splitting matrices of the isosceles case, when $p \geq 7$ and $q \geq 3$.

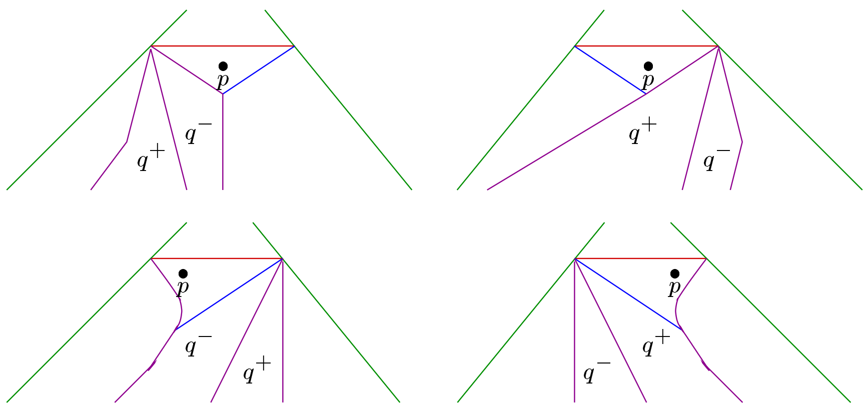
As it can be seen from the figures 9, we have the same expression depending on whether the reflection of T on the side of the splitting into (q)'s yields $(q)^-$ or $(q)^+$. When $(\overset{\bullet}{p})$ is in the left hand in T or when $(\overset{\bullet}{p})$ is in the middle of T and (q)'s are taken in the left side, then the reflection of T yields $(q)^-$. When $(\overset{\bullet}{p})$ is in the right side in T or when $(\overset{\bullet}{p})$ is in the middle of T and (q)'s are taken in the right side, then the reflection of T yields $(q)^+$.

At last, in the isosceles case, consider the case when $p = 5$ and $q \geq 4$. A basis of splitting is then given by the sets:

$(\overset{\bullet}{p})$, $(q)^-$, $(q)^+$, $[p,q]$, $[q,q]$, $[p,2*q]$, $[\overset{\bullet}{p}+p,q]$,
$[q,\overset{\bullet}{p}+p]$, $[q,2*q]$, $[2*q,\overset{\bullet}{p}+p]$, $[2*q,q]$, $[3*q,q]$.

$(\overset{\bullet}{p})$ to left:	$(\overset{\bullet}{p})$ in the middle and (q)'s to left:
$\lfloor\frac{q-a}{2}\rfloor\Big((q)^-(q)^+\Big)+\epsilon\,(q)^-$	$\epsilon\,(q)^- + \lfloor\frac{q-a}{2}\rfloor\Big((q)^+(q)^-\Big)$
$(\overset{\bullet}{p})$ to right:	$(\overset{\bullet}{p})$ in the middle and (q)'s to right:
$\epsilon\,(q)^+ + \lfloor\frac{q-a}{2}\rfloor\Big((q)^-(q)^+\Big)$	$\lfloor\frac{q-a}{2}\rfloor\Big((q)^+(q)^-\Big)+\epsilon\,(q)^+$

Table 5. The expressions in $(q)^-$ and $(q)^+$.



Figures 9 The splittings with $(q)^-$ and $(q)^+$.

We make use of the same notations as in the previous case for $b0$, $b\epsilon$ and $b1$. We notice that $\lfloor\frac{q-4}{2}\rfloor = \lfloor\frac{q-2}{2}\rfloor - 1$ and that in the considered splitting, the (q)'s are on the left side and that T has the angle $\overset{\bullet}{p}$ in the middle. As $p=5$, we have here that $a+1=0$.

Accordingly, we obtain the following matrix:

0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0
0	$b1$	$b\epsilon$	0	0	0	1	0	0	0	0	0
0	$b1$	$b\epsilon$	0	0	1	0	0	0	0	0	0
0	$b\epsilon$	$b0$	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0
0	$b0$	$b\epsilon$	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	1
0	$b\epsilon$	$b0$	0	0	1	0	0	0	0	0	0
0	$b1-1$	$b\epsilon-1$	0	0	1	0	0	0	0	0	0

Table 6. The matrix for the isosceles case,
when $p = 5$ and $q \geq 4$.

