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Continuous Gromov–Hausdorff distance

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Abstract. This paper is devoted to the study of the properties of a continuous analogue of the Gromov–Hausdorff distance.

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1 Introduction

The concept of a metric space is one of the fundamental concepts of mathematics. Many constructions preserve the property of space metrizability. An example is the space $\mathcal{K}(X)$ of all non-empty compact subsets of a given metric space X.

Let (X,ρ) be a metric space and x and y be its points. Then $|xy| = \rho(x,y)$ denotes the distance between these points. If A and B are non-empty subsets of X, then we set $|AB| = |BA| = \inf\bigl\{|ab| : a \in A,\, b \in B\bigr\}$. If $A = \{a\}$, then instead of $\bigl|\{a\}B\bigr| = \bigl|B\{a\}\bigr|$ we write $|aB| = |Ba|$. For a positive real s and a nonnegative real r, we define

- the *open ball with center $a \in X$ and radius s* as $U_s(a) = \{x \in X : |xa| < s\}$;
- the *open s-neighborhood of non-empty $A \subset X$* as $U_s(A) = \{x \in X : |xA| < s\}$;
- the *closed ball with center $a \in X$ and radius r* as $B_r(a) = \{x \in X : |xa| \le r\}$;
- the *closed r-neighborhood of non-empty $A \subset X$* as $B_r(A) = \{x \in X : |xA| \le r\}$.

For non-empty subsets $A, B \subset X$, the value

$$d_H(A,B) = \max\bigl\{\sup_{a\in A}|aB|,\, \sup_{b\in B}|Ab|\bigr\}$$

is called the *Hausdorff distance between A and B*. An equivalent definition is:

$$d_H(A,B) = \inf\bigl\{r : A \subset B_r(B) \ \text{ and } \ B \subset B_r(A)\bigr\}.$$

It is well known that d_H is a generalized pseudometric on the set of all non-empty subsets of a metric space X. Here, the word "generalized" means that d_H may

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be equal to infinity on some pairs of subsets (for example, on a bounded and an unbounded set), and the word "pseudometric" means that d_H may be equal to zero on some pairs of distinct subsets (for example, on a non-closed set and its closure).

M. Gromov defined the distance between two non-empty metric spaces [1–3]. *The Gromov–Hausdorff distance* $d_{GH}(X,Y)$ is the greatest lower bound of the Hausdorff distances between the images of the spaces X and Y under their isometric embeddings in all possible metric spaces.

In this paper, we study a continuous analogue of the Gromov–Hausdorff distance $d^c_{GH}(X,Y)$ (Formula (7)). We show that

- the continuous Gromov–Hausdorff distance $d^c_{GH}(X,Y)$ is a generalized pseudometric[1] (Proposition 5) and it dominates the Gromov–Hausdorff metric $d_{GH}(X,Y) \le d^c_{GH}(X,Y)$ (Property (2) of Proposition 6);
- for zero-dimensional metric spaces these distances coincide (Corollary 3), and the continuous distance from a connected space X to any zero-dimensional space is not less than the diameter of this connected space (Proposition 12);
- although every continuum in the Euclidean space $\mathbb{R}^n$, $n \ge 2$, is at zero distance from the set of all pseudo-arcs (Bing's Theorem [6]), its continuous Gromov–Hausdorff distance to the set of all pseudo-arcs is equal to half the diameter of this continuum (Proposition 19);
- Cook's continuum provides a contrasting example of the difference between the Hausdorff and Gromov–Hausdorff metrics on one hand and the continuous Gromov–Hausdorff metric on the other (Corollary 10);
- a metric space Y at zero continuous Gromov–Hausdorff distance from a sphere S^n, $n \ge 1$, with an arbitrary metric, is isometric to this sphere (Corollary 4). But for every metric on the sphere S^n, $n \ge 1$, there are at least continuum pairwise non-isometric metric spaces at zero classical Gromov–Hausdorff distance from this sphere S^n (for example, spaces obtained from the sphere by discarding of finite sets of points that are pairwise nonisometric to each other).

We also show that, unlike the classical Gromov–Hausdorff distance [7–9], the continuous Gromov–Hausdorff distance is not complete (Theorem 11). Here it should be emphasized that all metric spaces do not form a set, and therefore do not form a topological space. For complete rigor of exposition, a transition to the theory of von Neumann–Bernays–Gödel classes [10] is necessary. Taking this into account, the concept of completeness of the (continuous) Gromov–Hausdorff distance is well defined [11]. The assertion we proved that the continuous Gromov–Hausdorff distance is intrinsic (Theorem 10) also has a correct meaning. However, in order to remain within the framework of set theory, one can only consider metric spaces of some cardinalities bounded from above by some fixed one.

