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Note on the n-cycles and their achromatic numbers

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Abstract

For an n-cycle C, with achromatic number $\psi(C)$, we show that

$$n \geq \begin{cases} \psi(C)[\psi(C)-1]/2, & \text{if } \psi(C) \text{ is odd,} \\ [\psi(C)]^2/2, & \text{if } \psi(C) \text{ is even.} \end{cases}$$

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1 Introduction

In this note, our graph-theoretic terminology is fairly standard, graphs considered here being finite and undirected. For a graph $G = (V, E)$, without loops or multiple edges, with V as the vertex set and E as the edge set, by an *achromatic k-colouring* of G, we mean a function $f : V \to \{1, 2, ..., k\}$ such that

(a) if $(x, y) \in E$, then $f(x) \neq f(y)$

(b) for every $1 \leq i < j \leq k$, there exist $x, y \in V$ such that $(x, y) \in E$ and $f(x) = i$, $f(y) = j$.

The *achromatic number* [1] of G, $\psi(G)$, is defined to be the greatest integer k for which an achromatic k-colouring of G exists. Obviously, by the definition of $\psi(G)$, we have

$$|E| \geq C^2_{\psi(G)} = \psi(G)[\psi(G)-1]/2. \tag{1}$$

From (1), by a straightforward calculus, we obtain

$$\psi(G) \leq (1+\sqrt{1+8|E|})/2. \tag{2}$$

Remark 1. Let k be a non-negative integer and C an n-cycle (an elementary undirected cycle of length n) such that $n = k(k-1)/2$. Obviously, there exists an achromatic k-colouring of C and, therefore, $\psi(C) \geq k$. On the other hand, by (1), we have $k \geq \psi(C)$ and, hence, $\psi(C) = k$.

2 Main result

The main result of this paper consists of the following

Theorem 1 *If C is an n-cycle with achromatic number $\psi(C)$, then*

$$n \geq \begin{cases} \psi(C)[\psi(C)-1]/2, & \text{if } \psi(C) \text{ is odd,} \\ [\psi(C)]^2/2, & \text{if } \psi(C) \text{ is even.} \end{cases}$$

Let f be an achromatic k-colouring of a graph $G = (V, E)$. We shall denote by $\mathbf{K}(f, k, G)$ the complete multigraph with vertices labelled $1, 2, ..., k$, such that $(x, y) \in E$ if and only if $(f(x), f(y))$ is an edge of $\mathbf{K}(f, k, G)$, and $(f(x), f(y))$ is a multiple edge of order t if there exist $(x_i, y_i) \in E$, $i = 1, 2, ..., t$, with $f(x_i) = f(x)$ and $f(y_i) = f(y)$, for every $i = 1, 2, ..., t$.

Remark 2. Obviously, if $C = (x_1, x_2, \ldots, x_n)$ is an n-cycle and f is an achromatic k-colouring of C, then $(f(x_1), f(x_2), ..., f(x_n))$ is an Euler cycle of $\mathbf{K}(f, k, C)$. Thus, every vertex of $\mathbf{K}(f, k, C)$ is of an even degree. Hence, if $\mathbf{K}(k)$ is a complete multigraph with k vertices, every one of an even degree, then there exists a cycle C and an achromatic k-colouring f of it, such that $\mathbf{K}(k) = \mathbf{K}(f, k, C)$.

Proof of the Theorem. Let k be a non-negative integer and $\mathbf{K}_k$ the complete simple graph on k vertices. If k is even, then, by adding to

$\mathbf{K}_k$ a set of edges that form a perfect matching, we obtain a complete multigraph with k vertices, $\mathbf{K}^*(k)$, for which every vertex is of an even degree, equal to $k^2/2$. Thus, by Remark 2, there exists an n-cycle ($n = k^2/2$), C and an achromatic k-colouring f of it, such that $\psi(C) \geq k$ and $\mathbf{K}^*(k) = \mathbf{K}(f, k, C)$. Hence, by (2), we have $\sqrt{2n} \leq \psi(C) \leq (1 + \sqrt{1 + 8n})/2$, that is, $\psi(C) = \sqrt{2n}$ or $n = [\psi(C)]^2/2$. Moreover, C is the smallest cycle having an achromatic k-colouring.

If k is odd, then $\mathbf{K}_k$ has all its vertices of an even degree. Moreover, for the n-cycle C with $n = k(k-1)/2$ and the achromatic k-colouring f, we have $\mathbf{K}_k = \mathbf{K}(f, k, C)$. But, for such a cycle, by Remark 1, $\psi(C) = k$.

So, summarizing and having in view (1), the theorem is proved.

Remark 3. In [2], is given a necessary and sufficient condition in which all the elementary cycles of a connected graph $G = (V, E)$ have lengths $\leq k$, where k is a non-negative integer with $2 < k \leq |E|/2$.

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