2.2 The polynomials and the languages

Theorem 1 was the first implicit application by the author of the splitting method and it appeared in the technical report [8], after which the paper [9] appeared in 2001. In [4], the author significantly improved the method by considering its algebraic consequences. This gave rise to the *Fibonacci technology* which gives a solution to the problem of locating the cells of a cellular automaton. In particular, this solution consists in attaching a language to the tiling. This allowed the author and its co-authors to devise cellular automata in the conditions which are indicated by Theorem 2 and Theorem 3. Here, we shall see what happens if we apply the same idea for the triangular tessellations of the hyperbolic plane.

First, as in [4, 5], number the nodes of $\mathcal{A}$ level by level, starting from the root and, on each level, from left to right. Second, consider the recurrent sequence of the splitting, $\{u_n\}_{n\geq 1}$: it is generated by the polynomial of the splitting. As we shall see, it turns out that the polynomial has a greatest real root β and that $\beta > 1$. The sequence $\{u_n\}_{n\geq 1}$ is increasing. Now, it is possible to represent any positive number n in the form $n = \sum_{i=0}^{k} a_i.u_i$, where $a_i \in \{0..\lfloor\beta\rfloor\}$, see [2], for instance. The string $a_k \ldots a_0$ is called a representation of n. In general, the representation is not unique and it is made unique by an additional condition: we take the representation which is maximal with respect to the lexicographic order on the words on $\{0..b\}$ where $b = \lfloor\beta\rfloor$. The set of these representations is called the **language of the splitting**.

We have the following results:

Theorem 7 – (Margenstern, [4]) *The splitting language of the pentagrid,* $\{5, 4\}$*, is regular.*

Theorem 8 – (Margenstern, Skordev [10]) *The splitting languages of the tilings* $\{s, 4\}$*,* $s \geq 5$*, are all regular.*

Theorem 9 – (Margenstern, [12]) *The splitting language of the tiling* $\{5, 3, 4\}$ *is neither regular nor context-free.*

For what is the tessellations of the hyperbolic plane which we consider and which are based on a triangle, the main result of this paper is the following theorem:

Theorem 10 – *In the conditions of the theorems* 4 *and* 5*, in all the cases when the tiling is combinatoric, the language of the splitting is not regular.*

In this sub-section, we give the polynomials of the splitting in the different cases and we prove the conclusion of the theorem 10 for the corresponding language.

The proof is based on the following consideration:

We know from [3] that a necessary and sufficient condition for the language of the splitting to be regular is that the splitting polynomial has as its roots the conjugates of a Pisot number and, possibly, the non real roots of $X^k - 1$.

And so, it will be enough for us to show that the polynomial of the splitting has always a real root, that the greatest one is positive, and that it is not a Pisot number.

It will be enough to check that in almost all cases, the polynomial of the splitting satisfies the following property: there is a number $a \geq 1$

such that $P(a) > 0$, and there is another number $b < -1$ such that $P(b) > 0$ if P has an odd degree and $P(b) < 0$ if P has an even degree. The first condition with a entails that there is a greatest real root x_0 with $x_0 > 1$. The second condition with b entails that there is a negative root x_1 with $x_1 < -1$. Later on, we shall refer to this condition as the *a and b lemma.*

For the general case, when $\boxed{p, q, r \geq 3}$, in which case we have the matrix of the table 1, the polynomial of the splitting is:

$$P(X) = X^9 - X^6 - (p+q+r-9)X^5 - (pq+qr+rp-5(p+q+r)+18)X^4 \\ -(p-2)(q-2)(r-2)X^3 + 1.$$

Notice that if we consider symmetric images of the same sector as different elements of the basis of splitting, the relations which we obtained in the section 3.2, namely from $(1.a)$ up to $(1.l)$, give rise to a square matrix of dimension 18 whose characteristic polynomial contains $P(X)$ as a factor.

