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HT-graphs: centers, connected r-domination and Steiner trees

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Abstract

HT-graphs have been introduced in [11] and investigated with respect to location problems on graphs. In this paper two new characterizations of these graphs are given and then it is shown that the central vertex, connected r-domination and Steiner trees problems are linear or almost linear time solvable in HT-graphs.
Keywords: HT-graph, vertex elimination ordering, center, connected domination, Steiner tree, linear-time algorithm.

1 Introduction

All graphs in this paper are connected and simple, i.e. finite, undirected, loopless and without multiple edges. In a graph $G = (V, E)$ the *length* of a path from a vertex v to a vertex u is the number of edges in the path. The *distance* $d(u, v)$ from vertex u to vertex v is the length of a minimum length path from u to v.

The *eccentricity* $e(v)$ of a vertex v is the maximum distance from v to any vertex in G. The *radius* $r(G)$ is the minimum eccentricity of a vertex in G and *diameter* $d(G)$ is the maximum eccentricity. The $Center(G)$ is both a set of all *central vertices* of G, i.e. vertices whose eccentricities are equal to $r(G)$, and the subgraph induced by this set. The well known location problem in graphs is to find a central vertex of graph G.

Suppose $G = (V, E)$ is a connected graph with n vertices and $(r(v_1), r(v_2), \ldots, r(v_n))$ is a n-tuple of nonnegative integers. A *connected r-domination set* of graph G is a set of vertices D such that the subgraph induced by D is connected and for every vertex u in V there exists some vertex v in D satisfying $d(v,u) \leq r(u)$. The *connected r-domination problem* is to find for a graph G the connected r-dominating set with minimum cardinality.

For a given graph G and set $R \subset V$ of terminal vertices, a *Steiner tree* of G (with respect to R) is a tree which is a subgraph of G containing R. The *Steiner tree problem* is to find the minimum cardinality Steiner tree in graph G.

A graph G is *triangulated (chordal)* if every cycle of length greater than three possesses a chord, i.e. an edge joining two nonconsecutive vertices on the cycle. A *clique* of G is a set of pairwise adjacent vertices. For a graph G and nonnegative integer r the *r-neighborhood* of a vertex v is the set

$$N_r[v] = \{u \in V : d(v,u) \leq r\}.$$

Usually, $N[v]$ is used for $N_1[v]$. A vertex v of G is called *simplicial* if its neighborhood $N[v]$ is a clique. It is well known that a graph G is triangulated iff it has a *perfect elimination ordering*, i.e. an ordering $v_1, v_2, \ldots, v_n$ of V such that v_i is a simplicial vertex of the subgraph G_i induced by vertices $v_i, \ldots, v_n$; see [15].

A *k-sun S_k* $(k \geq 3)$ is a graph whose vertex set V can be partitioned into $X = (x_1, x_2, \ldots, x_k)$ and $Y = (y_1, y_2, \ldots, y_k)$ such that X is a clique in G, $0 < d(y_i, y_j) \leq 2$ iff $i = j \pm 1 (mod\ k)$ and $(x_i, y_j) \in E$ if and only if $i = j$ or $i = j + 1 (mod\ k)$. For triangulated graphs, this definition coincides with the usual definition of k-sun [13]. Triangulated graphs without induced k-suns ($k \geq 3$) are called *strongly chordal graphs* by Farber [13]. A vertex v is *simple* if the set $\{N[u] : u \in N[v]\}$ is linearly ordered by inclusion. A simple vertex is simplicial, but the converse is not necessarily true. Farber [13] proved that a graph G is strongly chordal iff it has *a simple elimination ordering*, i.e. an ordering $v_1, v_2, \ldots, v_n$ of V such that v_i is a simple vertex of G_i.

To find a central vertex of a strongly chordal graph we used the simple elimination ordering [8]. Also, using the same ordering, several authors gave efficient algorithms for solving the Steiner tree problem [23] and the connected r-domination problem [4] in a strongly chordal graph. All these algorithms are linear if we do not count the time of finding a simple elimination ordering. Note that this ordering can be obtained by algorithm for doubly lexical ordering of graph [19] with running time $O(|E|log|V|)$.

