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The splitting method and Poincaré's theorem: (I) – the geometric part

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Abstract

In this paper we revisit Poincaré's theorem in the light of the *splitting method* which was introduced by the author in [3]. This led to the definition of *combinatoric* tilings.

We show that all tessellations which are constructed on a triangle with interior angles $\frac{\pi}{p}$, $\frac{\pi}{q}$ and $\frac{\pi}{p}$ with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1$ are combinatoric, except when $p = 2$ and $q = 3$. At the price of a small extension of the definition of a combinatoric tiling, which we call *quasi-combinatoric*, we show that all tessellations with the above numbers p, q and r are quasi-combinatoric for all possible values of p, q and r, the case when $p = 2$ and $q = 3$ being included.

As a consequence, see [3, 8], there is a bijection of the tiling being restricted to an angular sector S_0 and a tree which we call the *spanning tree of the splitting*. Accordingly, there is also a polynomial $P_{p,q,r}$ which allows us to compute the number of triangles which are associated with the nodes of the $n^{\rm th}$ level in the tree: this will be examined in the second part of the paper.

We also show that the tessellations which are constructed on an isosceles triangle with interior angles $\frac{2\pi}{p}$, p odd, for the vertex angle and $\frac{\pi}{q}$ for the basis angles with $\frac{1}{p}+\frac{1}{q} < \frac{1}{2}$ are quasi-combinatoric and indeed combinatoric for $p \geq 5$. However, all the tessellations which are constructed on an equilateral triangle with interior angle $\frac{2\pi}{p}$ are combinatoric tilings.

*Key-words:*hyperbolic tessellations, algorithmic approach

1 Introduction

Poincaré's theorem is a famous result about tessellations in the hyperbolic plane by triangles.

A tessellation of a polygon is a tiling which is obtained by recursively reflecting it in its sides and the images in their sides: this defines the tiles. The tiling property requires that the interior of the tiles are pairwise disjoint and that any point of the plane belongs to the closure of at least one tile. We say that the considered polygon generates a tiling by tessellation.

Poincaré's theorem, being established in the late 19$^{\text{th}}$ century, says that a triangle T generates a tiling by tessellation if the interior angles of T are of the form $\frac{\pi}{p}$, $\frac{\pi}{q}$ and $\frac{\pi}{p}$ and if p, q and r satisfy the inequality: $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$. Notice that the latter inequality simply says that T is indeed a triangle in the hyperbolic plane.

Several proofs of this result where given, among them elementary ones, see, for instance, [1].

In this paper, we revisit the proof of this theorem in the light of the new method which was introduced by the author in [3] and which we call the **splitting method**.

2 The splitting method

It lies on the following notion which is a generalisation of [3]:

Definition 1 – *Consider finitely many sets S_0, …, S_k of some geometric metric space X which are supposed to be closed with non-empty interior, unbounded and simply connected. Consider also finitely many closed simply connected bounded sets P_1, …, P_h with $h \leq k$. Say that the S_i's and P_ℓ's constitute a* **basis of splitting** *if and only if:*

(*i*) *X splits into finitely many copies of S_0,*

(*ii*) *any S_i splits into one copy of some P_ℓ and finitely many copies of S_j's,* *where* **copy** *means an* **isometric image***, and where, in the condition* (ii)*, the copies may be of different* S_j*'s,* S_i *being possibly included.*

As usual, it is assumed that the interiors of the copies of $\mathcal{T}$ *and the copies of the* S_j*'s are pairwise disjoint.*

The set S_0 *is called the* **head** *of the basis and the* P_ℓ*'s are called the* **generating tiles***.*

Consider a basis of splitting of X, if any. We recursively define a tree A which is associated with the basis as follows. First, we split S_0 according to the condition (ii) of Definition 1. This gives us a copy of say P_0 which we call the *root* of A and which we call also the *leading tile* of S_0. In the same way, by the condition (ii) of Definition 1, the splitting of each S_i provides us with a copy of some P_ℓ which we call the *leading tile* of S_i. The splitting provides us also with k_i *regions*, S_{i_1}, …, $S_{i_{k_i}}$ which enter the splitting of S_i. The regions which enter the splitting of S_0 according to the condition (ii) of Definition 1 are called the *regions* of the first generation. Assume that we have all the regions of the n^{th} generation: S_{n_1}, …, $S_{n_{m_n}}$. By definition, their leading tiles constitute the nodes of the n^{th} generation. We split again these S_j's according to the condition (ii). We obtain m_n tiles which are called the tiles of the $n{+}1^{\text{th}}$ generation and, for each S_{n_h} which is some S_i, we have a splitting which is the isometric image of the splitting of S_i as it is above indicated. We say that the leading tiles of these copies of the splitting of S_i are called the *sons* of the leading tile of S_{n_h}. By definition, the sons of the leading tile of S_{n_h} belong to the $n{+}1^{\text{th}}$ generation. The union of all the sons of the nodes of the n^{th} generation constitutes the nodes of the $n{+}1^{\text{th}}$ generation.

This recursive process generates an infinite tree with finite branching. This tree, A, is called the **spanning tree of the splitting**, where the *splitting* refers to the basis of splitting $S_0, \ldots, S_k$.

Definition 2 – *Say that a tiling of* X *is* **combinatoric** *if it has a basis of splitting and if the spanning tree of the splitting yields exactly the restriction of the tiling to* S_0*, where* S_0 *is the head of the basis.*

In this paper, we consider only the case when we have a single generating tile, *i.e.* when $h = 1$.

