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Discrete Control Processes, Dynamic Games and Multicriterion Control Problems*

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Abstract

The discrete control processes with state evaluation in time of dynamical system is considered. A general model of control problems with integral-time cost criterion by a trajectory is studied and a general scheme for solving such classes of problems is proposed. In addition the game-theoretical and multicriterion models for control problems are formulated and studied.

1 Introduction

We study the discrete control processes with state evaluation in time of dynamical system. We consider that the state evaluation of the system at every moment of time is determined uniqualy by the state evaluation and control parameters of the system at the previous moment of time. Such processes conduct to a generalization of the control problems with integral-time cost criterion by a trajectory from [1–3]. We formulate the states evaluation control problems (SECP), which represent a general model for mentioned class of problems. A general scheme for solving the SECP is proposed and some details concerning the computational complexity of the algorithms for different classes of problems are discussed. In addition the game theoretical and multicriterion models for considered class of problems are formulated and studied.

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2 The State Evaluation Control problem and algorithm for its solving

Let L be the dynamical system with the set of states $X \subseteq R$ where at every moment of time $t = 0, 1, 2, \dots$ the state of L is $x(t) \in X$, $x(t) =$ $= (x_1(t), x_2(t), \dots, x_n(t))$. The dynamics of the system L is described as follows

$$x(t+1) = g_t(x(t), u(t)),\ t = 0, 1, 2, \dots \tag{1}$$

where

$$x(0) = x_s \tag{2}$$

is the starting point of system L and $u(t) = (u_1(t), u_2(t), \dots, u_m(t)) \in$ $\in \mathbb{R}^m$ represent the vector of control parameters [1–3]. For vectors of control parameters $u(t)$, $t = 0, 1, 2, \dots$ the admissible sets $U_t(x(t))$ are given by

$$u(t) \in U_t(x(t)),\ t = 0, 1, 2, \dots \quad . \tag{3}$$

We assume that in (1) the vector functions

$$g_t(x(t), u(t)) = \Big(g_t^1(x(t), u(t)), g_t^2(x(t), u(t)), \dots, g_t^n(x(t), u(t))\Big)$$

are determined uniquely by $x(t)$ and $u(t)$ at every moment of time $t = 0, 1, 2, \dots$. So, $x(t+1)$ is determined uniquely by $x(t)$ and $u(t)$.

Let

$$x(0), x(1), \dots, x(t), \dots \tag{4}$$

be a process generated by (1) – (3) with given vectors of control parameters

$$u(0), u(1), \dots, u(t-1), \dots$$

For each state $x(t)$, $t = 0, 1, 2, \dots$ of the process (4) we define the numerical evaluation $F_t(x(t))$ by using the following recurrent formula

$$F_{t+1}(x(t+1)) = f_t\Big(F_t(x(t)), x(t), u(t)\Big),\ t = 0, 1, 2 \dots$$

where

$$F_0(x(0)) = F_0$$

is a given evaluation of the starting state $x(0)$ of the system L; $f_t(\cdot,\cdot,\cdot)$; $t = 0, 1, 2, \ldots$ are arbitrary functions.

We consider the following two problems:

Problem 1. For given T to find the vectors of control parameters $u(0), u(1), \ldots, u(T-1)$, which satisfy the conditions

$$\begin{cases} x(t+1) = g_t(x(t), u(t)),\ t = 0, 1, 2, \ldots, T-1; \\ x(0) = x_0,\ x(T) = x_T, \\ u(t) \in U_t(x(t)),\ t = 0, 1, 2, \ldots, T-1; \\ F_{t+1}(x(t+1)) = f_t\Big(F_t(x(t)), x(t), u(t)\Big), t = 0, 1, 2, \ldots, T-1; \\ F_0(x(0)) = 0 \end{cases} \tag{5}$$

and minimize the object function

$$I_{x_0 x(T)}(u(t)) = F_T(x(T)) \tag{6}$$

Problem 2. To find T and $u(0), u(1), \ldots, u(T-1)$ which satisfy the condition (5) and minimize the function (6).