In this paper we investigate a generalization of strongly chordal graphs, namely HT-graphs (Sections 2,3). They have a weaker vertex elimination ordering (called extremal elimination ordering) and are in general even not triangulated. This ordering can be obtained in linear time. In Sections 4,5, we show that if graph G has an extremal elimination ordering then both the connected r-domination problem and central vertex problem are linear-time or almost linear-time solvable. We also show that the Steiner tree problem is a connected r-domination problem for some special n-tuple $(r(v_1), r(v_2), \ldots, r(v_n))$. This generalizes the results for strongly chordal graphs.

Notations and terminology not explained here may be found in [3,15,24].

2 HT-graphs

Let $\mathcal{N} = \{N_r[v] : v \in V, 0 \leq r \leq d(G)$ and r is integer$\}$, $\mathcal{N}_1 = \{N_1[v] : v \in V\}$, and $\mathcal{C}$ is the family of all maximal cliques of graph G. The hypergraphs $C(G) = (V, \mathcal{C})$ and $N(G) = (V, \mathcal{N}_1)$ are called respectively the *clique hypergraph* and the *neighborhood hypergraph* of graph G.

Intersecting graph $L(\mathcal{M})$ of a family of sets $\mathcal{M} = \{M_1, \ldots, M_k\}$ is defined in the following way: elements of $\mathcal{M}$ are the vertices of $L(\mathcal{M})$, and two vertices are adjacent if and only if the corresponding sets intersect. A family $\mathcal{M}$ of sets is said to have a *Helly property* if every subfamily of $\mathcal{M}$ having empty intersection contains two disjoint members.

A graph G is *Helly graph* if the family $\mathcal{N}$ of G has the Helly property. A Helly graph G is *HT-graph* if the graph $L(\mathcal{N})$ of G is triangulated [11].

The *kth power* of a graph $G = (V, E)$ is the graph $G^k = (V, E^k)$ with $(v, w) \in E^k$ if and only if $1 \leq d(v, w) \leq k$ in graph G. Clearly, $G^2 = L(\mathcal{N}_1)$.

A vertex v of G is *extremal* if there exists a vertex $u \neq v$ in V, called maximal neighbor, such that $N_2[v] = N_1[u]$. A simple vertex is extremal, but the converse is not necessarily true.

A hypergraph $H = (V, \mathcal{E})$ is a *hypertree* or *arboreal* if there exists a tree T on V such that each edge of H induces a subtree in T [3].

The following characterizations of HT-graphs are obtained in [11,12]. For other characterizations see [9].

Theorem 1 *Let $G = (V, E)$ be a graph. Then the following statements are equivalent:*

1. *G is HT-graph;*
2. *G^2 is triangulated graph and the family $\mathcal{N}_1$ of G has the Helly property;*
3. *Neighborhood hypergraph $N(G)$ of G is hypertree;*
4. *Clique hypergraph $C(G)$ of G is hypertree;*
5. *G has an extremal elimination ordering, i.e. an ordering $v_1, v_2, \ldots, v_n$ of V such that v_i $(i < n)$ is an extremal vertex in G_i.*

The dual hypergraph of $H = (V, \mathcal{E})$ is the hypergraph $H^* = (V', \mathcal{E}')$ where the vertices and the edges in H^* correspond to the edges and the vertices of H respectively, and a vertex v in H^* belongs to an edge E in H^* if and only if the edge corresponding to v in $\mathcal{E}$ contains the vertex corresponding to E in V.

Theorem 2 *Let $G = (V, E)$ be a graph. Then the following statements are equivalent:*

1. *G is triangulated HT-graph;*
2. *both its clique hypergraph $C(G)$ and the dual hypergraph $C(G)^*$ are hypertrees;*
3. *G has a perfect elimination ordering which is also an extremal elimination ordering.*

We note that since the triangulated graphs and hypertrees are recognized in linear time [21] then HT-graphs and triangulated HT-graphs are also recognized in linear time.

In the remaining part of this section we consider two new characterizations of HT-graphs. The first of them is closely related to cyclomatic number of hypertree; see [3].