In previous works by the author and some of its co-authors, a lot of partial corollaries of that result were already proved as well as the extension of this method to other cases, all in the case when X is the hyperbolic plane or the hyperbolic $3D$ space. Notice that Definitions 1 and 2 are meaningful also when X is an euclidean space and that they can be applied not only to tessellations but also on tilings being generated by a single tile in another way. As an example, take the square grid of the euclidean plane. Define odd columns and even columns by suitable coordinates for the centres of the squares. Then shift all the odd columns vertically with an amplitude of half the length of the square. We get a tiling which is generated by a single tile and which is not a tessellation. It is not very difficult to see that this tiling is combinatoric.

Here, we state the results which were established for $I\!H^2$ and $I\!H^3$:

Theorem 1 – (Margenstern-Morita, [4, 5]) *The tiling* $\{5, 4\}$ *of the hyperbolic plane is combinatoric.*

Theorem 2 – (Margenstern-Skordev, [6]) *The tilings* $\{s, 4\}$ *of the hyperbolic plane are combinatoric, with* $s \geq 5$.

Theorem 3 – (Margenstern-Skordev, [7, 8]) *The tiling* $\{5, 3, 4\}$ *of the hyperbolic* $3D$ *space is combinatoric.*

Theorem 1 is the first implicit application by the author of the splitting method and it appeared in the technical report [4], after which the paper [5] appeared in 2001. In [2], the author significantly improved the method by considering its algebraic consequences. This gave rise to the *Fibonacci technology* which gives a solution to the problem of locating the cells of a cellular automaton. This allowed the author and its co-authors to devise cellular automata in the conditions which are indicated by Theorem 2 and Theorem 3. We turn to this aspect of the question in the second part of the paper to be published with the next issue of the journal.

Turning now to the theorem of Poincaré, recall that this result considers that the angles of the triangle are of the form $\dfrac{\pi}{p}$, $\dfrac{\pi}{q}$ and $\dfrac{\pi}{p}$.

For the tiling property by tessellation, it is needed that they are of the form $\frac{2\pi}{h}$, $\frac{2\pi}{k}$ and $\frac{2\pi}{\ell}$ with the condition $\frac{1}{h}+\frac{1}{k}+\frac{1}{\ell}<\frac{1}{2}$. If h, k and ℓ are all even, we find again the condition of Poincaré's theorem. As announced before, in most cases, the tiling which is generated by the triangle by tessellation is combinatoric. But it is not always the case and we need a weaker notion:

Definition 3 – *Say that a tiling is* **quasi-combinatoric** *if it has a sub-tiling which is combinatoric.*

Recall that a *sub-tiling* of a tiling is a partition of the same set where the members of the partition are unions of tiles of the initial tiling. We also can view a sub-tiling as a partition over the partition which is defined by the tiling.

From the definition of a combinatoric tiling, it is not difficult to see that a sub-tiling of a tiling $\mathcal{T}$ is generated by *super-tiles* which split into finitely many tiles of $\mathcal{T}$.

In the following, we shall see that in the cases when we are not able to prove whether the tiling is combinatoric, it turns out that the tiling is always quasi-combinatoric.

3 The splitting for the triangular tessellations

In this section, we describe the splitting which we associate to each tessellation of $I\!H^2$ being defined by a triangular tile which obeys the condition of Poincaré's theorem. We also describe the splitting in conditions which go a bit beyond the traditional statement of the theorem.

We begin by indicating the *regions* which we take as a basis for the splitting. Then, we shall proceed to the splitting in the general case. In a third sub-section, we introduce new regions which are adapted to several special cases and we produce the corresponding splittings.

In the first subsection, we also fix the notations which will be followed later on, in the whole paper.

3.1 Angular sectors and cut angular sectors

In this sub-section, we deal with the conditions of the theorem of Poincaré as it is stated in the introduction. Namely, the angles of the triangle T under consideration are $\frac{\pi}{p}$, $\frac{\pi}{q}$ and $\frac{\pi}{r}$ where the positive integers p, q and r satisfy $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$.

Consider the pictures which are displayed by the figures 1 and 2.