Obviously that if for the problem 2 the interval $[T_1, T_2]$ for parameter T is given, i.e. $T \in [T_1, T_2]$, then the optimal solution of problem 2 can be obtained by reducing to problem 1 fixing each time $T = T_1$, $T = T_1 + 1$, ..., $T = T_2$. Choosing the best of the solutions of the problems of type 1 with $T = T_1$, $T = T_1 + 1$, ..., $T = T_2$ we obtain the solution of the problem 2 with $T \in [T_1, T_2]$.

It is easy to observe that if

$$f_t\Big(F_t(x(t)), x(t), u(t)\Big) = F_t(x(t)) + c_t(x(t), u(t)),$$

where

$$F_0(x_0) = 0$$

and $c_t(x(t), u(t))$ represent the cost of system's passage from the state $x(t)$ to state $x(t+1)$, then we obtain the discrete control problems with integral-time criterion by a trajectory [1–3]. Some classes of control problems from [1–2] may be obtained if

$$F_0(x_0) = 1,$$
$$F_t\Big(F_t(x(t)), x(t), u(t)\Big) = F_t(x(t)) \cdot c_t(x(t), u(t)),\ t = 1, 2, \ldots$$

and if

$$F_0(x_0) = 0,$$
$$F_t\Big(F_t(x(t)), x(t), u(t)\Big) = \max\{F_t(x(t)), c_t(x(t), u(t))\}.$$

3 The main result

We propose a general scheme for finding the optimal solution of the formulated problems in the case when $f_t(F, x, u),\ t = 0, 1, 2, \ldots$ are non-decreasing functions with respect to first argument F. So, we shall consider that for fixed x and u the functions $f_t(F, x, u),\ t = 0, 1, 2, \ldots$ satisfy the condition

$$f_t(F', x, u) \le f_t(F'', x, u) \quad \text{if} \quad F' \le F''. \tag{7}$$

Then the following algorithm finds the optimal solution of problem 1.

Algorihm

1. Set $F_0^*(x(0)) = F_0$; $F_t^*(x(t)) = \infty$; $x(t) \in X, t = 1, 2, \ldots$; $X_0 = \{x_0\}$.

2. For $t = 1, 2, \ldots, T$ find:

$$X_{t+1} = \{x(t+1) \in X \mid x(t+1) = g_t(x(t), u(t)),\ x(t) \in X_T,\ u(t) \in U_t(x(t))\}$$

and

$$F_{t+1}^*(x(t+1)) = \min_{x \in X_t, u(t) \in U_t\left(x(t)\right)} \Big\{f_t\Big(F_t^*(x(t)), x(t), u(t)\Big)\Big\}, \quad \forall\, x(t+1) \in X_{t+1};$$

3. Find the sequence

$$x_T = x^*(T),\ x^*(T-1), x^*(T-2),\ \ldots,\ x^*(1), x(0)) = x_0$$

and

$$u^*(T-1),\ u^*(T-2),\ \ldots,\ u^*(1),\ u(0),$$

which satisfy the conditions

$$F^*_{T-t}(x^*(T-1)) = f_{T-t-1}\Big(F^*_{T-t-1}(x(T-t-1)), x^*(T-t-1), u^*(T-t-1)\Big),\ t = 0, 1, 2, \ldots, T.$$

Then $u^*(0), u^*(1), u^*(2), \ldots, u^*(T-1)$ represent the optimal solution of the problem 1.

Theorem 1 *If $f_t(F, x, u)$, $t = 0, 1, 2, \ldots, T$ are non-decreasing functions with respect to first argument F, i.e. the functions $f_t(F, x, u)$, $t = 0, 1, 2, \ldots, T$ satisfy the condition (7), then the algorithm finds the optimal solution of problem 1. Moreover, an arbitrary leading part $x^*(0)$, $x^*(1)$, $\ldots$, $x^*(k)$ of the optimal trajectory $x^*(0)$, $x^*(1)$, $\ldots$ $\ldots$, $x^*(k)$, $\ldots$, $x^*(T)$ is optimal one.*

Proof. We proove the theorem by using the induction principle on number of stages T. In the case $T \leq 1$ the theorem is evident. We consider that the theorem holds for $T \leq k$ and let us prove it for $T = k+1$.