Proposition 1 – *In the case when p, q, $r \geq 3$, the language of the splitting which is associate to the tessellation of the triangle with angles $\frac{\pi}{p}$, $\frac{\pi}{q}$ and $\frac{\pi}{p}$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ is not regular.*

Proof. It is enough to prove that P satisfies the a and b lemma. Let us introduce the following notations:

$$s = p + q + r - 6,$$
$$t = (p-2)(q-2)+(q-2)(r-2)+(r-2)(p-2),$$
$$w = (p-2)(q-2)(r-2).$$

Then $P(X) = X^9 - X^6 - (s-3)X^5 - (t-s)X^4 - wX^3 + 1$.

We have: $P(1) = -(s-3) - (t-s) - w + 1 = -t - w + 4$. And so, as $s \geq 3$, $t \geq 3$ and $w \geq 1$, $P(1) \geq 0$. Indeed, when $w = 1$, necessarily, $p = q = r = 3$, $s = 3$ and $t = 3$ and we get that $P(1) = 0$. Indeed, we get that $P(X) = (X+1)(X-1)^2(X^2-X+1)(X^2+X+1)^2$. Now, from [3], as the polynomial has $X+1$ and $X-1$ as factors, the language of fthe splitting is not regular for this case.

For the other cases, $w > 1$ and so, $P(1) < 0$.

Next, $P(-1) = -1-1+(s-3)-1)-(t-s)+w+1 = w+2s-t-4$. Notice that when p, q, $r \geq 5$, $\frac{1}{p-2} + \frac{1}{q-2} + \frac{1}{r-2} \leq 1$. Accordingly,

$t = w.(\frac{1}{p-2} + \frac{1}{q-2} + \frac{1}{r-2})$ and so, we obtain $t \leq w$ under these conditions. As a consequence, $P(-1) = (w-t) + 2(s-2) > 0$.

We have now to examine the cases when one of the numbers p, q, r is less than 5. We may assume that $p \leq q \leq r$ and so $p \geq 4$.

We already dealt with the case when $p = q = r = 3$.

First assume that $p = 3$ and $p = q = r$ is ruled out, we may assume $s \geq 4$. We have $w = (q-2)(r-2)$, $t = (q-2)(r-2) + (q-2) + (r-2)$ and so, $P(-1) = (w-t) + 2(s-2) = s + (p-2) - 4 = s + 3 - 6 \geq 1$.

Assume now that $p = 4$ and that q, $r \geq 4$. This time, we obtain: $w = 2(q-2)(r-2)$, $t = (q-2)(r-2) + 2((q-2) + (r-2))$. And so, $P(-1) = (w-t) + 2(s-2) = (q-2)(r-2) + 2(p-2) - 4 \geq 4$.

As all cases were examined, we obtain the conclusion of the theorem. ■

Consider again the general case of Poincaré's theorem, this time when $\boxed{p = 2 \text{ and } q, r \geq 4}$ which corresponds to the matrix of the table 2. The polynomial of the splitting is now:

$$P(X) = X^{10} - X^6 - (q+r-8)X^5 - (q-3)(r-3)X^4 + 1.$$

Proposition 2 – *In the case when $p = 2$ and q, $r \geq 4$, the language of the splitting which is associate to the tessellation of the triangle with angles $\frac{\pi}{p}$, $\frac{\pi}{q}$ and $\frac{\pi}{p}$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ is not regular.*

Proof. It is enough to prove that P satisfies the assumptions of the a and b lemma for a polynomial with an even degree.

We get that $P(1) = -(q+r-8) - (q-3)(r-3) + 1$, and as $q, r \geq 4$ and $q + r \geq 9$, we obtain that $P(1) < 0$. At the same time, $P(-1) \geq 2(r-4) - (q-3)(r-3) + 1 = (r-3)(2-(q-3)) - 2 + 1 = -(r-3)(q-5) - 1$. This gives us $P(-1) < 0$ for $q \geq 5$. If $q = 4$, then $r \geq 5$ and we get: $P(1) = -(r-4) - (r-3) + 1 = -2r + 8 \leq -2$. ■

Notice that in the case when $q = 4$ and $r = 5$, which corresponds to a triangle which generates the pentagrid, we obtain the following polynomial:

$$P(X) = X^5 - X - 1,$$

as long as X becomes a factor. This new form of P is irreducible and its greatest root is not a Pisot number. Consequently, the corresponding language is not regular, see [3]. However, as it is indicated by the theorem 8, the splitting which is directly associated to the regular rectangular pentagon is combinatoric and its associated language is regular.