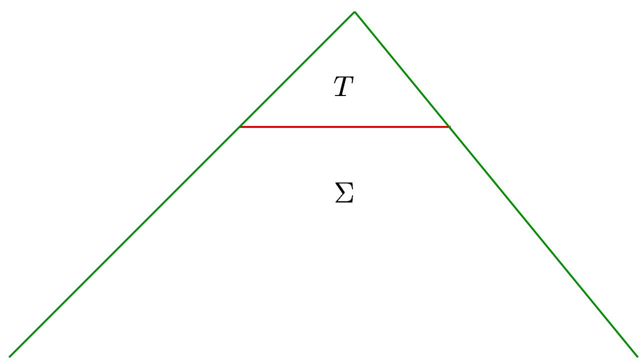

Figure 1. Splitting an angular sector,
Notice the splitting into $T+\Sigma$

In the figure 1, we have an angular sector of $\frac{\pi}{p}$, for instance. The triangle T is inscribed in the sector, and the closure of the complement of the convex hull of T in the sector is a convex region which we call a truncated angular sector, *truncated sector* for short. A truncated sector is always determined as the intersection of two angles which have a common side. The vertices of the angles are at a distance which is one of the sides of T. With respect to the common side, each angle has an absolute value which is $\pi-\frac{h\pi}{v}$ on one side and $\pi-\frac{k\pi}{w}$, where $v, w \in \{p, q, r\}$. As we want that the rays which start from the respective vertices do not meet, neither in the hyperbolic plane nor at infinity, we may assume that $\pi-\dfrac{h\pi}{v}+\pi-\dfrac{k\pi}{w}\geq\pi$ or, in other terms, that $\dfrac{h\pi}{v}+\dfrac{k\pi}{w}\leq 1$. Notice that when the equality holds, the rays are still supported by lines which have a common perpendicular and so, they are non secant. In general, we shall denote such a truncated sector as $[\dfrac{h\pi}{v},\dfrac{k\pi}{w}]$ and also $[h{*}v,k{*}w]$ for short. In the case of Figure 1, we have $\Sigma=[v,w]$ for the truncated sector Σ. On another hand, we shall denote by $\Big(\dfrac{h\pi}{v}\Big)$ the angular sector of angle $\dfrac{h\pi}{v}$, $(h*v)$ for short.

Below, the figures 2 and 3 indicate the way in which a truncated sector $[h*v,k*w]$ can be split. It can be split in different ways as it is stated by the following statement:

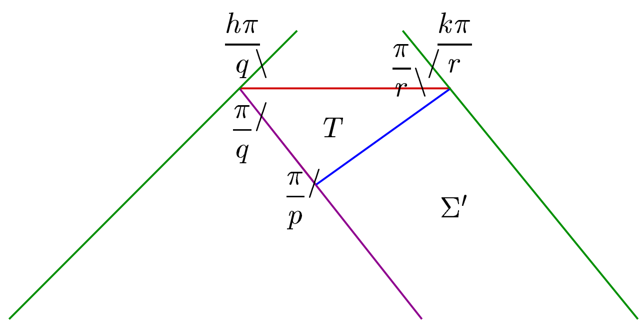

Figure 2. Splitting a truncated angular sector,
The splitting is $T+(q-(h+1))\Big(\dfrac{\pi}{q}\Big)+\Sigma'$

Basic lemma – *Consider three positive integers p, q and r such that $\dfrac{h}{q}+\dfrac{k}{r}\leq 1$. If we have $\dfrac{1}{p}+\dfrac{(k+1)}{r}\leq 1$ then,*

$$(1)\qquad [\frac{h\pi}{q},\frac{k\pi}{r}]=T+(q-(h+1))\Big(\frac{\pi}{q}\Big)+[\frac{\pi}{p},\frac{(k+1)\pi}{r}].$$

On another hand, if we have $\dfrac{(h+1)}{q}+\dfrac{1}{p}\leq 1$, then:

$$(2) \qquad [\frac{h\pi}{q},\frac{k\pi}{r}] = T + [\frac{(h{+}1)\pi}{q},\frac{\pi}{p}] + (r{-}(k{+}1))\Big(\frac{\pi}{r}\Big).$$

Proof. Easy computation being left to the reader. ■

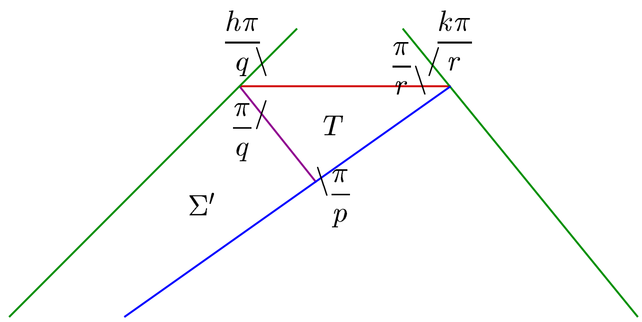

Figure 3. Splitting a truncated angular sector another way,
The splitting is now $T{+}\Sigma'{+}(r{-}(k{+}1))\Big(\frac{\pi}{q}\Big)$

Making use of the short notation which we introduced, we rewrite the relations (1) and (2) as:

$$(1) \qquad [h*q,k*r] \Rightarrow (q{-}(h{+}1))(q) + [p,(k{+}1)*r].$$

$$(2) \qquad [h*q,k*r] \Rightarrow [(h{+}1)*q,p] + (r{-}(k{+}1))(r).$$

Notice that in the new formulation, T is omitted: indeed, we have to consider that it is included in the symbol $\Rightarrow$ which replaces the symbol $=$ of the former formulation.

In the notations (q) and $[h*q,k*r]$, if q or r are replaced by an explicit integer, say n and, possibly, m, we write $(\overline{n})$ and $[h*\overline{n},k*\overline{n}]$.

Notice also, in the statement of the lemma, the different order of the terms between formulas (1) and (2): this corresponds to the orientation of the picture which is settled by a fixed global orientation of $I\!\!H^2$. This is also due to the reflection process which underlies in the construction of the tessellation and which requires that we should be able to split a truncated sector as well as its reflected image.