Let $G = (V, E)$ be a connected graph; n and m are the numbers of vertices and edges of G respectively. Put to any edge $e \in E$ the weight $w(e)$ which equals the number of triangles (cycle of length 3) containing e. Denote by $\Delta(G)$ the total weight of maximum spanning tree of weighted graph G. From the definition of cyclomatic number of a simple connected graph G we conclude

$$2(m - n + 1) \geq \Delta(G).$$

For the HT-graphs we have

Theorem 3 *G is HT-graph if and only if* $\Delta(G) = 2(m - n + 1)$

Proof. The inequality $2(m - n + 1) \leq \Delta(G)$ for HT-graph G we shall prove by induction on the vertex number of G. Let x be the first vertex in extremal elimination ordering of HT-graph G, and y is the vertex such that $N_2[x] = N_1[y]$. By the induction's assumption for the graph $G' = G - x$ we have

$$\Delta(G') \geq 2((m - (|N[x]| - 1)) - (n - 1) + 1) = 2(m - n + 1 - |N[x]| + 2).$$

To prove the inequality $\Delta(G) \geq \Delta(G') + 2|N[x]| - 4$, we show that among all spanning trees of graph $G - x$ with maximum weight $\Delta(G')$ there exists a spanning tree T' for which any vertex from $N[x] \setminus \{x, y\}$ is adjacent to y. Assume that there is a vertex z in $N[x] \setminus \{x, y\}$, which is not adjacent to y. Let $P = (y, v_1, \ldots, v_k, z)$ be a path connecting the vertices y and z in tree T'. Since

$$N[z] \bigcap N[v_k] \subset N[z] \bigcap N[y],$$

then weight of edge (z, v_k) is no more than weight of edge (y, z) in graph $G - x$. Therefore the edge (z, v_k) can be replaced with edge (y, z) to obtain a new spanning tree with total weight $\Delta(G')$.

So, each vertex from $N[x] \setminus \{x, y\}$ is adjacent to y in T'. Then we obtain the spanning tree T from T' by adding vertex x and edge (x, y). Total weight of T is equal to $\Delta(G') + |N[x]| - 2 + |N[x]| - 2$. Hence

$$\Delta(G) \geq \Delta(G') + 2|N[x]| - 4 \geq 2(m - n + 1).$$

Conversely. Let for a graph G we have $\Delta(G) = 2(m - n + 1)$. By Theorem 1 it is sufficient to prove that the neighborhood $N[v]$ of every vertex $v \in V$ induces the connected subgraph in spanning tree T of weight $\Delta(G)$. The proof proceeds by induction on the number of vertices of G.

Let x be a pendant vertex of T, and y is its neighbor. Obviously, the tree $T' = T - x$ is a spanning tree of graph $G' = G - x$. Let $w(T')$ be the total weight of T'. Then

$$w(T') \leq \Delta(G') \leq 2(m' - n' + 1) = 2(m - |N[x]| + 1 - n + 1 + 1) =$$

$$2(m - n + 1) - 2(|N[x]| - 2) = \Delta(G) - 2(|N[x] - 2).$$

On the other hand, $\Delta(G) \leq w(T') + 2(|N[x]| - 2)$. Hence,

$$w(T') = \Delta(G') = 2(m' - n' + 1) = \Delta(G) - 2(|N[x] - 2).$$

This is possible only in case when in tree T all vertices from $N[x] \setminus \{x, y\}$ ($N[x]$ is taken in G) are adjacent to y. From the induction's assumption and the form of tree T it follows that the neighborhood $N[v]$ of every vertex $v \in V$ induces connected subgraph in T.[]

An *isometric subgraph* of a graph G is an induced subgraph in which the distance between any two vertices v, w equals the distance $d(v, w)$ taken in G. Let C_k be the induced cycle of length k.

For another characterization of HT-graphs we need a lemma:

Lemma 1 *Let $G = (V, E)$ be a Helly graph. Then the following statements are equivalent:*

1. *G^2 has no cycles C_k ($k \geq 4$), i.e. G^2 is triangulated graph;*

2. G does not contain a sun S_k $(k \geq 4)$ as an isometric subgraph.

Proof. 1) $\Rightarrow$ 2) Suppose that graph G contains a sun S_k $(k \geq 4)$ with vertices $\{x_1, x_2, \ldots, x_k\} \bigcup \{y_1, y_2, \ldots, y_k\}$ as an isometric subgraph. Then the vertices $y_1, y_2, \ldots, y_k$ induce in graph G^2 a cycle C_k $(k \geq 4)$.