Consider now the equilateral case, when p is even. The corresponding matrix is given by one of the tables 3. The polynomial of the splitting is now:

$$P(X) = X^3 - X^2 - (a+1)X + 1,$$

where $p = 2(a+3)$.

Proposition 3 – *In the case of an equilateral triangle T with a vertex angle of $\frac{\pi}{k}$, with $k \geq 4$, the language of the splitting which is associate to the tessellation of T is not regular.*

Proof. Notice that $p = 2k$, which allows to make the correspondence with the right expression of P. It is enough to prove that P satisfies the assumptions of the a and b lemma for a polynomial with an odd degree.

Notice that $P(1) = -a$ and so, it is negative. Next, we obtain that $P(-1) = -1 + a + 1 = a > 0$. ■

In the equilateral case, when p is odd, the matrix is given by the other array in the tables 3. The polynomial of the splitting is:

$$P(X) = X^4 - (a+2)X^2 - (a+1)X + 1.$$

where $a = \lfloor \frac{p}{2} \rfloor - 3$.

Proposition 4 – *In the case of an equilateral triangle T with a vertex angle of $\frac{2\pi}{p}$, with $p \geq 7$, the language of the splitting which is associate to the tessellation of T is not regular.*

Proof. It is enough to prove that P satisfies the assumptions of the a

and b lemma for a polynomial with an even degree.

$$P(1) = 1 - (a+2) - (a+1) + 1 = -2a - 1 < 0.$$

Next, $P(-1) = 1 - (a+2) + (a+1) + 1 = 1 > 0$. However, an easy computation shows that $P(-\frac{3}{2}) = \frac{49}{16} - \frac{3a}{4}$, and so, $P(-\frac{3}{2}) < 0$ for $a \geq 5$. Clearly, this modification of the assumptions of the lemma is valid for its conclusion. For $a \in \{0..4\}$, a computation under *Maple*8 shows that there is a single real root x_0 which satisfies $x_0 > 1$, and that there is always a complex root z_0 with $|z_0| > 1$ and so, x_0 is neither a Pisot nor a Salem number. ■

Consider now the case of the isosceles triangle, when p is odd, $p \geq 7$. The corresponding matrix is given by the table 4 and the polynomial of the splitting is:

$$P(X) = X^{11} - b\epsilon X^9 - b\epsilon X^8 + a_7X^7 + a_6X^6 + a_5X^5 \\ + a_4X^4 + a_3X^3 + a_2X^2 + a_1X + a_0$$

the polynomial being obtained from the characteristic polynomial by a division by X. The coefficients of the polynomial are given by the following expressions:

$$a_7 = -((a+3)b0 + 2(a+1) + 2\epsilon)$$
$$a_6 = -2(a+2)b0 + 2(b0 - (a+1))\epsilon$$
$$a_5 = -(2b0+1) + (2b0 + a + 2)\epsilon$$
$$a_4 = -(2(a+2)b0 + 2a + 1) + (2(a+2)b0 + 2a + 3)\epsilon$$
$$a_3 = -2(a+1)b0 + (2(a+3)b0 + a + 4)\epsilon$$
$$a_2 = -b0 + (2(a+2)b0 + 2a + 1)\epsilon$$
$$a_1 = 1 - (a+1)b0 + (-1 + 2(a+1)b0)\epsilon$$
$$a_0 = b0 - (2b0+1)\epsilon$$

Proposition 5 – *In the case of an isosceles triangle T with a vertex angle of $\frac{2\pi}{p}$, with $p \geq 7$, and a basis angle of $\frac{\pi}{q}$ with $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$, the language of the splitting which is associate to the tessellation of T is not regular.*

Proof. It is enough to prove that P satisfies the assumptions of the a and b lemma for a polynomial with an odd degree.

We shall distinguish two cases, depending on whether $\epsilon = 0$ or $\epsilon = 1$. First consider the $\boxed{\text{case } \epsilon = 0}$.