Assume toward contradiction that

$$u'(0), u'(1), \ldots, u'(T-2), u'(T-1)$$

is an optimal solution of problem 1, where

$$F'_{t+1}(x'(t+1)) = f_t\Big(F'_t(x'(t)), x'(t), u'(t)\Big),\ t = 0, 1, 2 \ldots, T-1;$$
$$x'(0) = x_0,\ F_0(x'(0)) = F_0,\ x'(T) = x(T)$$

and

$$F'_T(x'(T)) < F^*_T(x'(T)). \tag{8}$$

According to induction principle for the problem 1 with $T-1$ stages the algorithm finds the optimal solution. So, for arbitrary $x(T-1) \in X$ we obtain the optimal evaluations $F^*_{T-1}(x(T-1))$ for $x(T-1) \in X$. Therefore

$$F^*_{T-1}(x'(T-1)) \leq F'_{T-1}(x'(T-1)).$$

According to algorithm

$$f_{T-1}\Big(F^*_{T-1}(x^*(T-1)), x^*(T-1), u^*(T-1)\Big) \leq$$
$$\leq f_{T-1}\Big(F^*_{T-1}(x'(T-1)), x'(T-1), u'(T-1)\Big). \qquad (9)$$

Since $f_t(F, x, u),\ t = 0, 1, 2, \ldots$ are non-decreasing functions with respect to F then

$$f_{T-1}\Big(F^*_{T-1}(x'(T-1)), x'(T-1), u'(T-1)\Big) \leq$$
$$\leq f_{T-1}\Big(F'_{T-1}(x'(T-1)), x'(T-1), u'(T-1)\Big). \qquad (10)$$

Using (9) and (10) we obtain

$$F^*_T(x(T)) = f_{T-1}\Big(F^*_{T-1}(x^*(T-1)), x^*(T-1), u^*(T-1)\Big) \leq$$
$$\leq f_{T-1}\Big(F^*_{T-1}(x'(T-1)), x'(T-1), u'(T-1)\Big) \leq$$
$$\leq f_{T-1}\Big(F'_{T-1}(x'(T-1)), x'(T-1), u'(T-1)\Big) = F'_T(x(T)),$$

i.e.

$$F^*_T(x(T)) \leq F'_T(x(T)).$$

This is in contradiction with (8). So the algorithm finds the optimal solution of the problem 1 with $T = k + 1$. □

Theorem 2 *Let X and $U_t(x)$, $x \in X, t = 0, 1, 2, \ldots, T-1$, be the finit sets, and $M = \max\limits_{x\in X,\, t=0,1,2,\ldots,T-1} |U_t(x)|$. Then the algorithm uses at most $M \cdot |X| \cdot T$ elementary operations (excluding the operations for calculation the values of functions $f_t(F, x, u)$ for given F, x and u).*

Proof. It is sufficient to prove that at step t the algorithm uses no more than $M \cdot |X|$ elementary operations. Indeed, for finding the value $F_{t+1}(x(t+1))$, for $x(t+1) \in X$ it is necessary to use $\sum\limits_{x\in X} |U_t(x)|$ operation. Since $\sum\limits_{x\in X} |U_t(x)| \leq |X| \cdot M$ then at step t the algorithm uses no more than $|X| \cdot M$ elementary operations. So in general the algorithm uses no more than $M \cdot |X| \cdot T$ elementary operations. □

4 The discrete optimal control problem on network

Let L be a dynamical system with a finite set of states X, $|X| = N$, and at every discrete moment of time $t = 0, 1, 2, \ldots$ the state of the system L is $x(t) \in X$. Note that here we associate $x(t)$ with an abstract element (in sections 1 and 2 $x(t)$ represents a vector from $\mathbb{R}^n$). Two states x_s and x_f are chosen in X, where x_s is a starting state of the system L, $x_s = x(0)$, and x_f is the final state of the system, i.e. x_f is the state to which the system must be brought. The dynamics of the system is described by a directed graph of passages $G = (X, E), |E| = m$, an edge $e = (x, y)$ which signifies the possibility of passage of system L from the state $x = x(t)$ to the state $y = x(t + 1)$ at any moment of time $t = = 0, 1, 2, \ldots$. That means that the edges $e = (x, y) \in E$ can be regarded as the possible values of the control parameter $u(t)$ when the state of the system is $x = x(t)$, $t = 0, 1, 2, \ldots$. The next state $y = x(t + 1)$ of the system L is determined uniquely by $x = x(t)$ at the moment of time t and an edge $e = (x, y) \in E(x)$, where $E(x) = \{(x, y) \in \in E \setminus y \in X\}$. So $E(x) = E(x(t))$ corresponds to the admissible set $U_t(x(t))$ for the control parameter $u(t)$ at every moment of time t. To each edge $e = (x, y)$ a function $c_e(t)$ is assigned, which reflects the cost of system's passage from the state $x(t) = x \in X$ to the state $x(t + 1) = y \in X$ at any moment of time $t = 0, 1, 2, \ldots$