3.2 The basic case of Poincaré's theorem

There is an easy consequence of the basic lemma:

Corollary – *For all positive integers p, q and r with p, q, $r \geq 3$, and $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1$, we have:*

$$[q, r] \Rightarrow (q{-}2)(q) + [p, 2*r].$$
$$[q, 2*r] \Rightarrow [2*q, p] + (r{-}3)(r).$$

If we have $r = 3$ in the latter relation, we obtain:

$$[q, 2*\overline{3}] \Rightarrow [2*q, p].$$

Proof. As $p, r \geq 3$, we have $\frac{1}{p}+\frac{2}{r} \leq 1$. Consequently, the lemma follows from the formulas (1) and (2) as long as p, q and r play here symmetrical rôles. The case when $r = 3$ can easily be checked by the reader, using the Figure 4, below. ■

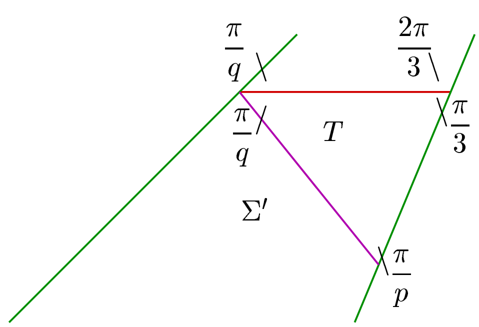

Figure 4. Splitting the truncated sector $[q, 2*\overline{3}]$
The splitting is now $T + (r{-}(k{+}1))\left(\frac{\pi}{q}\right)$

We are now in the position to prove the following property:

Theorem 4 – *The recursive replication of a triangle with angles $\frac{\pi}{p}$, $\frac{\pi}{q}$ and $\frac{\pi}{r}$ which satisfy the relation $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1$ gives rise to a combinatoric tiling for all p, q and r with p, q, $r \geq 3$ and for all q and*

r with q, $r \geq 4$ for $p = 2$. When $p = 2$ and $q = 3$, the tiling is quasi-combinatoric.

Proof. We first consider the $\boxed{\text{case when } p,\ q,\ r \geq 3}$.

We may assume that $p \leq q \leq r$. From the condition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, we obtain that $r \geq 4$.

The Figure 1 gives us the first relation:

$$(1.a) \qquad (p) \Rightarrow [q, r].$$

Then, applying the basic lemma and its corollary, we get, successively:

$$(1.b) \qquad [q, r] \Rightarrow (q{-}2)(q) + [p, 2 * r],$$
$$(1.c) \qquad [p, 2 * r] \Rightarrow [2 * p, q] + (r{-}3)(r),$$
$$(1.d) \qquad [2 * p, q] \Rightarrow (p{-}3)(p) + [r, 2 * q],$$
$$(1.e) \qquad [r, 2 * q] \Rightarrow [2 * r, p] + (q{-}3)(q),$$

At this point, we notice that $[2 * r, p]$ is symmetric to $[p, 2 * r]$ and so, it is a copy of that latter set. By symmetry, from the relations $(1.c)$ $-$ $(1.e)$ we obtain the following ones:

$$(1.f) \qquad [2 * r, p] \Rightarrow (r{-}3)(r) + [q, 2 * p],$$
$$(1.g) \qquad [q, 2 * p] \Rightarrow [2 * q, r] + (p{-}3)(p),$$
$$(1.h) \qquad [2 * q, r] \Rightarrow (q{-}3)(q) + [p, 2 * r].$$

And so, the relations $(1.a)$ $-$ $(1.e)$ are enough to prove the recursion, provided that we give a similar splitting for the other angular sectors (q) and (r) which are involved in the considered formulas.

For the first sector, we get easily that:

$$(1.i) \qquad (q) \Rightarrow [r, p],$$

and then:

$$(1.j) \qquad [r, p] \Rightarrow (r{-}2)(r) + [q, 2 * p],$$

and from that point, by symmetry, we are led to $(1.d)$.

Next for the remaining angular sector we obtain that:

$$(1.k) \qquad (r) \Rightarrow [p, q],$$

and then:

$$(1.\ell) \qquad [p,q] \Rightarrow (p{-}2)(p) + [r, 2*q],$$

and from that point, we are led to $(1.e)$.

We have now to consider the $\boxed{\text{case when } p = 2}$.

The condition on the angles provides us with the following relation:

$$\frac{1}{q} + \frac{1}{r} < \frac{1}{2}.$$

This entails that q, $r \geq 3$ and, in the case when $q = 3$, that $r \geq 7$. As previously, we assume that $q \leq r$.

First we assume $q \neq 3$. Accordingly, $\boxed{q,\ r \geq 4}$.

Applying the technique of the basic lemma and its corollary, we obtain, successively:

$$(2.a) \qquad (\overline{2}) \Rightarrow [q,r].$$

$$(2.b) \qquad [q,r] \Rightarrow [2*q, \overline{2}] + (r{-}2)(r),$$

$$(2.c) \qquad [2*q, \overline{2}] \Rightarrow [3*q, r],$$

$$(2.d) \qquad [3*q, r] \Rightarrow (q{-}4)(q) + [\overline{2}, 2*r],$$

$$(2.e) \qquad [\overline{2}, 2*r] \Rightarrow [q, 3*r],$$

$$(2.f) \qquad [q, 3*r] \Rightarrow [2*q, \overline{2}] + (r{-}4)(r).$$

and from that point, we are led to $(2.c)$.

We notice that the condition on the angles is satisfied: $\dfrac{2}{q} + \dfrac{1}{2} \leq 1$ as far as $q \geq 4$ and $\dfrac{3}{q} + \dfrac{1}{r} \leq 1$ as long as q, $r \geq 4$.