2) $\Rightarrow$ 1) The proof proceeds by induction on the number of vertices of G. Since G is Helly graph, then there exists a pair of vertices $v, y \in V$ such that $N[y] \subset N[v]$; see [1,10]. It is easy to see that $G - y$ is also a Helly graph without isometric suns S_k $(k \geq 4)$. Therefore, by induction, the graph $(G - y)^2$ has no cycles C_k $(k \geq 4)$.

Now let G^2 contains a cycle C_k with vertices $y_1, y_2, \ldots, y_k$ $(k \geq 4)$ and k be as small as possible. It is enough to consider only the case when $y_1 = y$. In graph $(G - y)^2$ the vertices $v, y_2, \ldots, y_k$ form a cycle of length k. Since $(G - y)^2$ does not contain C_k $(k \geq 4)$, this cycle is divided by chords into triangles. Note that one of the ends of these chords is v.

Hence, in graph G we have

$$
\begin{array}{ll}
d(y_i, v) = d(y_i, y) - 1 = 2 & \text{for all} \quad i = 3, 4, \ldots, k-1; \\
d(y_i, y_j) \leq 2 & \text{if and only if} \quad j = i \pm 1 (mod k); \\
d(y_2, v) \leq 2; \quad d(y_k, v) \leq 2. &
\end{array}
$$

By Helly property for every three vertices v, y_i, y_{i+1} $(i = 2, 3, \ldots, k-2)$ there is a common neighbor x_i. In the similar way for vertices y, v, y_2, x_2 there exists a vertex $x_1 \in N[y] \bigcap N[v] \bigcap N[y_2] \bigcap N[x_2]$. Let us suppose, without loss of generality, that neither v nor x_2 coincides with x_1. Now we consider the vertices y, y_k, y_{k-1}, x_{k-2}. They form in graph G^2 a cycle of length 4. Since G has no isometric sun S_4, this cycle is not induced; see [9]. Hence, $d(y_k, x_{k-2}) = 2$. Then, by Helly property, for the vertices v, y_{k-1}, y_k, x_{k-2} there is a common neighbor x_{k-1} too. Among all sets $M = \{x_1, x_2, \ldots, x_{k-1}\}$ we choose a set inducing a subgraph with a maximal number of edges.

We claim that M is a clique. Assume the contrary and let x_i be the vertex with the smallest index for which there is a vertex x_j such that x_i and x_j are not adjacent, and $j < i < k$. Since $d(x_i, y) = 2$, the vertices $y, y_2, \ldots, y_i, x_i$ in case $i < k-1$ or the vertices $y, y_2, \ldots, y_{i-1}, x_i$ in case $i = k-1$ form in graph G^2 a cycle of length $< k$. By virtue of minimality of k, this cycle is not induced. Hence, in graph G we have $d(x_i, y_t) = 2$ for every $t = 1, 2, \ldots, i-1$ and so the distance between any two vertices from $A = \{N[x_j] \bigcap M\} \bigcup \{x_i, y_j, y_{j+1}\}$ is ≤ 2. By Helly property there exists a vertex w adjacent to every vertex from A. Then the subgraph induced by set $M \setminus \{x_j\} \bigcup \{w\}$ has more edges than the subgraph induced by M; a contradiction.

Thus, the vertices $x_1, x_2, \ldots, x_{k-1}$ induce a clique. Further, consider the family of neighborhoods $\{N[w] : w \in M \bigcup \{y, y_k\}\}$. Since these neighborhoods pairwise intersect, then, by Helly property, there exists some vertex x_k adjacent to every vertex from $M \bigcup \{y, y_k\}$. Now we conclude that the vertices $\{x_1, \ldots, x_k\} \bigcup \{y_1, \ldots, y_k\}$ induce an isometric sun S_k.[]

From Theorem 1 and Lemma 1 we deduce one more characterization of HT-graphs.

Theorem 4 *G is HT-graph if and only if G is Helly graph without isometric suns S_k ($k \geq 4$).*

From the preceding theorems it follows that G is strongly chordal graph if and only if every induced subgraph of G is HT-graph. In other words, strongly chordal graphs are exactly hereditary HT-graphs.

3 Some aspects of extremal ordering

Since G is HT-graph if and only if $N(G)$ is hypertree, we can apply the maximum cardinality search [21] on hypergraph $N(G)$ to obtain an extremal elimination ordering of graph G. So, we have the following variant of maximal cardinality search, which operates directly on G.