We consider the discrete optimal control problem on network for which the sequence of system's passages $(x(0), x(1))$, $(x(1), x(2))$, $\ldots$ $\ldots$, $(x(T-1), x(T)) \in E$ which transfers the system L from the state $x_s = x(0)$, to the state $x_f = x(T)$ with minimal integral-time cost of the passages by a trajectory $x_s = x(0), x(1), x(2), \ldots, x(T) = x_f$ must be found.

Here may be two variants of the problem:

1) the number of the stages (time T) is fixed;

2) T is unknown and it must be found.

It is easy to observe that for solving these problems we could use the algorithm from section 3.

We put $F_0(x(0)) = 0$ and $F_{t+1}(x(t+1)) = F_t(x(t)) + c_{\left(x(t),x(t+1)\right)}(t)$

for $(x(t), x(t+1)) \in E$. So we obtain the algorithm of dynamic programing, which solves the problem in time n^2T.

5 The game control model with p players

In this section we formulate the game control models with p players, which represent a generalization of the problem from section 2. We shall give the game theoretical model in the cases when T is fixed and in the case when T is not fixed. First we formulate the discrete control process with p players.

Let L be a time-discrete system with the set of states $X \subseteq I\!R^n$. We consider that the dynamics of the system L is controled by p players and it is described as follows

$$x(t+1) = g_t(x(t), u^1(t), u^2(t), \dots, u^p(t)),\ t = 0, 1, 2, \dots \tag{11}$$

where

$$x(0) = 0$$

is the starting state of the system L and $u^i(t) \in I\!R^{m_i}$ represents the vector of control parameters of player i, $i = \overline{1,p}$. So, the state $x(t+1)$ of the system L at the moment of time $t+1$ is obtained uniquely if the state $x(t)$ at the moment of time is known and the players $1, 2, \dots, p$ fix their vectors of control parameters $u^1(t), u^2(t), \dots, u^p(t)$, respectively. We shall consider that the players fix there vectors of control parameters independently and for each player $i = \overline{1,p}$ the admissible sets $U_t^i(x(t))$ are given by

$$u^i(t) \in U_t^i(x(t)),\ t = 0, 1, 2, \dots$$

We shall consider the sets $U_t^i(x(t)),\ i = \overline{1,p},\ t = 0, 1, 2, \dots$ non-empty and $U_t^i(x(t)) \cap U_t^j(x(t)) = \emptyset$ for $i \neq j,\ t = 0, 1, 2, \dots$

Let

$$x(0), x(1), \dots, x(t), \dots \tag{12}$$

be a process generated by (9) – (11) with given vectors of control parameters

$$\begin{cases} u^1(0), u^1(1), \dots, u^1(t), \dots \\ u^2(0), u^2(1), \dots, u^2(t), \dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ u^p(0), u^p(1), \dots, u^p(t), \dots \end{cases} \tag{13}$$

For each state $x(t)$, $t = 0, 1, 2, \dots$ of the process (12) we define the numerical evaluation $F_t^i(x(t))$ for the player i, $i = \overline{1, p}$, by using the following recurrent formula

$$F_{t+1}^i(x(t+1)) = f_t^i\Big(F_t^i(x(t)), x(t), u(t)\Big), \ t = 0, 1, 2, \dots \tag{14}$$

where

$$F_0^i(x(0)) = F_0^i \tag{15}$$

is a given evaluation of the starting state $x(0)$ for the player i, $i = \overline{1, p}$, here $f_t^i(\cdot, \cdot, \cdot)$, $t = 0, 1, 2, \dots,$ $i = \overline{1, p}$, are arbitrary functions.

