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On the invariance of the Pareto optimal set

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Abstract

In the previous paper presented at the 11th SOR, Darmstadt 1986, the authors had shown that the Pareto optimal set of a multicriteria problem remains invariable if on the objective vector function operates a strictly monotone operator of strictly monotone kind. In the present paper we give new proofs of two main theorems of this result and we show that the invariance of Pareto optimal set takes place also in the case of a topological transformation of the problem's variables. A possible interpretation of the above results in decision theory is suggested.
Keywords: Pareto invariance, monotone operators and topological maps in multicriteria optimization.

1 Introduction

Let R^n, R^k be numerical finite dimensional metric spaces, and $F^t(x) = (f_1(x), \ldots, f_k(x))$ a vector function $F : R^n \rightarrow R^k$, upper semicontinuous on the compact set $X \subset R^n$.

Consider the multicriteria optimization problem:

$$\max_{x \in X} F(x) \tag{1}$$

and let denote by $P(F) = \{x_F^* \in X | x_F^* = \arg\max_{x \in X} F(x)\}$ the set of all nondominated, efficient or Pareto optimal points, i.e. the set of points with the property that there is no $x' \in X$, $x' \neq x_F^*$, such that $F(x') > F(x_F^*)$, where the sign $>$ has the meaning that $F(x') > F(x_F^*)$ is equivalent to $F(x') \geq F(x_F^*)$ and $F(x') \neq F(x_F^*)$.

In [2] and [5] it is shown that given two unicriterion problems

$$(I) \quad \max_{x \in X} f(x)$$

$$(II) \quad \max_{u \in U} g(u)$$

such that $U = \tau(X)$, where τ is one-to-one map of the constraint set X of (I) on to the constraint set U of (II), and $f(x) = g(\tau(x))$ then holds the following:

Theorem 1 *If the problem (I) attains a maximum at $x^* \in X$, then the problem (II) attains a maximum at $\tau(x^*) \in U$. If the problem (II) attains a maximum at $u^* \in U$ then the problem (I) attains a maximum at $\tau^{-1}(u^*) \in X$. The problems (I) and (II) in this case are called τ – equivalent problems.*

In the first part of the paper it will be shown that the above result remains true in the case of the multicriteria problem.

If instead of applying a map to the problem's variables we apply an operator to the objective vector function, we have shown in previous papers, [6, 7, 8], that the Pareto optimal set remains invariable. After the presentation, in a slight different new proofs, of two main theorems regarding this invariance property, we make some applications and we suggest a possible use of the above results in management science.

2 The invariance of Pareto optimal set under bijective maps

Let us consider now, together with the vectorial problem (1), the following vectorial problem:

$$\max_{u \in U} G(u) \tag{2}$$

where $G : R^m \to R^k$, and $U \subset R^m$ such that $U = \tau(X)$ and τ is an injective map on X such that $G(\tau(x)) = F(x)$.

Theorem 2 *Let P(F) and P(G) denote the Pareto optimal set to problem (1), respectively to problem (2). Then*

$$P(G) = \tau P(F) \tag{3}$$

and if τ is a continuous map then problem (2) has solutions and

$$P(F) = \tau^{-1}[P(G)] \tag{4}$$

Proof: τ is an injective map on X, and surjective on U, by the definition of U, therefore τ is a bijective map (one-to-one), and the inverse mapping τ^{-1} with $x = \tau^{-1}(u)$ exists.

As X is compact and F(x) is upper semicontinuous on X then $P(F) \neq \emptyset$. Let be $x^* \in P(F)$ and $u^* = \tau(x^*)$. Suppose that $u^* \notin P(G)$. Then there exists $\bar{u} \in U$, $\bar{u} \neq u^*$ such that $G(\bar{u}) > G(u^*)$ or, equivalently, $F(\tau^{-1}(\bar{u})) > F(\tau^{-1}(u^*))$ and, as τ in one-to-one map, there exists $\bar{x} = \tau^{-1}(\bar{u})$, $\bar{x} \in X$, $\bar{x} \neq x^*$ such that $F(\bar{x}) > F(x^*)$ which contradicts the Pareto optimality hypothesis of x^*. Therefore $u^* \in P[G]$ and $\tau[P(F)] \subset P(G)$.