We have now that the polynomial is

$$P(X) = X^{11} - b0X^9 - b0X^8 + a_{7_0}X^7 + a_{6_0}X^6 + a_{5_0}X^5 + a_{4_0}X^4 + a_{3_0}X^3 + a_{2_0}X^2 + a_{1_0}X + a_{0_0}$$

with, this time:

$$\begin{aligned} a_{7_0} &= -((a+3)b0 + 2(a+1)) & a_{3_0} &= -2(a+1)b0 \\ a_{6_0} &= -2(a+2)b0 & a_{2_0} &= -b0 \\ a_{5_0} &= -(2b0+1) & a_{1_0} &= 1-(a+1)b0 \\ a_{4_0} &= -(2(a+2)b0 + 2a + 1) & a_{0_0} &= b0 \end{aligned}$$

Notice that almost all coefficients of P are negative.

Now, $P(1) = -(9b0 + 4ab0 + 2a + 1) < 0$ and so, P has a greatest real root x_0 with $x_0 > 1$. A simple computation shows that $P(-1) = 0$, whatever $b0$ and a are. As -1 is a root of P, x_0 cannot be a Pisot number.

Now, consider the $\boxed{\text{case } \epsilon = 1}$.

We have now that the polynomial is

$$P(X) = X^{11} - b1X^9 - b1X^8 + a_{7_1}X^7 + a_{6_1}X^6 + a_{5_1}X^5 + a_{4_1}X^4 + a_{3_1}X^3 + a_{2_1}X^2 + a_{1_1}X + a_{0_1}$$

with, this time:

$$\begin{aligned} a_{7_1} &= -((a+3)b0 + 2(a+2)) & a_{3_1} &= 4b0 + a + 4 \\ a_{6_1} &= -2(a+1)(b0+1) & a_{2_1} &= (2a+3)b0 + 2a + 1 \\ a_{5_1} &= a+1 & a_{1_1} &= (a+1)b0 \\ a_{4_1} &= 2 & a_{0_1} &= -(b0+1) \end{aligned}$$

We have that $P(1) = 0$ for all values of $b0$ and of a and we have that $P(-1) = -2(b0+1) < 0$. The fact that 1 is a root shows that the greatest real root is not a Pisot number if that root is greater than 1. But we have that $P(2) = 739 - 336a - (246a + 1235)b0$. This means that when $b0 \geq 1$, $P(2) < 0$ and so the greatest real root x_0 exists and it satisfies $x_0 > 2$. When $b0 = 0$, we have now that $P(2) = 739 - 336a$ and so, $P(2) < 0$ as soon as $a \geq 3$. And so, we remain with three cases, according to whether a is 0, 1 or 2.

Next, we notice that $P(-2) = (6a + 489)b0 + 96a - 1437$. This means that $P(-2) > 0$ when $b0 \geq 3$. When $0 \leq b0 < 3$, we have only finitely many cases when $P(-2) \leq 0$. Indeed, $P(-2) > 0$ for $b0 = 0$ and $a \geq 15$, $b0 = 1$ and $a \geq 10$, $b0 = 2$ and $a \geq 5$.

It remains to look at the particular cases:

We remained with the cases when $b0 = 0$ and $0 \leq a \leq 14$ for $P(-2) > 0$, which contains also a particular case which we considered for $P(2) < 0$. We have also to consider the case when $b0 = 1$ with $0 \leq a \leq 9$ and the case when $b0 = 2$ with $0 \leq a \leq 4$. In all these cases, the particular values of a and $b0$ provide us with an explicit polynomial for which we may compute the values. Using *Maple*8 to that purpose, we obtain the data which are displayed by the tables 7, 8 and 9. The tables indicate the polynomial, its greatest real root x_0, the smallest real root x_1, and a complex root z with $|z| > 1$. ■

In the case of the isosceles triangle, when $p = 5$, in which the corresponding matrix is given by the table 6, the polynomial of the splitting is:

$$\begin{aligned} P(X) = X^{10} &- b\epsilon X^8 - b\epsilon X^7 - 2b\epsilon X^6 + (-(1+b0)+(2b0+1)\epsilon)X^5 \\ &+(-3b0+1+2b0\epsilon)X^4 + 2(b0+1)\epsilon X^3 \\ &+(-b0+(2b0+1)\epsilon)X^2 + (-b0+2b0\epsilon)X + 1 - 2\epsilon, \end{aligned}$$

this polynomial being obtained from the characteristic polynomial of the matrix by a division by X^2.