Algorithm 1. Find an extremal elimination ordering of HT-graph

Input. An HT-graph G.

Output. An extremal elimination ordering $v_1, v_2, \ldots, v_n$ of G.
Initially all vertices of G are unnumbered and unmarked;
Using 1–5 number the vertices from n to 1 in decreasing order:

1. number an arbitrary vertex $v \in V$;
2. among all unmarked vertices select a numbered vertex u such that $N[u]$ contains as many numbered vertices as possible;
3. number all unnumbered vertices from $N[u]$ consecutively;
4. mark vertex u;
5. repeat steps 2–4 if the unnumbered vertices exist.

The correctness and linearity of this algorithm follows from [21]. Note that as a result of algorithm we have also for every vertex v_i $(i < n)$ its maximal neighbor in graph G_i. To obtain for a triangulated HT-graph an extremal elimination ordering which is also a perfect elimination ordering, we must implement step 3 of Algorithm 1 as follows:

a) among all unnumbered vertices from $N[u]$ choose a vertex w such that $N[w]$ has as many numbered vertices as possible;

b) number vertex w;

c) repeat steps a), b) if the unnumbered vertices in $N[u]$ exist.

Obviously the time bound for this new algorithm is also linear if a suitable implementation is chosen. For a proof of correctness assume that the ordering generated by algorithm is not a perfect elimination ordering. Then similar to [20] we establish that for some vertices v_i, v_j with $i < j < n$ the strict inclusion $N[v_j] \bigcap \{v_{j+1}, \ldots, v_n\} \subset N[v_i] \bigcap \{v_{j+1}, \ldots, v_n\}$ holds. So in step a) we could not have selected vertex v_j before v_i.

As a result we have

Theorem 5 *Algorithm 1 correctly finds in time $O(|E|)$ an extremal elimination ordering of HT-graph. For triangulated HT-graph this ordering is also a perfect elimination ordering.*

In section 5 we will use the following two lemmas. Let $v_1, v_2, \ldots, v_n$ be an extremal elimination ordering of graph G, generated by Algorithm 1.

Lemma 2 *If in graph G for vertices v_i, v_j with $i < n$ the inclusion $N[v_i] \subset N[v_j]$ holds, then $v'_1, v'_2, \ldots, v'_{n-1}$ is an extremal elimination ordering of graph $G - v_i$, where $v'_k = v_{k+1}$ when $k > i$ and $v'_k = v_k$ otherwise.*

Proof. Obviously the inclusion $N[v_i] \subset N[v_j]$ holds for every subgraph G_l of graph G induced by vertices $v_l, v_{l+1}, \ldots, v_n$ where $l \leq min\{i, j\}$. So it is enough to prove that the vertex v_k with $j \leq k < i$ is extremal in graph $G_k - v_i$.

Let v_s be a maximal neighbor of vertex v_i in graph G_i found by Algorithm 1. Since the vertices v_j and v_s are adjacent in G, then every vertex v_k with $j \leq k \leq i$ gets its number while the numbered and unmarked vertex v_s is examined (see Algorithm 1). Hence the vertex v_s is also a maximal neighbor of vertex v_k in graph G_k, i.e. $N_2[v_k] = N_1[v_s]$. It remains to note that the equality $N_2[v_k] = N_1[v_s]$ holds in graph $G_k - v_i$ too.[]

Lemma 3 *If in graph G for vertices v_i, v_k, v_j the inclusions $N[v_i] \subset N[v_j]$ and $N[v_k] \subset N[v_j]$ hold, then $v_1, v_2, \ldots, v_n$ is an extremal elimination ordering of graph G', obtained from G by adding an edge (v_i, v_k).*

This statement has a similar proof with Lemma 2.

4 Centers of HT-graphs

In this section we will present a linear algorithm for finding a central vertex of an HT-graph. Note that for general graphs with n vertices and m edges the upper bound on the time complexity of this problem is $O(nm)$ and the lower bound is $\Omega(m)$. Hence the presented algorithm is optimal. Other linear algorithms for finding central vertices are known for trees [16], 2-trees and maximal outerplanar graphs [14] and strongly chordal graphs [8].