Now, assume that $\bar{x} \notin P(F)$, $\bar{x} \in X$, then $\bar{x}$ is a dominated point and therefore exists $\bar{\bar{x}} \in X$, $\bar{\bar{x}} \neq \bar{x}$ such that $F(\bar{\bar{x}}) > F(\bar{x})$. It follows that $G(\bar{\bar{x}}) > G(\bar{u})$, $\bar{\bar{u}} \neq \bar{u}$, where $\bar{\bar{u}} = \tau(\bar{\bar{x}})$ and $\bar{u} = \tau(\bar{x})$ i.e. the image of a dominated point $\bar{x}$ on X is a dominated point $\bar{u}$ on U.

Suppose now that $u^* \in P(G)$, $u^* \notin \tau[P(F)]$. It follows that $u^* \in \tau[C_X P(F)]$ which contradicts the above. Then $P(G) = \tau[P(F)]$.

For the last part of the Theorem 2, let τ be a continous map. Then $U = \tau(X)$ is compact.

As R^k is a metric space, then R^k is a Haussdorff space, i.e. a separable space. Then, [9], τ^{-1} is also continous on U, i.e. τ is a homeomorphism or a topological transformation. It follows that $G(u) = F(\tau^{-1}(u))$ is uppersemicontinous on the compact U and problem (2) is well defined and has solutions. Then (4) holds. Theorem 2 is proved.

Theorem 2 can be stated also as it follows:
The topological transformations of the feasible compact set of a multicriteria problem leave invariable the Pareto optimal set in the sense

of (3) and (4). Multicriteria problems for which conditions of Theorem 2 are satisfied will be called also, as in the case of a single criterion, multicriteria τ - equivalent problems.

3 Monotone operators and the invariance of Pareto optimal set

Let $R^{(\cdot)}$ and $R^{(*)}$ be partially ordered linear metric spaces, and T an operator, $T : D \to W$, $D \subset R^{(\cdot)}$, $W \subset R^{(*)}$.

Definition 1 [4]. *The operator T is called isotone if*

$$v \leq w \Rightarrow Tv \leq Tw \tag{5}$$

for all $v, w \in D$, and antitone if

$$v \leq w \Rightarrow Tv \geq Tw \tag{6}$$

We say that T is monotone if T is either isotone or antitone. The operator T is said to be of isotone kind if

$$Tv \leq Tw \Rightarrow v \leq w \tag{7}$$

for all $v, w \in D$.

If the inequality (5), (6) or (7) are satisfied with the sign $<$ ($>$) then the operator T is strictly isotone (antitone) and/or of strictly isotone (antitone) kind.

Let us consider also, together with problem (1), the problem

$$\max_{x \in X} TF(x) \tag{8}$$

where $T : R^k \to R^p$, is an operator defined on $F(x), R^p$ a numerical real space, and let us denote by $P(TF)$ the set of all Pareto optimal points to problem (8).

It is easy to see that if T is an isotone operator and F(x) is a m-vector function uppersemicontinuous on X, then TF(x) is also uppersemicontinuous on X. Then, if X is compact, problem (8) is well defined and $P(TF) \neq \phi$. Moreover, the following invariance theorem [8], holds:

Theorem 3 *If P(F) and P(TF) denote the sets of all Pareto optimal points to problem (1) and to problem (8), respectively, and T is a strictly isotone operator defined on $F(X)$, i.e.*

$$F^1 < F^2 \Rightarrow TF^1 < TF^2 \tag{9}$$

then

$$P(TF) \subseteq P(F) \tag{10}$$

If problem (8) is well defined and T is an operator of strictly isotone kind, i.e.

$$TF^1 < TF^2 \Rightarrow F^1 < F^2 \tag{11}$$

then

$$P(TF) \supseteq P(F) \tag{12}$$

If T satisfies both (9) and (11), then

$$P(TF) = P(F) \tag{13}$$

Proof: As we have mentioned, in the stated conditions for both problems, (1) and (8) are well defined and $P(F) \neq \phi$ as well as $P(TF) \neq \phi$.

Let $\bar{x} \in C_X P(F) \Leftrightarrow \bar{x} \in X, \bar{x} \notin P(F)$ i.e. $\bar{x}$ is a dominated point in X with respect to F. If follows that $\exists \hat{x} \in X, \hat{x} \neq \bar{x}$, such that $F(\hat{x}) > F(\bar{x})$. If T is a strictly isotone operator, then from (9) we have that $TF(\hat{x}) > TF(\bar{x})$, i.e. $\bar{x}$ is a dominated point in X with respect to TF, hence $\bar{x} \in C_X P(TF)$. It follows that

$$C_X P(F) \subseteq C_X P(TF) \Leftrightarrow P(TF) \subseteq P(F)$$

and (10) holds.