Proposition 6 – *In the case of an isosceles triangle T with a vertex angle of $\dfrac{2\pi}{p}$, with $p = 5$, and a basis angle of $\dfrac{\pi}{q}$ with $q \geq 4$, the language of the splitting which is associate to the tessellation of T is not regular.*

Proof. It is enough to prove that P satisfies the assumptions of the a and b lemma for a polynomial with an even degree.

As in the case when $p \geq 7$, we shall distinguish two cases, depending on whether $\epsilon = 0$ or $\epsilon = 1$.

First consider the case $\epsilon = 0$.

We have now that the polynomial is

$$P(X) = X^{10} - b0X^8 - b0X^7 - 2b0X^6 - (1 + b0)X^5$$
$$-(3b0 - 1)X^4 - b0X^2 - b0X + 1,$$

Now, $P(-1) = 4 - 4b0 < 0$ when $b0 \geq 2$. We have that

$$P(-\frac{21}{20}) = \frac{52435788178201}{10240000000000} - \frac{132436057341b0}{25600000000},$$

and it turns out that $\frac{52435788178201}{10240000000000} < \frac{132436057341}{25600000000}$.

This means that $P(-\frac{21}{20}) < 0$ when $b0 \geq 1$.

Also, $P(1) = 2 - 10b0 < 0$ when $b \geq 1$.

Notice that when $\epsilon = 0$ we have necessarily that $b0 \geq 1$ and so, the theorem is proved in that case.

Now, consider the $\boxed{\text{case } \epsilon = 1}$.

We have now that the polynomial is

$$P(X) = X^{10} - (b0+1)X^8 - (b0+1)X^7 - 2(b0+1)X^6 + b0X^5$$
$$-(b0-1)X^4 + 2(b0+1)X^3 + (b0+1)X^2 + b0X - 1,$$

Notice that $P(1) = 0$ for any $b0$, which already rules out a greatest real root as a Pisot number if the greatest real root has its value greater than 1. But, $P(2) = 547 - 474b0$ and so, for $b0 \geq 2$, $P(2) < 0$. This leaves us with $b0 = 0$ and $b0 = 1$ to be examined. Notice that as $\epsilon = 1$, the value $b0 = 0$ is possible here.

On another hand, $P(-1) = -6b0 - 2 < 0$. And so, when $b \geq 2$, the greatest real root x_0 satisfies $x_0 > 1$ (even $x_0 > 2$). There is also a smallest real root x_1 with $x_1 < -1$.

When $b0 = 1$ the polynomial is:

$$P(X) = X^{10} - 2X^8 - 2X^7 - 4X^6 + X^5 + 4X^3 + 2X^2 + X - 1,$$

whose greatest real root is 1.941130349, whose smallest real root is 0.4079364160 and which has several complex roots with a modulus being greater than 1.