[1] In [4], some analogue of the continuous Gromov–Hausdorff distance is also considered, but there this distance does not satisfy the triangle inequality, which leads to numerous technical difficulties. Instead of that, we follow the approach of [5].

2 Definitions

The proper class of all metric spaces considered up to isometry is denoted by $\mathcal{GH}$, and the subset of $\mathcal{GH}$ consisting of all compact metric spaces is denoted by $\mathcal{M}$.

A more technical definition of the Gromov–Hausdorff distance is often used. For non-empty metric spaces X and Y, the set of non-empty relations between X and Y, i.e., non-empty subsets of the Cartesian product $X\times Y$, is denoted by $\mathcal{P}_0(X\times Y)$. For $\sigma\in\mathcal{P}_0(X\times Y)$, we define the *distortion* by setting

$$\operatorname{dis}\sigma=\sup\Bigl\{\bigl||xx'|-|yy'|\bigr|:(x,y),\,(x',y')\in\sigma\Bigr\}. \tag{1}$$

In particular, if $f\colon X\to Y$ is an arbitrary mapping, then we define the *distortion* $\operatorname{dis} f$ for it as the distortion of its graph

$$\operatorname{dis} f=\sup\Bigl\{\bigl||xx'|-|f(x)f(x')|\bigr|:x,x'\in X\Bigr\}. \tag{2}$$

Let us note a well-known property of distortion of the relations composition.

Proposition 1. *If $\sigma\in\mathcal{P}_0(X\times Y)$, $\tau\in\mathcal{P}_0(Y\times Z)$ and $\tau\circ\sigma\neq\emptyset$ then* $\operatorname{dis}(\tau\circ\sigma)\le\operatorname{dis}\sigma+\operatorname{dis}\tau$.

A multivalued surjective mapping R from X to Y is called a *correspondence between X and Y*. The set of all correspondences between X and Y is denoted by $\mathcal{R}(X,Y)$. The Gromov–Hausdorff distance is calculated by the formula [3, Theorem 7.3.25]

$$d_{GH}(X,Y)=\frac12\inf\bigl\{\operatorname{dis}R:R\in\mathcal{R}(X,Y)\bigr\}. \tag{3}$$

A special case of a correspondence can be constructed for any pair of mappings $f\colon X\to Y$ and $g\colon Y\to X$ by setting $R_{f,g}=f\cup g^{-1}$, where g^{-1} denotes the inverse relation to the relation g. On the other hand, in each correspondence $R\in\mathcal{R}(X,Y)$, a subcorrespondence $R_{f,g}$ can be extracted if $f\colon X\to Y$ and $g\colon Y\to X$ are chosen such that $f\subset R$ and $g\subset R^{-1}$. It is easy to see that the distortion of each subcorrespondence does not exceed the distortion of the correspondence, in particular, $\operatorname{dis}R_{f,g}\le\operatorname{dis}R$. To calculate the distortion of the correspondence $R_{f,g}$, we need the *codistortion* $\operatorname{codis}(f,g)$, defined as follows:

$$\operatorname{codis}(f,g)=\sup\Bigl\{\bigl||x\,g(y)|-|f(x)\,y|\bigr|:x\in X,\,y\in Y\Bigr\}. \tag{4}$$

Proposition 2. *For any $X,Y\in\mathcal{GH}$, a correspondence $R\in\mathcal{R}(X,Y)$ and any two mappings $f\colon X\to Y$ and $g\colon Y\to X$ such that $f\subset R$ and $g\subset R^{-1}$, the following holds*:

$$\operatorname{dis}R_{f,g}=\max\bigl\{\operatorname{dis}f,\,\operatorname{dis}g,\,\operatorname{codis}(f,g)\bigr\}\le\operatorname{dis}R. \tag{5}$$

From Equality (3) and Proposition 2 the following result is immediately obtained.

Corollary 1 (see [5, p. 3]). *For any $X, Y \in \mathcal{GH}$, the following holds*:

$$d_{GH}(X,Y) = \frac{1}{2} \inf_{\substack{f\colon X\to Y\\ g\colon Y\to X}} \max\bigl\{\operatorname{dis} f,\, \operatorname{dis} g,\, \operatorname{codis}(f,g)\bigr\}. \tag{6}$$

It follows from Proposition 1 that the distortion of the composition of mappings does not exceed the sum of the distortions of these mappings.