We have to split (q) and (r). As q and r play symmetrical rôles as far as we do not use the hypothesis $q \leq r$, it is enough to decompose (q). We have:

$$(2.g) \qquad (q) \Rightarrow [\overline{2}, r].$$

$$(2.h) \qquad [\overline{2}, r] \Rightarrow [q, 2*r],$$

$$(2.i) \qquad [q, 2*r] \Rightarrow [2*q, \overline{2}] + (r{-}3)(r),$$

Of course, from the relations $(2.g)$–$(2.i)$, we obtain the symmetrical ones which correspond to another displaying of the angular sector (q), namely when the right angle is on the other side with respect to the side being opposite to the angle $\dfrac{\pi}{q}$.

Let us now consider the $\boxed{\text{case when } p=2 \text{ and } q=3}$. From the condition on p, and r, we get that $r \geq 7$.

The above relations become:

$$(3.a) \qquad (\overline{2}) \Rightarrow [\overline{3}, r].$$

$$(3.b) \qquad [\overline{3}, r] \Rightarrow [2*\overline{3}, \overline{2}] + (r-2)(r).$$

But $[2*\overline{3}, \overline{2}]$ is not a region: it is a copy of T. And so, what we obtain does not match the definition of a combinatoric tiling. Accordingly, we have to split $[\overline{3}, r]$ in the other way:

$$(3.b_1) \qquad [\overline{3}, r] \Rightarrow (\overline{3}) + [\overline{2}, 2*r].$$

Next, we necessarily obtain:

$$(3.c) \qquad [\overline{2}, 2*r] \Rightarrow [\overline{3}, 3*r],$$

which, in its turn, generates

$$(3.d) \qquad [\overline{3}, 3*r] \Rightarrow (\overline{3}) + [\overline{2}, 4*r].$$

As long as $\dfrac{h}{r} \leq \dfrac{1}{2}$, we have an infinite region, and the only way to split the truncated sector in order to be conformal to the definition of a combinatoric tiling, is to arrive to $[\overline{2}, (h+1)*r]$.

When $\dfrac{h}{r} > \dfrac{1}{2}$, it is not difficult to see that we cannot extract $(\overline{3})$ from the truncated sector because, starting from that point, the ray of angle $\dfrac{\pi}{3}$ with the basis of the truncated sector cuts the opposite side of the sector which can be easily split into finitely many copies of T. But the other way to split the sector generates immediately two copies of T and so, the definition of a combinatoric tiling is not satisfied.

This is why this tiling is not combinatoric, at least, by defining the regions in this way.

However, we can prove that the tiling is *quasi-combinatoric*. Indeed, six copies of T can be glued around their vertex with angle $\dfrac{\pi}{3}$ in order to constitute the equilateral triangle with angle $\dfrac{2\pi}{r}$. As in the last sub-section of this section we show that the tiling being based on the equilateral triangle is combinatoric, we obtain that the tiling being generated by T is quasi-combinatoric. Accordingly, we postpone the end of the proof to the study of the case of the equilateral triangle. ∎

3.3 The particular cases

As announced in the introduction, now we consider the situation when the triangle which is the basis of the tessellation has its angles of the form $\frac{2\pi}{p}$, $\frac{2\pi}{q}$ and $\frac{2\pi}{r}$. The condition for T to be a triangle in the hyperbolic plane is now $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<\frac{1}{2}$. Notice that the form $\frac{2\pi}{k}$ for the angles is needed in order to obtain a tiling: roughly speaking, when we turn around a vertex, we must go back exactly on the initial position.

Let us look closer at this argument. If ℓ is a line, denote by ρ_ℓ the reflection in ℓ. Now, let us start from T, and consider ℓ and m the sides of T which meet on the vertex with angle $\frac{2\pi}{p}$. We perform the reflection $\sigma_1 = \rho_\ell$ on T, then the reflection $\sigma_2 = \rho_{\sigma_1(m)}$ on $\sigma_1(T)$. More generally, we perform $\sigma_{2k+1} = \rho_{\sigma_{2k}(\ell)}$ on $\sigma_{2k}(\sigma_{2k-1}(\ldots(\sigma_1(T))\ldots))$. If p is even, it is plain that $\sigma_p(\sigma_{p-1}(\ldots(\sigma_1(T))\ldots)) = T$. If p is odd, it is not difficult to see that $\sigma_p(\sigma_{p-1}(\ldots(\sigma_1(T))\ldots)) = T$ if and only if the other angles are equal, *i.e.* if and only if $q = r$.

Accordingly, if T tiles the plane, either p, q and r are all even, or if one of them is odd, p is odd and q and r are even with $q = r$, or $p = q = r$.

In other words, if one of the three numbers is odd, either the triangle is equilateral, or it is isosceles with, as angle in the vertex $\frac{2\pi}{p}$ with p odd and, as the basis angle, $\frac{\pi}{q}$.

The theorem 4 has solved the case when p, q and r are even. We shall now consider the other cases.

The case of an equilateral triangle

Theorem 5 – *The recursive replication of an equilateral triangle with vertex angle* $\frac{2\pi}{p}$*,* $p \geq 7$*, gives rise to a combinatoric tiling for all values of* p*,* $p \geq 7$*.*

Proof. Notice that the case when p is even is already dealt with: this means that the vertex angle is $\frac{\pi}{h}$ with $h \geq 4$ and $p = 2h$. In the proof of the theorem 4, we made no hypothesis on the differences between p, q and r, so that the same proof holds also in the present case. Indeed, a specific simpler proof works here:

We easily see that:

$$(4.a) \qquad (h) \Rightarrow [h, h].$$

$$(4.b) \qquad [h, h] \Rightarrow (h{-}2)(h) + [h, 2 * h],$$

$$(4.c) \qquad [h, 2 * h] \Rightarrow [2 * h, h] + (h{-}3)(h),$$

and we loop on that point. Notice that by taking the symmetric relation to (4.c) we obtain the following one:

$$(4.d) \qquad [2 * h, h] \Rightarrow (h{-}3)(h) + [h, 2 * h],$$

which leads back to (4.c).