It is known [9,10] that the central subgraph $Center(G)$ of Helly graph G is isometric and also a Helly graph. Moreover, we have

Lemma 4 *If G is Helly graph and G^2 does not contain C_4, then $r(Center(G)) \leq 1$.*

Proof. Since for Helly graph G the inequality $d(G) \geq 2r(G) - 1$ holds [9], it is enough to prove, that $d(Center(G)) \leq 2$. Assume the contrary and let x, y be the vertices from $Center(G)$, for which $d(x, y) = 3$. We note that the distance between any two vertices from $N_2[x] \bigcap N_2[y]$ is ≤ 2. Otherwise, if $d(v, w) \geq 3$ for some $v, w \in N_2[x] \bigcap N_2[y]$, then the vertices x, v, y, w induce a cycle C_4 in G^2; contradiction. By Helly property, there exists a common neighbor z of vertices from $N_2[x] \bigcap N_2[y]$.

Let v be a vertex such that $d(v, z) = e(z)$. Since $x, y \in Center(G)$, then $d(v, x) \leq d(v, z)$ and $d(v, y) \leq d(v, z)$. Hence, the neighborhoods $N_2[x], N_2[y], N_{d(v,z)-2}[v]$ pairwise intersect. Therefore, there is a vertex $u \in N_2[x] \bigcap N_2[y]$, such that $d(u, v) \leq d(v, z) - 2$, i.e. $d(u, z) \geq 2$. This is in conflict with $N[z] \supset N_2[x] \bigcap N_2[y]$. So we have $d(Center(G)) \leq 2$.[]

The following theorem gives a complete description of central subgraph of HT-graphs.

Theorem 6 *A graph G is central subgraph of HT-graph H, i.e. $G \cong Center(H)$, if and only if $r(G) \leq 1$.*

Proof. *Necessity* follows from Lemma 4.

Sufficiency. Let $r(G) \leq 1$. Graph H is obtained from G by adding to G four new vertices u_1, u_2, v_1, v_2 so that: v_1 is adjacent to u_1 and to any vertices of graph G; v_2 is adjacent to u_2 and also to any vertices of graph G; and the vertices u_1, u_2 are pendant in H. From the construction of graph H we have $e(v_1) = e(v_2) = 4, e(u_1) = e(u_2) = 5$ and $e(x) = 3$ for all vertices x of G. Hence, G is central subgraph of graph H.[]

Denote by $D(v)$ the set of all farthest from v vertices, i.e. $D(v) = \{w \in V : d(v, w) = e(v)\}$.

Lemma 5 *For any vertex v of HT-graph G and any farthest vertex $u \in D(v)$ we have $e(u) \geq 2r(G) - 2$.*

Proof. It is known [9,10] that in Helly graph G the eccentricity of any vertex v is computed by formula $e(v) = d(v, Center(G)) + r(G)$. Let x and y be the closest to v and $u \in D(v)$, respectively, vertices of central subgraph $Center(G)$. From the inequalities

$$d(v,u) \leq d(v,x) + d(x,u) \leq d(v,x) + r(G) = e(v) = d(v,u)$$

we conclude that $d(x,u) = r(G)$. Since, by Lemma 4, $d(x,y) \leq 2$, then

$$e(u) = d(u,y) + r(G) \geq d(x,u) - 2 + r(G) = 2r(G) - 2.$$

Now we give the algorithm.

Algorithm 2. Find the central vertex and radius of HT-graph.

Input. An HT-graph G with an extremal elimination ordering $v_1, v_2, \ldots, v_n$ $(n \geq 2)$.

Output. A central vertex and radius of G.

```
for i := 1 to n do a(v_i) = 0;
w := the farthest vertex from v_1;
compute the eccentricity e(w);
k := [(e(w) + 1)/2];
for i := 1 to n do
    begin
       v_j := a maximal neighbor of v_i;
       if a(v_i) = k - 1 then
         begin
            compute the eccentricity e(v_j);
            if a(v_j) > k then k := k + 1
              else begin
                        v_j is a central vertex of G;
                        r(G) := k;
                        stop
                   end
         end
       a(v_j) := max{a(v_i) + 1, a(v_j)}
    end
```

The correctness of Algorithm 2 follows from Lemma 5 and lemmas described below. These lemmas are from [8,9], but instead of a simple vertex we use an extremal vertex.