Proposition 3. *Let three metric spaces X, Y and Z be given, as well as mappings $X \underset{g}{\overset{f}{\rightleftarrows}} Y \underset{k}{\overset{h}{\rightleftarrows}} Z$. Then*

$$\operatorname{codis}(h\circ f, g\circ k) \le \operatorname{codis}(f,g) + \operatorname{codis}(h,k).$$

Proof. Let's choose arbitrary $x \in X$, $z \in Z$ and let's put $y_1 = f(x)$, $y_2 = k(z)$. Then

$$\begin{aligned}\Bigl|\bigl|x\,(g\circ k)(z)\bigr| - \bigl|(h\circ f)(x)\,z\bigr|\Bigr| &= \Bigl|\bigl|x\,g(y_2)\bigr| - \bigl|h(y_1)\,z\bigr|\Bigr| = \\ &= \Bigl|\bigl|x\,g(y_2)\bigr| - |y_1y_2| + |y_1y_2| - \bigl|h(y_1)\,z\bigr|\Bigr| \le \\ \le \Bigl|\bigl|x\,g(y_2)\bigr| - \bigl|f(x)y_2\bigr|\Bigr| &+ \Bigl|\bigl|y_1k(z)\bigr| - \bigl|h(y_1)\,z\bigr|\Bigr| \le \operatorname{codis}(f,g) + \operatorname{codis}(h,k).\end{aligned}$$

The result follows from the arbitrariness of x and z. □

If in Corollary 1 we restrict ourselves to continuous mappings f and g, then we obtain the definition of the *continuous Gromov–Hausdorff distance*:

$$d^c_{GH}(X,Y) = \frac{1}{2} \inf_{\substack{f\in C(X,Y)\\ g\in C(Y,X)}} \operatorname{dis} R_{f,g} = \frac{1}{2} \inf_{\substack{f\in C(X,Y)\\ g\in C(Y,X)}} \max\bigl\{\operatorname{dis} f,\, \operatorname{dis} g,\, \operatorname{codis}(f,g)\bigr\}, \tag{7}$$

where $C(Z,W)$ denotes the set of all continuous mappings from the metric space Z to the metric space W. For brevity, we will also call this distance the *continuous GH-distance.*

For a (non-single-point) metric space X and a point $x \in X$, we define the circumscribed and inscribed radii (centered at x) as $R_x = \sup\bigl\{|xx'| : x' \in X\bigr\}$ and $r_x = \inf\bigl\{|xx'| : x' \in X,\ x' \ne x\bigr\}$, respectively. We introduce four values:

$$\begin{aligned}\operatorname{diam}(X) &= \sup\bigl\{R_x : x\in X\bigr\} = \sup\bigl\{|xx'| : x,x'\in X\bigr\}\ (\textit{diameter}),\\ R(X) &= \inf\bigl\{R_x : x\in X\bigr\}\ (\textit{Chebyshev radius}),\\ d(X) &= \sup\bigl\{r_x : x\in X\bigr\},\\ s(X) &= \inf\bigl\{r_x : x\in X\bigr\} = \inf\bigl\{|xx'| : x,x'\in X,\ x'\ne x\bigr\}.\end{aligned}$$

For a one-point metric space we reserve the notation Δ_1. It is natural to assume $\operatorname{diam}(\Delta_1) = R(\Delta_1) = d(\Delta_1) = s(\Delta_1) = 0$. It is known [3, p. 255] that

$2d_{GH}(\Delta_1, X) = \operatorname{diam} X$. Note that for an unbounded space, it holds $\operatorname{diam}(X) = R(X) = \infty$.

It is clear that for a non-single-point metric space, $d(X) = 0$ if and only if X has no isolated points. For a non-single-point finite metric space, $s(X) > 0$. When $s(X) > 0$ we say that the metric is bounded away from zero. For the standard sphere $S^m \subset \mathbb{R}^{m+1}$, equipped with both the metric induced from $\mathbb{R}^{m+1}$ and the intrinsic metric, we have $0 = s(S^m) = d(S^m) < R(S^m) = \operatorname{diam}(S^m)$; for a ball $B^m \subset \mathbb{R}^m$, we have $0 = s(B^m) = d(B^m) < 2R(B^m) = \operatorname{diam}(B^m)$.

The next result is obvious.