Consider now that p is odd.

In this case, we change the definition of the regions which are used for the splitting.

The reason of this change is the fact that π is not an integer multiple of $\frac{2\pi}{p}$. Now, in order to obtain the splitting, in a relation as (4.c), for instance, we need to split $\left(\pi - \frac{2\pi}{p}\right)$ in an integral number of copies of $\left(\frac{2\pi}{p}\right)$. This is not possible when p is odd.

At least, this not possible with angular sectors. And so, we have to define new regions if a solution is possible. An angular sector, respectively a truncated sector, has two infinite borders which are straight rays. Here, we define figures which can be seen as angular sectors, respectively, as truncated sectors up to a distortion: it consists in replacing the rays by a broken line which is defined in an appropriate way. Above, the figure 5 indicates the basic principle which lies in the construction of the new borders.

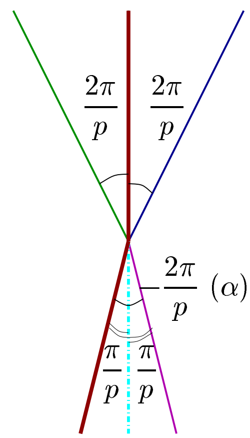

Figure 5. Constructing the border of a region
The border is in bold: notice the deviation to left.

The broken line is constructed as follows: it starts from a vertex of a copy of T and, running along one side, it arrives at a vertex V where the interior angle of the triangle is $\frac{2\pi}{p}$. By the tessellation process, this side is shared by two copies of T and the side is also shared by two angles of $\frac{2\pi}{p}$ with V as a vertex. As p is odd, the continuation of the side is not a side of one of the angles of $\frac{2\pi}{p}$ which lie around V: the continuation of the side is the bisector of the angle (α) of $\frac{2\pi}{p}$ which is opposite to the side with respect to V, see the figure 5. We assume that an orientation of the plane is fixed and the border is continued by taking the side of (α) which is reached clockwise from the inner bisector of (α).

Notice that if $p \geq 5$, this process defines broken lines which goes to infinity by taking lines of the tiling only. This does not hold for $p = 3$: whatever side we take for the next element of the initial side σ, we obtain another side of this copy of T or of its reflection in σ and, by fixing the choice according to the orientation, the broken line goes cyclically around the considered triangle.

Now, we consider that the new regions are defined from the angular sector and from the truncated one by replacing the straight rays by the broken line which we just define: see Figure 6, below.

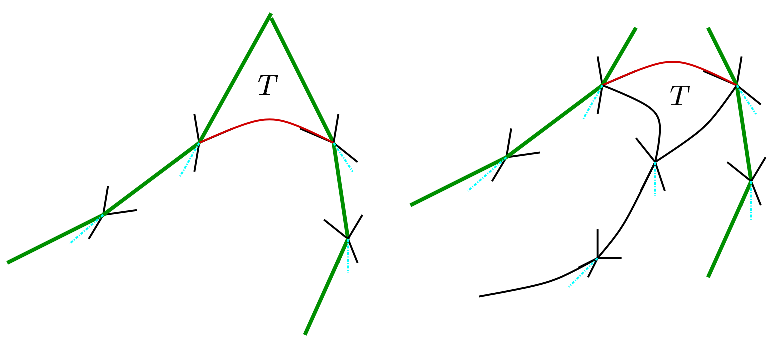

Figure 6. Two examples of the new regions: an angular one and a truncated one. Notice the deviation to left.

As the borders of the regions are clearly defined, we make use of the same symbolism as previously in order to represent the splittings of the regions. However, in the symbolism for angular sectors and truncated ones we shall denote the angle $\dfrac{2\pi}{p}$ by $\overset{\bullet}{p}$, while p still represents the angle $\dfrac{\pi}{p}$.

The previous splitting cannot be used with the new regions which are associate with an odd value for p: the main difference is that the definition of the border, making use of a fixed orientation of the space breaks the symmetry. Indeed, $[\overset{\bullet}{p}, 2 * \overset{\bullet}{p}]$ is not a copy of $[2 * \overset{\bullet}{p}, \overset{\bullet}{p}]$.