Let us assign to every vertex $v \in V$ a nonnegative integer $a(v)$. We consider the following notations:

$$\begin{aligned}
&e(v,a) = max\{a(u) + d(v,u) : u \in V\},\\
&r(G,a) = min\{e(v,a) : v \in V\},\\
&Center(G,a) = \{v \in V : e(v,a) = r(G,a)\}.
\end{aligned}$$

Obviously, if $a(v) = 0$ for all $v \in V$, then $e(v,a) = e(v)$ and $r(G,a) = r(G)$.

Suppose v is an extremal vertex with a maximal neighbor u in a graph G and $G' = G - v$ is a graph obtained from G by deleting the vertex v.

Lemma 6 *If $a(v) + 1 < r(G,a)$, then for every vertex w of G' holds $e'(w,a') = e(w,a)$, where $e'(w,a')$ is the eccentricity of w in graph G' with $a'(x) = a(x)$ when $x \neq u$, and $a'(u) = max\{a(v)+1, a(u)\}$. In particular, $r(G',a') = r(G,a)$ and $Center(G',a') \subset Center(G,a)$.*

Lemma 7 *If $a(v) + 1 = r(G,a)$ and $a(w) < r(G,a)$ for all $w \in N[v]$, then $u \in Center(G,a)$.*

Proofs of these lemmas are straightforward and omitted. We note also that according to Lemma 5, the initial k in our algorithm equals $r(G)$ or $r(G) - 1$. Therefore, there is at most one improvement of the value of k.

Summarizing the results of this section, we have the following theorem.

Theorem 7 *Algorithm 2 correctly finds a central vertex of HT-graph in time $O(|E|)$.*

5 Connected r-domination and Steiner trees

The connected domination problem (connected r-domination problem when $r(v) = 1$ for all $v \in V$) is NP-complete for planar bipartite graphs [23], chordal bipartite graphs [18], $2-CUBs$ [6] and split Helly graphs [11]. Efficient algorithms have been found for the connected domination problem in strongly chordal graphs [23], 2-trees [23], series-parallel graphs [23], permutation graphs [5] and distance-hereditary graphs [7]. For connected r-domination problem the polynomial algorithm is known only in strongly chordal graphs [4].

The Steiner tree problem is also NP-complete for planar bipartite graphs [23], chordal bipartite graphs [18] and split Helly graphs (it follows from the results of [11,23]). However, efficient algorithms have been found in strongly chordal graphs [23], 2-trees and series-parallel graphs [22], permutation graphs [5] and distance-hereditary graphs [7].

In this section we generalize the algorithm from [4], constructing the connected r-dominating set in strongly chordal graph, for HT-graphs. We also show that the Steiner tree problem is a connected r-domination problem for some special n-tuple $(r(v_1), r(v_2), \ldots, r(v_n))$.

To establish an algorithm we need the following lemmas; some of them are from [4] (but instead of a simple vertex we use an extremal vertex); others are similar. Proofs of these lemmas are straightforward and omitted.

Lemma 8 *If v is an extremal vertex of a graph with at least two vertices, then there exists a minimum connected r-dominating set D in which $v \notin D$ if and only if $r(v) \neq 0$.*

Suppose v is an extremal vertex with a maximal neighbor u in graph $G = (V, E)$.

Lemma 9 *Let $r(v) > 0$. A subset $D \subset V \setminus \{v\}$ is a minimum connected r-dominating set of G if D is a minimum connected r'-dominating set of $G - v$ with $r'(x) = r(x)$ when $x \neq u$, $r'(u) = r(u)$ when $r(w) = 0$ for some $w \in N[v]$ and $r'(u) = min\{r(u), r(v) - 1\}$ otherwise.*

Lemma 10 *Let $r(v) = 0, r(x) = 0$ for some vertex $x \in V \setminus \{v\}$, and $r(w) \neq 0$ for every $w \in N[v] \setminus \{v\}$. A set D is a minimum connected r-dominating set of G if $D = D' \bigcup \{v\}$ where D' is a minimum connected r'-dominating set of $G - v$ with $r'(x) = r(x)$ when $x \neq u$ and $r'(u) = 0$.*

Lemma 11 *Let $r(v) = 0$, v is also a simplicial vertex of G, and $r(w) = 0$ for some vertex $w \in N[v] \setminus \{v\}$. A set D is a minimum connected r-dominating set of G if $D = D' \bigcup \{v\}$ where D' is a minimum connected r-dominating set of $G - v$.*

Unfortunately, Lemma 11 holds only for such extremal vertex, which is also simplicial. Similar to Lemma 11 for arbitrary extremal vertex v we have a more general result as follows.