Proposition 4. *For an arbitrary metric space X, the following chains of inequalities are valid*

$$0 \le s(X) \le d(X) \le \operatorname{diam}(X) \le 2R(X) \quad \textit{and} \quad s(X) \le R(X) \le \operatorname{diam}(X).$$

For a metric space (X, ρ) and a number $\lambda \ge 0$, we will denote by λX the set X with (pseudo)metric $\lambda\rho$.

3 General properties of the continuous Gromov–Hausdorff distance

Proposition 5. *The distance d^c_{GH} satisfies the triangle inequality.*

Proof. Let X, Y, and Z be arbitrary metric spaces. We will show that

$$d^c_{GH}(X, Z) \le d^c_{GH}(X, Y) + d^c_{GH}(Y, Z).$$

If one of the distances $d^c_{GH}(X, Y)$ or $d^c_{GH}(Y, Z)$ is infinite, then the inequality holds. Now let both of these distances be finite. For any $\varepsilon > 0$, choose $f \in C(X, Y)$ and $g \in C(Y, X)$, as well as $h \in C(Y, Z)$ and $k \in C(Z, Y)$, for which $d^c_{GH}(X, Y) \ge \max\bigl\{\operatorname{dis} f,\, \operatorname{dis} g,\, \operatorname{codis}(f, g)\bigr\} - \varepsilon$ and $d^c_{GH}(Y, Z) \ge \max\bigl\{\operatorname{dis} h,\, \operatorname{dis} k,\, \operatorname{codis}(h, k)\bigr\} - \varepsilon$. Then

$$\begin{gathered}
\operatorname{dis}(h \circ f) \le \operatorname{dis} f + \operatorname{dis} h \le d^c_{GH}(X, Y) + d^c_{GH}(Y, Z) + 2\varepsilon,\\
\operatorname{dis}(g \circ k) \le \operatorname{dis} g + \operatorname{dis} k \le d^c_{GH}(X, Y) + d^c_{GH}(Y, Z) + 2\varepsilon,\\
\operatorname{codis}(h \circ f, g \circ k) \le \operatorname{codis}(f, g) + \operatorname{codis}(h, k) \le d^c_{GH}(X, Y) + d^c_{GH}(Y, Z) + 2\varepsilon,
\end{gathered}$$

where

$$\begin{aligned}
d^c_{GH}(X, Z) \le \max\bigl\{\operatorname{dis}(h \circ f),\, \operatorname{dis}(g \circ k),\, \operatorname{codis}(h \circ f, g \circ k)\bigr\} \le \\
\le d^c_{GH}(X, Y) + d^c_{GH}(Y, Z) + 2\varepsilon.
\end{aligned}$$

It remains to use the arbitrariness of ε. $\square$

The proper class of all non-empty metric spaces, considered up to isometry, endowed with the continuous Gromov–Hausdorff distance is denoted by $\mathcal{GH}^c$. It follows from the above that d^c_{GH} is a generalized pseudometric on $\mathcal{GH}^c$. The subset of $\mathcal{GH}^c$ consisting of all compact metric spaces is denoted by $\mathcal{M}^c$. As in the case of the ordinary Gromov–Hausdorff distance, the restriction of d^c_{GH} to $\mathcal{M}^c$ is a metric (Theorem 1).

The basic properties of the ordinary Gromov–Hausdorff distance also hold to its continuous analogue.

Proposition 6. *For any $X, Y \in \mathcal{GH}^c$, the following hold*:

(1) *if metric spaces X and Y are isometric, then $d^c_{GH}(X,Y) = 0$;*

(2) $d_{GH}(X,Y) \le d^c_{GH}(X,Y)$;

(3) $2d^c_{GH}(\Delta_1, X) = \operatorname{diam} X$;

(4) $2d^c_{GH}(X,Y) \le \max\{\operatorname{diam} X, \operatorname{diam} Y\}$;

(5) *if the diameter of X or Y is finite, then $2d^c_{GH}(X,Y) \ge \bigl|\operatorname{diam} X - \operatorname{diam} Y\bigr|$;*

(6) *if the diameter of X is finite, then for any $\lambda \ge 0$, $\mu \ge 0$, we have $2d^c_{GH}(\lambda X, \mu X) = |\lambda - \mu| \operatorname{diam} X$, whence it follows immediately that the curve $\gamma(t) := t\,X$ is shortest between any of its points, and the length of such a segment equals the distance between its ends*;

(7) *for any $\lambda > 0$, we have $d^c_{GH}(\lambda X, \lambda Y) = \lambda\, d^c_{GH}(X,Y)$, and if the spaces X and Y are bounded, then the equality also holds for $\lambda = 0$;*

(8) *if X_1, $X_2, \ldots$ is a sequence of metric spaces converging in the metric d^c_{GH}, then it also converges in the metric d_{GH}.*

Proof. (1) The isometries $h\colon X \to Y$ and h^{-1} are continuous and $\operatorname{dis} h = \operatorname{dis} h^{-1} = \operatorname{codis}(h, h^{-1}) = 0$.