b	a	polynom	$x0$	$x1$	z
0	0	$X^{11}-X^9-X^8-4X^7-2X^6+X^5+2X^4+4X^3+X^2-1$	1.70989..	.534984..	$-1.193..-i.213..$
	1	$X^{11}-X^9-X^8-6X^7-4X^6+2X^5+2X^4+5X^3+3X^2-1$	1.86365..	.428794..	$-1.228..-i.232..$
	2	$X^{11}-X^9-X^8-8X^7-6X^6+3X^5+2X^4+6X^3+5X^2-1$	1.97968..	.366369..	$-1.270..-i.241..$
	3	$X^{11}-X^9-X^8-10X^7-8X^6+4X^5+2X^4+7X^3+7X^2-1$	2.07470..	.324561..	$-1.314..-i.235..$
	4	$X^{11}-X^9-X^8-12X^7-10X^6+5X^5+2X^4+8X^3+9X^2-1$	2.15604..	$-.72347..$	$-1.356..-i.219..$
	5	$X^{11}-X^9-X^8-14X^7-12X^6+6X^5+2X^4+9X^3+11X^2-1$	2.22762..	$-.79371..$	$-1.395..-i.193..$
	6	$X^{11}-X^9-X^8-16X^7-14X^6+7X^5+2X^4+10X^3+13X^2-1$	2.29184..	$-.83286..$	$-1.430..-i.156..$
	7	$X^{11}-X^9-X^8-18X^7-16X^6+8X^5+2X^4+11X^3+15X^2-1$	2.35027..	$-.85879..$	$-1.462..-i.100..$
	8	$X^{11}-X^9-X^8-20X^7-18X^6+9X^5+2X^4+12X^3+17X^2-1$	2.40402..	$-1.5642..$	$.1574..+i2.074..$
	9	$X^{11}-X^9-X^8-22X^7-20X^6+10X^5+2X^4+13X^3+19X^2-1$	2.45389..	$-1.6652..$	$.1611..+i2.128..$
	10	$X^{11}-X^9-X^8-24X^7-22X^6+11X^5+2X^4+14X^3+21X^2-1$	2.50048..	$-1.7390..$	$.1645..+i2.179..$
	11	$X^{11}-X^9-X^8-26X^7-24X^6+12X^5+2X^4+15X^3+23X^2-1$	2.54425..	$-1.8020..$	$.1674..+i2.226..$
	12	$X^{11}-X^9-X^8-28X^7-26X^6+13X^5+2X^4+16X^3+25X^2-1$	2.58559..	$-1.8582..$	$.1701..+i2.270..$
	13	$X^{11}-X^9-X^8-30X^7-28X^6+14X^5+2X^4+17X^3+27X^2-1$	2.62478..	$-1.9095..$	$.1726..+i2.312..$
	14	$X^{11}-X^9-X^8-32X^7-30X^6+15X^5+2X^4+18X^3+29X^2-1$	2.66208..	$-1.9569..$	$.1748..+i2.352..$

Table 7. The polynomials for $b0 = 0$, case $p \geq 7$

b	a	polynom	$x0$	$x1$	z
1	0	$X^{11}-2X^9-2X^8-7X^7-4X^6$ $+X^5+2X^4+8X^3+4X^2+X-2$	2.15549..	−1.3765..	.1766.. + i1.202..
	1	$X^{11}-2X^9-2X^8-10X^7-8X^6$ $+2X^5+2X^4+9X^3+8X^2+2X-2$	2.29137..	.342216..	−1.401..−i.0458..
	2	$X^{11}-2X^9-2X^8-13X^7-12X^6$ $+3X^5+2X^4+10X^3+12X^2+3X-2$	2.40030..	.279165..	−1.443..−i.0361..
	3	$X^{11}-2X^9-2X^8-16X^7-16X^6$ $+4X^5+2X^4+11X^3+16X^2+4X-2$	2.49241..	−1.5422..	.1513.. + i1.8207..
	4	$X^{11}-2X^9-2X^8-19X^7-20X^6$ $+5X^5+2X^4+12X^3+20X^2+5X-2$	2.57287..	−1.6328..	.1669.. + i1.9278..
	5	$X^{11}-2X^9-2X^8-22X^7-24X^6$ $+6X^5+2X^4+13X^3+24X^2+6X-2$	2.64467..	−1.7145..	.1791.. + i2.0196..
	6	$X^{11}-2X^9-2X^8-25X^7-28X^6$ $+7X^5+2X^4+14X^3+28X^2+7X-2$	2.70976..	−1.7895..	.1890.. + i2.100..
	7	$X^{11}-2X^9-2X^8-28X^7-32X^6$ $+8X^5+2X^4+15X^3+32X^2+8X-2$	2.76947..	−1.8588..	.1972.. + i2.173..
	8	$X^{11}-2X^9-2X^8-31X^7-36X^6$ $+9X^5+2X^4+16X^3+36X^2+9X-2$	2.82475..	−1.9232..	.2041.. + i2.239..
	9	$X^{11}-2X^9-2X^8-34X^7-40X^6$ $+10X^5+2X^4+17X^3+40X^2+10X-2$	2.87630..	−1.9831..	.2101.. + i2.300..