Lemma 12 *Let $r(v) = 0$ and $r(w) = 0$ for some vertex $w \in N[v] \setminus \{v\}$. A set D is a minimum connected r-dominating set of G if $D = D' \bigcup \{v\}$ where D' is a minimum connected r-dominating set of graph G^* obtained from $G - v$ by deleting every vertex $x \in N[v]$ such that $r(x) \neq 0$ and $x \neq u$, and by adding some new edges (in case of need) so that a subgraph induced by set $F = \{x \in N[v] \setminus \{v\} : r(x) = 0\}$ becomes connected.*

Basing on the above lemmas and Lemmas 2, 3, we have the following algorithm (compare to [4]). In the algorithm, we will only use $G := G - v$ for deleting v from G and $G := G_F$ for adding new edges to G between some vertices of F without detailed implementations, and degrees of vertices are considered updated automatically.

Algorithm 3. Find a minimum connected r-dominating set of HT-graph.

Input. An HT-graph G and n-tuple $(r(v_1), r(v_2), \ldots, r(v_n))$.

Output. A minimum connected r-dominating set D of G.

$D := \emptyset$;
if $r(v) > 0 \ \forall v \in V$ **then**

```
begin
  using Algorithm 1 find an extremal elimination ordering
  v_1, v_2, ..., v_n of G;
  for i := 1 to n do
    begin
      v_j := a maximal neighbor of v_i;
      r(v_j) := min{r(v_i) - 1, r(v_j)};
      G := G - v_i;
      if r(v_j) = 0 then goto outloop
    end
end;
outloop: { now r(v) = 0 for some v ∈ V and suppose G has p vertices.}
using Algorithm 1 find an extremal elimination ordering
 v_1, v_2, ..., v_p of G with r(v_p) = 0;
 for i := 1 to p do a(v_i) = 0;
 for i := 1 to p - 1 do
   if a(v_i) = 0 then
     begin
       v_j := a maximal neighbor of v_i;
       if r(v_i) = 0 then D := D ⋃ {v_i};
       if r(x) > 0 ∀x ∈ N[v_i] \ {v_i}
       then
         if r(v_i) = 0 then r(v_j) := 0
         else r(v_j) := min{r(v_i) - 1, r(v_j)};
       else
         if r(v_i) = 0 then
           begin
             F := ∅;
             for x ∈ N[v_i] \ {v_i} do
               if r(x) = 0 then F := F ⋃ {x}
                 else if x ≠ v_j then
                   begin
                     a(x) := 1;
                     G := G - x
                   end
             G := G - v_i;
             G := G_F {add some new edges (in case of need) so
                       that a subgraph induced by set F became
                       connected }
           end
         else G := G - v_i
     end
     D := D ⋃ {v_p}
```

The running time of this algorithm is $O(|E| + |E'|)$, where $|E'|$ is a number of added edges. It is easy to see that $|E| + |E'|$ is less than an edge number of graph G^2.

So, we have

Theorem 8 *Algorithm 3 correctly finds a minimum cardinality connected r-dominating set of HT-graph in time $O(|E| + |E'|)$. For triangulated HT-graph the running time of this algorithm is $O(|E|)$.*

Finally, we consider the problem of finding a minimum cardinality Steiner tree in HT-graph. For the given graph $G = (V, E)$ and set $R \subset V$ we have the following obvious lemma.

Lemma 13 *A set $D \subset V$ is a minimum connected r-dominating set of G with $r(v) = 0$ when $v \in R$ and $r(v) = 2r(G)$ otherwise if and only if a spanning tree of a subgraph induced by D is a minimum cardinality Steiner tree of G (with respect to R).*

Thus, there are quite efficient algorithms, solving the central vertex, connected r-domination and Steiner tree problems in HT-graphs. Note that the r-domination problem (not necessary connected) is efficiently solvable in HT-graphs too [11,12] (see also [2] for 1-domination).

After the initial version of this paper was submitted, we learned the paper by M. Moscarini [17], in which she develops an $O(|V|^3)$ algorithm for connected domination and Steiner tree problems in triangulated HT-graphs (called in [17] doubly chordal graphs). Our results improve and generalize the results of M. Moscarini.

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