(2) The inequality holds because the distance $d_{GH}(X,Y)$ is the infimum over a larger family of mappings than that needed to calculate $d^c_{GH}(X,Y)$.

(3) For any continuous $\Delta_1 \underset{g}{\overset{f}{\rightleftarrows}} X$, we have $\operatorname{dis} f = 0$, $\operatorname{dis} g = \operatorname{diam} X$, and if $\Delta_1 = \{p\}$, then

$$\operatorname{codis}(f,g) = \sup_{x \in X} \Bigl| \bigl|p\, g(x)\bigr| - \bigl|f(p)\, x\bigr| \Bigr| = \sup_{x \in X} \bigl|f(p)\, x\bigr| \le \operatorname{diam} X,$$

which is what was required.

(4) For any continuous $X \underset{g}{\overset{f}{\rightleftarrows}} Y$, we have

$$\operatorname{dis} f \le \max\{\operatorname{diam} X, \operatorname{diam} Y\}, \quad \operatorname{dis} g \le \max\{\operatorname{diam} X, \operatorname{diam} Y\},$$
$$\operatorname{codis}(f,g) = \sup_{x \in X,\, y \in Y} \Bigl| \bigl|x\, g(y)\bigr| - \bigl|f(x)\, y\bigr| \Bigr| \le \max\{\operatorname{diam} X, \operatorname{diam} Y\},$$

which is what was required.

(5) This follows from the corresponding inequality for d_{GH} and the fact that $d_{GH} \le d^c_{GH}$.

(6) By property (5) we have

$$2d^c_{GH}(\lambda X, \mu X) \ge \bigl|\operatorname{diam}(\lambda X) - \operatorname{diam}(\mu X)\bigr| = |\lambda - \mu| \operatorname{diam} X.$$

To prove the converse inequality, we choose the identity mapping as continuous $X \underset{g}{\overset{f}{\rightleftarrows}} X$, then if we denote by $\operatorname{dis}_{\lambda,\mu}$ and $\operatorname{codis}_{\lambda,\mu}$ the distortion and codistortion of the mappings between the spaces λX and μX, respectively, then

$$\operatorname{dis}_{\lambda,\mu} f = \sup_{x,x' \in X} \Bigl|\lambda|xx'| - \mu|xx'|\Bigr| = |\lambda - \mu| \operatorname{diam} X,$$
$$\operatorname{codis}_{\lambda,\mu}(f,g) = \sup_{x,x' \in X} \Bigl|\lambda|xx'| - \mu\bigl|xx'\bigr|\Bigr| = |\lambda - \mu| \operatorname{diam} X,$$

from which the required result follows.

(7) We choose arbitrary continuous $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ and denote by $\operatorname{dis}_\lambda$ and $\operatorname{codis}_\lambda$ the distortion and co-distortion of these mappings, but between λX and λY, respectively. Then

$$\operatorname{dis}_\lambda f = \sup_{x,x' \in X} \Bigl|\lambda|xx'| - \lambda\bigl|f(x)f(x')\bigr|\Bigr| = \lambda \operatorname{dis}_\lambda f,$$
$$\operatorname{codis}_\lambda(f,g) = \sup_{x \in X,\, y \in Y} \Bigl|\lambda|x\,g(y)| - \lambda\bigl|f(x)y\bigr|\Bigr| = \lambda \operatorname{codis}(f,g),$$

from which the declared statement follows.

(8) This follows from the inequality $d_{GH} \le d^c_{GH}$. □

4 Comparison of metrics (on zero-dimensional spaces)

Proposition 7. *If X and Y are discrete metric spaces, then $d^c_{GH}(X,Y) = d_{GH}(X,Y)$.*

Proof. This follows from the fact that all mappings between X and Y are continuous. □

To prove the following proposition, we need a sufficient condition for the continuity of the mapping of metric spaces.