Table 8. The polynomials for $b0 = 1$, case $p \geq 7$

When $b0 = 0$ the polynomial is:

$$P(X) = X^{10} - X^8 - X^7 - 2X^6 + X^4 + 2X^3 + X^2 - 1.$$

As being computed by *Maple*8, its greatest real root is 1.429660343, and its smallest real root is 0.6743321665. Moreover, it has several complex roots with a modulus being greater than 1.

Accordingly, we proved that in all cases, the greatest real root of the polynomial of the splitting is neither a Pisot number nor a Salem number and so, the language of the splitting is not regular. ■

b	a	polynom	$x0$	$x1$	z
2	0	$X^{11}-3X^9-3X^8-10X^7-6X^6$ $+X^5+2X^4+12X^3+7X^2+2X-3$	2.47521..	$-1.7686..$	$.1983..+i1.199..$
	1	$X^{11}-3X^9-3X^8-14X^7-12X^6$ $+2X^5+2X^4+13X^3+13X^2+4X-3$	2.60351..	$-1.8155..$	$.0964..+i1.046..$
	2	$X^{11}-3X^9-3X^8-18X^7-18X^6$ $+3X^5+2X^4+14X^3+19X^2+6X-3$	2.70931..	$-1.8685..$	$.1343..+i1.762..$
	3	$X^{11}-3X^9-3X^8-22X^7-24X^6$ $+4X^5+2X^4+15X^3+25X^2+8X-3$	2.80029..	$-1.9257..$	$.1615..+i1.903..$
	4	$X^{11}-3X^9-3X^8-26X^7-30X^6$ $+5X^5+2X^4+16X^3+31X^2+10X-3$	2.88066..	$-1.9850..$	$.1812..+i2.0193..$

Table 9. The polynomials for $b0 = 2$, case $p \geq 7$

In the tables 7, 8 and 9, we notice that the smallest real root, x_1, satisfies the condition $|x_1| < x_0$ in twenty cases while this is not the case in ten other ones. Indeed, it can be proved that the condition is satisfied in all the other cases which are stated in the propositions 1 up to 6.

It is now plain that the theorem 10 is an immediate corollary of the propositions from 1 to 6.

Notice also that, in the case when $p = 2$ and $q = 4$, $r \geq 5$, it is possible to define a sub-tiling of the tessellation by grouping together two copies of T in such a way that one side of the right angle is continued by the side of the same length of the other triangle: this gives us an isosceles triangle with the angle $\frac{\pi}{4}$ as the basis angle. Now, grouping r such triangles around the vertex with the angle $\frac{\pi}{r}$, we obtain a tiling with the regular rectangular polygon with r sides. Now, we now, from the theorem 8, that the language being associated with this new tiling is regular. And so, a combinatoric tiling for which the language is not regular may contain a combinatoric sub-tiling whose associated language is regular.

Notice that the computations of this section, in particular the computation of the polynomials of the splitting from the splitting matrix have been performed or checked by *Maple*8.

3 Conclusion

As the conclusion of this second part of the paper, we would like to draw the attention on the possible continuation of this work on its algebraic side.

Is it possible to obtain more information on the languages which are associated to the splittings which we considered for the tessellations being based on triangles? We proved that these languages are not regular when the tiling is combinatoric. However, when the tiling is only quasi-combinatoric, the combinatoric sub-tiling which we obtained is associated with a splitting whose language *is* regular. On another side, the result of the author which is obtained for the hyperbolic three-dimensional tiling $\{5, 3, 4\}$ points to another direction: indeed, for that tiling, the language of the splitting is not only not regular, it is not context-free. Now, if we consider the polynomial of the splitting, its greatest real root is not a Pisot number, but it is a Salem number and not very far from being a Pisot number, as 1 is one if its root. Here, almost all the polynomials which we met have the property that they have at least two real roots which are in modulus greater than 1, the greatest one being positive. And so, this property seems to point at the conjecture that these languages are also not context-free. A first step could be to show that the greatest root of the polynomial which we met in this paper is always a real Perron number as it seems to be the case.

Also, since the time of the publication of the first part of the paper, the author could continue the work in the direction of the implementation of cellular automata in the triangular grids of the hyperbolic plane. Some interesting details which lie on the ground of this work and on the developments of [4] are given in [7], to appear.

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