Let now $x^* \in P(F)$ and suppose that $x^* \notin P(TF)$. Thent there exists $\hat{x} \neq x^*$ such that $TF(\hat{x}) > TF(x^*)$. As T is of monotone kind then by (11) we have that $F(\hat{x}) > F(x^*)$ which contradicts the Pareto optimality hypothesis of x^*. Therefore $x^* \in P(TF)$ and $P(F) \subseteq P(TF)$, i.e. if T is an operator of strictly monotone kind then every nondominated point in respect to F is also a nondominated point in respect of

TF and (12) holds. From the above, it follows that if T satisfies simultaneously (9) and (11) then we have $P(TF) = P(F)$, and the Theorem 3 is proved.

An example of a strictly isotone operator of strictly monotone kind is provided by the following:

Theorem 4 *Let $F : R^n \to R^k$ be a vector function on $X \subset R^n$ and $\varphi : R^k \to R$ a function defined on $F(X)$.*

Then the operator $T_+ : R^k \to R^{k+1}$ defined by

$$(T_+F)^t = (F^t, \varphi(F)) = F_+^t \tag{14}$$

is a strictly isotone kind operator, i.e.

$$T_+F^1 < T_+F^2 \Rightarrow F^1 < F^2 \tag{15}$$

If in addition φ is an isotone functional on F(X), then T_+ is a strictly isotone operator. i.e.

$$F^1 < F^2 \Rightarrow T_+F^1 < T_+F^2. \tag{16}$$

Proof: If

$$((F^1)^t, \varphi(F^1)) < ((F^2)^t, \varphi(F^2)) \tag{17}$$

and $\varphi(F)$ is a function of F, then either $F^1 < F^2$ and therefore we have (15) or $F^1 = F^2$ and therefore $\varphi(F^1) = \varphi(F^2)$ which contradicts (17).

On the other hand, if φ is an isotone functional then (16) is obvious. This completes the proof of Theorem 4.

Remark 1 *The first part of the Theorem 4 does not remain true if φ is a point to set map or if φ is function of x but not a function of F(X).*

Corollary 1 *If $\varphi(F) : R^k \to R$ is any real valued function of F and $F_+^t = (F^t, \varphi(F))$ then $P(F) \subseteq P(F_+)$.*

4 Some applications of the invariance theorems

Let us consider the nonlinear multicriteria fractional problem

$$\min_{x \in X} \Psi(x) \tag{18}$$

where X is a compact set and $\Psi(x)$ is upper semicontinuous on X, of the form

$$[\Psi(x)]^t = \left(\frac{f_1(x)}{h_1(x)}, \frac{f_2(x)}{h_2(x)}, \ldots, \frac{f_k(x)}{h_k(x)} \right) \tag{19}$$

with $h_j(x) > 0, \forall x \in X, \forall j \in 1, \ldots, k$, and the problem

$$\max_{x \in X} N_+ \Psi(x) \tag{20}$$

where

$$[N_+ \Psi(x)]^t = (f_1(x), f_2(x), \ldots, f_k(x), -h_1(x), -h_2(x), \ldots, -h_k(x))$$

Proposition 1 *Let $P(\Psi)$ and $P(N_+\Psi)$ denote the Pareto optimal sets to problem (18), respectively to problem (20). Then $P(\Psi) \subseteq P(N_+\Psi)$.*

Proof: The Proposition 1 is a consequence of Theorem 3 and of the following:

Lemma 1 *The operator $N_+ : R^k \to R^{2k}$ defined by $[N_+\Psi]^t = (f_1, f_2, \ldots, f_k, -h_1, -h_2, \ldots, -h_k)^t$ with $\Psi^t = \left(\frac{f_1}{h_1}, \frac{f_2}{h_2}, \ldots, \frac{f_k}{h_k} \right) \in R^k$ $h_j > 0, j = 1, \ldots, k$, is a strictly monotone kind operator, i.e. $N_+\Psi^1 < N_+\Psi^2 \Rightarrow \Psi^1 < \Psi^2$*

Proof of Lemma 1: Denote by $F^t = (f_1, f_2, \ldots, f_k)$ and by $H^t = (h_1, h_2, \ldots, h_k)$. Now, if $N_+\Psi^1 < N_+\Psi^2$ then either $F^1 < F^2$ and $H^1 \geq H^2$, or $F^1 \leq F^2$ and $H^1 > H^2$. In both situations, taking into account that $H > 0$, then $\Psi^1 < \Psi^2$. The Lemma 1 is proved.