Lemma 1. *Let $f\colon X \to Y$ be an arbitrary mapping of metric spaces, and there exists a positive number $\varepsilon \in \mathbb{R}$ such that for any distinct $y_1, y_2 \in Y$ with non-empty preimages and any $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$, it holds $|x_1x_2| > \varepsilon$. Then the mapping f is continuous.*

Proof. Indeed, let $y \in Y$ have a non-empty preimage, then for any point $x \in f^{-1}(y)$, we have $U_\varepsilon(x) \subset f^{-1}(y)$, therefore $f^{-1}(y)$ is open and, hence, for any open $V \subset Y$, the set $f^{-1}(V) = \cup_{y \in V} f^{-1}(y)$ is also open in X. □

Proposition 8. *Let X and Y be arbitrary metric spaces such that $2d_{GH}(X,Y) < s(X)$, then $d^c_{GH}(X,Y) = d_{GH}(X,Y)$.*

Proof. Let $s := s(X)$ and let $\varepsilon > 0$ such that $2d_{GH}(X,Y) < s-\varepsilon$. Then there exists a correspondence $R \in \mathcal{R}(X,Y)$ such that $\operatorname{dis} R < s-\varepsilon$. The set of all such correspondences will be denoted by $\mathcal{R}_\varepsilon(X,Y)$.

According to formula (3), the equality $d_{GH}(X,Y) = \frac{1}{2}\inf\big\{\operatorname{dis} R : R \in \mathcal{R}_\varepsilon(X,Y)\big\}$ holds. We choose an arbitrary $R \in \mathcal{R}_\varepsilon(X,Y)$. We show that $\big|R(x_1)R(x_2)\big| > \varepsilon$ for $x_1 \ne x_2$. Indeed, for any (x_1,y_1), $(x_2,y_2) \in R$, $x_1 \ne x_2$, the inequality $|y_1y_2| \ge |x_1x_2| - \operatorname{dis} R > s-(s-\varepsilon) = \varepsilon > 0$ holds. Therefore, the sets $\{R(x) : x \in X\}$ form an open partition of the space Y.

We construct mappings $f\colon X \to Y$ and $g\colon Y \to X$ as follows: for each $x \in X$, we choose an arbitrarily point $f(x)$ in $R(x)$, and for each $y \in R(x)$, we set $g(y) = x$. Since $s(X) > 0$, X is discrete, so f is continuous. The continuity of g follows from Lemma 1. By Proposition 2, we have $\operatorname{dis} R_{f,g} \le \operatorname{dis} R$. Thus, we have constructed a mapping $\mathcal{R}_\varepsilon(X,Y) \to \mathcal{R}_\varepsilon(X,Y)$, $R \mapsto R_{f,g}$, therefore, by the definition of d^c_{GH},

$$d^c_{GH}(X,Y) \le \frac{1}{2}\inf\big\{\operatorname{dis} R_{f,g} : R \in \mathcal{R}_\varepsilon(X,Y)\big\} \le$$
$$\le \frac{1}{2}\inf\big\{\operatorname{dis} R : R \in \mathcal{R}_\varepsilon(X,Y)\big\} = d_{GH}(X,Y).$$

It remains to use Inequality (2) of Proposition 6. □

Proposition 9. *Let X be a metric space and* $\dim X = 0$. *Then for every non-empty $A \subset X$, we have $d^c_{GH}(A,X) \le d_H(A,X)$.*

Proof. For an arbitrary $\varepsilon > 0$, we put $r = d_H(A,X)+\varepsilon$, then, by the definition of the Hausdorff distance, $\lambda = \big\{U_r(a) : a \in A\big\}$ is an open covering of X. Since $\dim X = 0$, by the theorems of Stone [15] (see also [16, Theorem 4.4.1]) and Dowker [17] (see also [16, Theorem 7.2.4]), there exists an open covering $\mu = \{U_\beta\}$ of multiplicity 1 inscribed in λ, i.e., μ is a partition of X by open sets. Therefore, for every U_β, there exists a point $a_\beta \in A$ such that $U_\beta \subset U_r(a_\beta)$.

Let $f\colon A \to X$ be the inclusion, and let the mapping $g\colon X \to A$ be defined by the formula $g(U_\beta) = a_\beta$. Since the sets U_β are open and pairwise disjoint, the mapping g is well-defined and continuous, and for every point $x \in X$, the inequality $\big|x\,g(x)\big| < r$ holds. Clearly, $\operatorname{dis} f = 0$,

$$\operatorname{dis} g = \sup_{x,x' \in X}\Big||xx'| - \big|g(x)g(x')\big|\Big| \le \sup_{x,x'\in X}\Big(\big|x\,g(x)\big| + \big|x'\,g(x')\big|\Big) \le 2r,$$

and finally

$$\operatorname{codis}(f,g) = \sup_{x\in X,\, a\in A}\Big|\big|x\,f(a)\big| - \big|g(x)a\big|\Big| = \sup_{x\in X,\, a\in A}\Big||xa| - \big|g(x)a\big|\Big| \le \sup_{x\in X}|x\,g(x)| \le r,$$

therefore $\operatorname{dis} R_{f,g} \le 2d_H(A,X) + 2\varepsilon$ and, because of arbitrariness of ε, we have $\operatorname{dis} R_{f,g} \le 2d_H(A,X)$, whence $d^c_{GH}(A,X) \le d_H(A,X)$, which is what was stated. □