We now obtain the following splitting:

$$(4.a_1) \qquad (\overset{\bullet}{p}) \Rightarrow [p, \overset{\bullet}{p}{+}p].$$

$$(4.b_1) \qquad [p, \overset{\bullet}{p}{+}p] \Rightarrow [p{+}\overset{\bullet}{p}, p{+}\overset{\bullet}{p}] + (\lfloor\frac{p}{2}\rfloor{-}2)(\overset{\bullet}{p}),$$

$$(4.c_1) \qquad [p{+}\overset{\bullet}{p}, p{+}\overset{\bullet}{p}] \Rightarrow (\lfloor\frac{p}{2}\rfloor{-}2)(\overset{\bullet}{p}) + [p, 2*\overset{\bullet}{p}{+}p],$$

$$(4.d_1) \qquad [p, 2*\overset{\bullet}{p}{+}p] \Rightarrow [p{+}\overset{\bullet}{p}, p{+}\overset{\bullet}{p}] + (\lfloor\frac{p}{2}\rfloor{-}3)(\overset{\bullet}{p}).$$

and, from that point, we go back to $(4.c_1)$. ■

The case of an isosceles triangle

Theorem 6 – *The recursive replication of an isosceles triangle with vertex angle* $\frac{2\pi}{p}$, $p \geq 3$, *and basis angle* $\frac{\pi}{q}$, *with*

$$(*) \qquad \frac{1}{p} + \frac{1}{q} < \frac{1}{2}$$

gives rise to a quasi-combinatoric tiling for all possible values of p *and* q *which satisfy* $(*)$. *The tiling is combinatoric when* $p \geq 5$.

Proof. As indicated by the statement of the theorem, first consider the case when $p \geq 5$.

In this case, we apply the technique of the broken lines which was introduced in the proof of the theorem 5, each time the line arrives on a vertex which is shared by p copies of the angle $\frac{2\pi}{p}$. When the border of a sector crosses vertices being shared by $2q$ copies of the angle $\frac{\pi}{q}$ and only them, we go on straightforward, see the figure 7, below.

Here also, the borders of the regions are clearly defined. Consequently, we make use of the same symbolism as previously in order to represent the splittings of the regions. However, in the symbolism for angular sectors and truncated ones we shall denote the angle $\frac{2\pi}{p}$ by $\overset{\bullet}{p}$, the angle $\frac{\pi}{p}$ by p, as in the equilateral case.

First, assume that $p \geq 7$ and $q \geq 4$ or that $p > 7$ and $g \geq 3$.

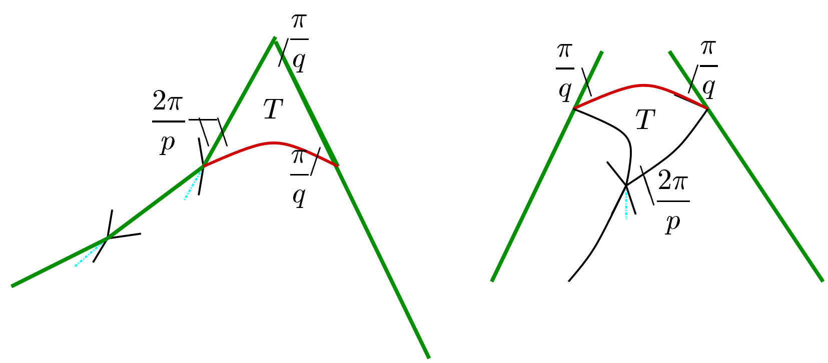

Figure 7. Two examples of the new regions in the case of an isosceles triangle: an angular one and a truncated one.

Accordingly, we have the following relations:

$$(5.a)\qquad (\overset{\bullet}{p}) \Rightarrow [q,q].$$
$$(5.b)\qquad [q,q] \Rightarrow (q{-}2)(q) + [p, 2*q],$$
$$(5.c)\qquad [p, 2*q] \Rightarrow [\overset{\bullet}{p}{+}p, q] + (q{-}3)(q),$$
$$(5.d)\qquad [\overset{\bullet}{p}{+}p, q] \Rightarrow (\lfloor\frac{p}{2}\rfloor{-}2)(\overset{\bullet}{p}) + [q, 2*q],$$
$$(5.e)\qquad [q, 2*q] \Rightarrow [2*q, \overset{\bullet}{p}{+}p] + (q{-}3)(q),$$
$$(5.f)\qquad [2*q, \overset{\bullet}{p}{+}p] \Rightarrow (q{-}3)(q) + [q, 2{*}\overset{\bullet}{p}{+}p],$$
$$(5.g)\qquad [q, 2{*}\overset{\bullet}{p}{+}p] \Rightarrow [2*q, q] + (\lfloor\frac{p}{2}\rfloor{-}3)(\overset{\bullet}{p}),$$
$$(5.h)\qquad [2*q, q] \Rightarrow (q{-}3)(q) + [p, 2*q],$$

and from that point, we are led back to $(5.c)$.

Notice that the sufficient conditions of the basic lemma are satisfied without problem from $(5.a)$ up to $(5.e)$. They are also satisfied for $[q, 2*\overset{\bullet}{p}{+}p]$ in $(5.f)$ when $p = 7$ if $q \geq 4$ and if $q = 3$ when $p > 7$. Notice that when $q = 3$, $[q, 2*q]$ is still an infinite region: the two infinite rays are non-secant.

When $p = 7$ and $q = 3$, the sufficient condition of the basic lemma is not satisfied for $[q, 2{*}\overset{\bullet}{p}{+}p]$. However, the splitting which is indicated by $(5.g)$ is still possible. As $[2*q,q]$ is an infinite region for $p=3$ as we already noticed, so is $[q,2*\overset{\bullet}{p}+p]$. And so, we conclude that the above splittings are valid also for the case when $p=7$ and $q=3$.

We have to go on with the splitting of (q). Indeed there are two cases, depending on which side is the angle $(\overset{\bullet}{p})$. As the border is defined only with respect to the orientation of the plane, the difference of sides is here relevent. See the difference on the figures 8.