Let us denote the denominator of the vector criteria (19) by $\eta(x) = \prod_{i=1}^{k} h_i(x)$ and consider the following change of variable:

$$\eta(x) = u_{n+1}^{-1} \tag{21}$$

$$x = \frac{1}{u_{n+1}} y$$

The change (21) of variables is a one-to-one map on X and if $\eta(x)$ is continuous on X, then (21) is topological. Let us denote by $u = (y, u_{n+1}) \in R^{n+1}$, by τ_1 the map defined by (21), and consider the problem:

$$max_{u \in U_1} G_1(u) = \max_{u \in U_1}(f_1^1(u), f_2^1(u), \ldots, f_k^1(u)) \tag{22}$$

where
$U_1 = \tau_1(X) = \{u | y/u_{n+1} \in X, \eta(y/u_{n+1} = u_{n+1}^{-1}\}$
and
$f_j^1(u) = f_j(y/u_{n+1})\eta_j(y/u_{n+1})u_{n+1}$
with
$\eta_j(y/u_{n+1}) = \prod_{i=1,i\neq j}^{k} h_i(x)$.

Proposition 2 *Let $P(\Psi)$ and $P(G_1)$ denote the Pareto optimal sets to problem (18), respectively to problem (22). Then $P(G_1) = \tau_1(P(\Psi))$ and $P(\Psi) = \tau_1^{-1}(P(G_1))$.*

Proof: The problems (18) and (22) are τ_1 - equivalent and according to Theorem 2. Proposition 2 holds.

Remark 2 *Problem (22) is not any more a fractional multicriteria problem but is not automatically more easier to solve. On the contrary, [5], if $\eta(x)$ is not linear, then U_1 is not convex even if X is convex. The conditions in which G_1 remains convex can be found in [1,11,12,13].*

5 A possible decision analysis application

Let us consider a state governed at time t_i by the vector valued function $F_i(x, t_i) = (f_1(x, t_i), f_2(x, t_i), \ldots, f_k(x, t_i))$, where $x \in H \subset R^n$, $H = \{$ the population of the state $\}$, an element x of H being considered as a point in the space of life's goods, such as income, education, health etc., and $f_j, j = 1, \ldots, k$ are some functions of this goods that govern the different aspects of the people's life. In respect to $F_i(x)$ at time t_i, we distinguish in H the P_i set, $P_i = \{the\ set\ of\ nondominated\ people\ in\ respect\ to\ F_i\}$.

We will say that the state is politically stable in period E if $P_i = P_{i+1}$ for all $t_i, t_{i+1} \in E$. It follows from our Pareto invariance theorems that a "kind government team" may allow to act in the society any activity whose output is an isotone functional of the components of the present governing vector function, without fear that the present political situation will change. More over, they may act to introduce by educational persuassion an infinity of such objectives in the people's life. The same kinds of speculations can be made if F is the caracteristic vector function of a product and the stability problem is seen in respect with the evolution of the competitive market products. Concluding, in order to change a situation, the producer or the govern must activate some nonisotone operators.

Regarding Theorem 2, one may interprete it as follows: any topological transformation of admissible goods of the state's people, leave the state politically stable in the sense of the above definition. Only nontopological transformations, consequences of the fusions, nationalizations, epidemic deseases, wars or revolutions may destabilize a society.

6 Notes and bibliographical references

Theorem 2 is an extension, given by V. Dumitru, to the multicriteria problem of Theorem 1, applying to the unicriterion optimization problem, presented in [2] and [5].

Theorem 3 and 4 were firstly presented in [8]. In the present paper a new proof to Theorem 3 is given and Theorem 4 is restated in a slight generalized form.

The results of section 4, refind in this general frame, as a consequence of the invariance theorems, known results regarding the nonlinear fractional programming problem, given in [3] and [5].

The results regarding the invariance of the Pareto optimal set under bijective maps had been noticed in the Abstract of the 12-SOR, Passau

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