Corollary 2. *For metric spaces X, Y, and any of their everywhere dense subsets $A \subset \overline{A} = X$, $B \subset \overline{B} = Y$ such that* $\dim A = \dim B = 0$, *we have*

$$d^c_{GH}(A, B) = d_{GH}(X, Y).$$

Proof. For a given $\varepsilon > 0$, we choose discrete ε-nets $A_\varepsilon \subset A$ and $B_\varepsilon \subset B$ in the sets A and B, respectively. According to the triangle inequalities for distances d^c_{GH} and d_{GH}, as well as by Propositions 9 and 7, the following chain of inequalities holds:

$$\begin{aligned} d^c_{GH}(A, B) &\le d^c_{GH}(A, A_\varepsilon) + d^c_{GH}(A_\varepsilon, B_\varepsilon) + d^c_{GH}(B_\varepsilon, B) \le \\ &\le 2\varepsilon + d^c_{GH}(A_\varepsilon, B_\varepsilon) = 2\varepsilon + d_{GH}(A_\varepsilon, B_\varepsilon) \le \\ &\le 2\varepsilon + d_{GH}(A_\varepsilon, A) + d_{GH}(A, X) + d_{GH}(X, Y) + d_{GH}(Y, B) + d_{GH}(B, B_\varepsilon) \le \\ &\le 2\varepsilon + \varepsilon + 0 + d_{GH}(X, Y) + 0 + \varepsilon = 4\varepsilon + d_{GH}(X, Y). \end{aligned}$$

From arbitrariness of ε and Inequality (2) of Proposition 6, we get

$$d^c_{GH}(A, B) \le d_{GH}(X, Y) = d_{GH}(A, B) \le d^c_{GH}(A, B),$$

which completes the proof. $\square$

Corollary 3. *For any zero-dimensional metric spaces X and Y, it holds*

$$d^c_{GH}(X, Y) = d_{GH}(X, Y).$$

Let us emphasize that Corollary 3 is a generalization of Proposition 7, which, however, we used in the proof of this more general statement.

Proposition 10. *Let X be a metric space and* $\operatorname{ind} X \ne 0$. *Then there exists $r > 0$ such that for every continuous mapping $f\colon X \to Y$ into a metric space Y with* $\operatorname{ind} Y = 0$, *the inequality* $\operatorname{dis} f \ge r$ *holds. In particular, we have $2d^c_{GH}(X, Y) \ge r$.*

Proof. By the condition $\operatorname{ind} X \ne 0$, there exist a closed subset $F \subset X$ and a point $x \in X \setminus F$ such that every clopen set containing the point x, intersects the set F. Put $r = |xF| > 0$. For every mapping $f\colon X \to Y$, the alternative holds: either $\bigl|f(x)f(F)\bigr| = 0$, or $\bigl|f(x)f(F)\bigr| > 0$.

In the first case, the estimate $\operatorname{dis} f \ge |xF| = r$ is valid.

In the second case, the condition $\operatorname{ind} Y = 0$ implies the existence of a clopen neighborhood $U^{f(x)}$ of the point $f(x)$ that lies outside the closure $\overline{f(F)}$ of the set $f(F)$. The preimage $U^x := f^{-1}\bigl(U^{f(x)}\bigr)$ is a clopen neighborhood of the point x, and $U^x \cap F = \emptyset$, which is a contradiction with the choice of the point x and the closed set F. $\square$

Proposition 11. *Let X be a metric space and* $\operatorname{Ind} X \ne 0$. *Then there exists $r > 0$ such that for every continuous mapping $f\colon X \to Y$ into a metric space Y with* $\operatorname{Ind} Y = 0$, *the inequality* $\operatorname{dis} f \ge r$ *holds. In particular, we have $2d^c_{GH}(X, Y) \ge r$.*

Proof. By the Nagami–Roberts Theorem [18, Theorem on p. 601] there exist closed subsets $F, H \subset X$ such that $r = |FH| > 0$ and every clopen set containing F intersects H. For every mapping $f\colon X \to Y$, the alternative holds: either $\bigl|f(F)f(H)\bigr| = 0$, or $\bigl|f(F)f(H)\bigr| > 0$.