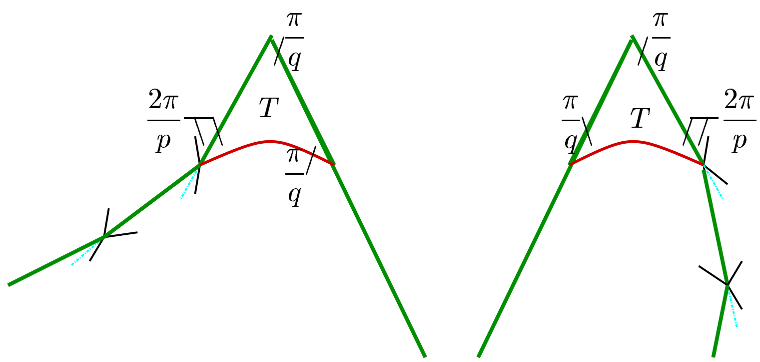

Figure 8. The two cases for (q):
on the right hand: $(q)^-$
on the left hand: $(q)^+$

First, let us consider the case when $(\overset{\bullet}{p})$ is on the left side of the angle (q). We have the following splitting:

$$(5.i)\qquad (q)^+ \Rightarrow [p,q].$$
$$(5.j)\qquad [p,q] \Rightarrow [\overset{\bullet}{p}+p,q] + (q-2)(q),$$

and so, we are led to the case of $(5.d)$.

Second, consider the case when $(\overset{\bullet}{p})$ is on the right side of the angle (q). Now, the splitting becomes the following:

$$(5.k)\qquad (q)^- \Rightarrow [q,\overset{\bullet}{p}+p],$$
$$(5.\ell)\qquad [q,\overset{\bullet}{p}+p] \Rightarrow [2*q,q] + (\lfloor\frac{p}{2}\rfloor-2)(\overset{\bullet}{p}),$$

and so, we are led to the case of $(5.h)$.

Notice that in both these cases, the sufficient condition of the basic lemma is satisfied when $p \geq 7$ and $q \geq 3$.

Now, we turn to the case when $p = 5$. This entails that $q \geq 4$.

Simple computations show us that the relations $(5.a)$ up to $(5.d)$ hold, the last one also being included. The relation $(5.e)$ gives a correct splitting if $[2 * q, \overset{\bullet}{p}{+}p]$ is infinite. We cannot use $(5.f)$ because the splitting has no meaning, due to the fact that $2 * \overset{\bullet}{p}{+}p = \pi$. However, when $p = 5$, it is not difficult to see that:

$$(5.f_1) \qquad [2 * q, \overset{\bullet}{p}{+}p] \Rightarrow [3 * q, q] + (\lfloor \frac{p}{2} \rfloor {-} 2)(\overset{\bullet}{p}),$$

and next, that:

$$(5.g_1) \qquad [3 * q, q] \Rightarrow (q{-}4)(q) + [p, 2 * q].$$

and this leads us to the case $(5.c)$

This completes the proof for the case when $p = 5$.

At last, we consider the case $p = 3$.

In this case, the splitting which we considered gives no answer. The first reason is that the way in which we defined the continuation of a broken line does not work for $p = 3$: the rule which we defined has as a consequence that the broken line takes a side of the originating angle. The second reason is that redefining the basis gives no answer: we may define the continuation of the line by taking the outer bisector of the angle. But in that case, we obtain first, that $(\overset{\bullet}{p}) \Rightarrow [3 * q, q]$, but $[3 * q, q]$ cannot be split in T plus infinite domains: we have additional copies of T because of the angle $\dfrac{2\pi}{3}$.

And so, the situation is very similar to the situation of the case $p = 2$ and $q = 3$ of the theorem 4. As in the theorem 4, we have an alternative solution by noticing that if we glue together three copies of T around the vertex with the angle $(\overset{\bullet}{p})$, we obtain an equilateral triangle with $\dfrac{2\pi}{q}$ as the interior angle. And so, from the theorem 5, the proof of the theorem 4 is completed. ■

4 Conclusion

It seems to me that there are a lot of possible continuations of the researches on which this paper is based.

A possible continuation deals with the implementation of cellular automata on triangular grids of the hyperbolic plane. This paper, together with its second part, provides basic tools for that but it is not complete. Indeed, when we deal with cellular automata, it is not enough to have a tiling. We have also to easily locate the *neighbours* of a cell. This means that we need also a convenient description or access to the dual graph of the tiling. In the cases of the pentagrid and of regular tilings with right angles, also in the case of the rectangular dodecahedral tiling of $I\!H^3$, $\{5,3,4\}$, it appears that the spanning tree of the splitting is a sub-graph of the dual graph of the tiling being restricted to S_0. In the planar cases which we quoted, it is easy to restore the dual graph from the spanning tree, see [2] and [6]. In the case of the tiling $\{5,3,4\}$, it is still easy, but not so straightforward, see [7]. In the case of the triangular tilings of $I\!H^2$, in most cases, the spanning tree is not a sub-graph of the dual graph of the tiling. However, it is also possible to restore the dual graph from the spanning tree, by using rather simple rules. A forthcoming paper which will be published after the second part of this paper, will give the details about this issue.

A second possible continuation is to investigate other combinatoric tilings. Indeed, I suspect that the spectrum of combinatoric tilings is very large and not especially attached to hyperbolic geometry. It is easy to provide examples of euclidean tilings with a single tile which are not tessellations but which are combinatoric: consider the example which is given in section 2. Also, it would be interesting to consider combinatoric tilings with more than a single generating tile.

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