The discrete process described by using (11) – (15) we name the discrete control process with p players.

Now let us state that each player i, $i = \overline{1, p}$ has the aim that the system L will reach a given final state $x_f \in X$, but each of them have the interest to minimize its final state evaluation $F_{T(x_f)}^i(x_f)$.

We denote by

$$I_{x_0, x(T(x_f))}^i(u^1(t), u^2(t), \dots, u^p(t) \,|\, F_0^i(x_0)), \ i = \overline{1, p}$$

the evaluation $F_{T(x_f)}^i(x_f)$ of the final state x_f when the players fix their vectors of control parameter $u^1(t), u^2(t), \dots, u^p(t),$ $t = 0, 1, 2, \dots T(x_f)$ and when the starting evaluations $F_0^i(x_0)$, $i = \overline{1, p}$ are given. Here $T(x_f)$ represent the time-moment when the state $x(f)$ is reached by system l when the players fix their vectors of control parameters $u^i(t)$, $i = \overline{1, p}$.

If $T(x_f) = \infty$ then we put

$$I_{x_0, x(T(x_f))}^i(u^1(t), u^2(t), \dots, u^p(t) \,|\, F_0^i(x_0)) = \infty.$$

So, the functions

$$\begin{array}{c} I^1_{x_0,x_f}(u^1(t),u^2(t),\dots,u^p(t)\,|\,F^1_0(x_0)),\\ I^2_{x_0,x_f}(u^1(t),u^2(t),\dots,u^p(t)\,|\,F^2_0(x_0)),\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ I^p_{x_0,x_f}(u^1(t),u^2(t),\dots,u^p(t)\,|\,F^p_0(x_0)), \end{array}$$

define a dynamic game. For this dynamic model we could formulate the problem of finding the optimal solution in the sense of Nash or in sense of Pareto.

Note that in the presented model we have considered that $T(x_f)$ is not fixed. We formulate the game model with fixed number of stages in the following way. We state that each player has the aim to reach the final state by using exactly T stages. Then we denote by

$$I^i_{x(0),x(T)}(u^1(t),u^2(t),\dots,u^p(t)\,|\,F^i_0(x_0)), i=\overline{1,p}$$

the evaluation $F^i(x_f)$ of the final state when the vectors of control parameters $u^i(t)$, $t=\overline{1,p}$, generate a trajectory, which contains exactly T stages; otherwise we shall consider

$$I^i_{x(0),x(T)}(u^1(t),u^2(t),\dots,u^p(t)\,|\,F^i_0(x_0))=\infty,\ i=\overline{1,p}.$$

The presented models generalize the problem from [2–4].

6 The multicriterion Control Problems

The problems 1 and 2 from section 2 can be extended as multicriterion models if for each state $x(t)$ of the process (1) – (3) we introduce p evaluations

$$F'_{t+1}(x(t+1))=f^i_t\Big(F^i_t(x(t)),x(t),u(t)\Big),\ t=0,1,2\dots,T-1,\ i=\overline{1,p}$$

with given starting evaluations

$$F^i_0(x(0))=F^i_0,\ i=\overline{1,p}.$$

So, we obtain the restrictions

$$\begin{cases} x(t+1) &= g_t(x(t),u(t)),\ t=0,1,2,\dots,T-1;\\ x(0)=x_0,\ x_f &= x(T);\\ & \quad u(t)\in U(x(t)),\ \ t=0,1,2,\dots,T-1;\\ F'_{t+1}(x(t+1)) &= f_t^i\Big(F_t^i(x(t)),x(t),u(t)\Big),\ \ t=0,1,2\dots,T-1;\\ F_0^i(x(0)) &= F_0^i,\ \ i=\overline{1,p} \end{cases}$$

and p object functions

$$I^i_{x_0,x_f}(u(t)) = F^i_T(x(t)),\ i=\overline{1,p}.$$

For this model we could formulate the problem of finding the optimal solution in the sense of Nash or in other sense [4].

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