In the first case, the estimate $\operatorname{dis} f \ge |FH| = r$ takes place.

In the second case, the condition $\operatorname{Ind} Y = 0$ implies the existence of a clopen neighborhood $U^{f(F)}$ of the closed set $\overline{f(F)}$ that lies outside the closure $\overline{f(H)}$ of the set $f(H)$. The preimage $U^F = f^{-1}\bigl(U^{f(F)}\bigr)$ is a clopen neighborhood of the set F and $U^F \cap H = \emptyset$, which is a contradiction with the choice of the closed sets F and H. $\square$

Recall that a topological space is called *totally disconnected* if all its connected components are singletons.

Proposition 12. *Let $K \subset X$ be a connected subset of a metric space X, and Y be a totally disconnected metric space. Then for every continuous mapping $f\colon X \to Y$, we have $\operatorname{dis} f \ge \operatorname{diam} K$ and therefore $2d^c_{GH}(X, Y) \ge \operatorname{diam} K$. In particular, if X is a connected metric space, and Y is a totally disconnected metric space such that $\operatorname{diam} X \ge \operatorname{diam} Y$, then $2d^c_{GH}(X, Y) = \operatorname{diam} X$. For example, this holds for any discrete ε-net $Y \subset X$.*

Proof. If $f\colon X \to Y$ is a continuous mapping, then, by virtue of the connectedness of K, the restriction of f to K is constant, so that $\operatorname{dis} f \ge \operatorname{diam} K$, therefore $2d^c_{GH}(X, Y) \ge \operatorname{diam} X$ by the definition of continuous Hausdorff distance. The second assertion follows from Item (2) of Proposition 6. $\square$

5 Zero distance

We have already pointed out that the (continuous) Gromov–Hausdorff distance is a pseudometric, i.e., it can be zero between non-isometric spaces. Therefore, it is important to describe certain classes of metric spaces that are at zero distance from each other.

Theorem 1. *For compact metric spaces X and Y, the following conditions are equivalent*:

(1) $d^c_{GH}(X, Y) = 0$;

(2) $d_{GH}(X, Y) = 0$;

(3) *spaces X and Y are isometric.*

Proof. The implications (3) $\Rightarrow$ (1) $\Rightarrow$ (2) follow from Properties (1) and (2) of Proposition 6, respectively.

Implication (2) $\Rightarrow$ (3) is the famous deep Theorem [3, theorem 7.3.30]. $\square$

For applications, we will need the following version of the previous theorem, which follows immediately from [3, Exercise 7.3.31].

Theorem 2. *If a metric space X is compact, then the following conditions on a metric space Y are equivalent*:

(1) $d_{GH}(X,Y)=0$;

(2) *the space Y is isometric to a dense subset of the space X (e.g., in the case of $X=S^n$ there are at least continuum number of pairwise non-isometric Y, see Introduction).*

The situation with the continuous Gromov–Hausdorff distance is significantly different.

We fix on the sphere S^n, $n\ge 1$, some metric ρ and some continuous free involution σ (for example, an antipodal mapping). Since each continuous function on a compact set attains its minimum, then

$$d(\rho,\sigma):=\inf\Big\{\rho\big(x,\sigma(x)\big):x\in S^n\Big\}=\min\Big\{\rho\big(x,\sigma(x)\big):x\in S^n\Big\}>0.$$

We put

$$d(\rho):=\sup\big\{d(\rho,\sigma):\sigma \text{ is a continuous free involution on } S^n\big\}>0.$$

Proposition 13. *For an arbitrary metric ρ on the sphere S^n and an arbitrary metric space Y that is topologically embeddable in $\mathbb{R}^n$, the following inequalities hold*:

$$d(\rho)\le 2d^c_{GH}(S^n,Y)\le\max\{\operatorname{diam}S^n,\operatorname{diam}Y\}.$$

Proof. Let $\nu\colon Y\to\mathbb{R}^n$ be a topological embedding, $f\colon S^n\to Y$ and $g\colon Y\to S^n$ be continuous mappings, and $h=\nu\circ f$. Consider an arbitrary continuous free involution σ on the sphere S^n. According to generalized theorem of Borsuk–Ulam type [12, Theorem 5], there exists a point $x\in S^n$ such that $h(x)=h\big(\sigma(x)\big)$, therefore, since ν is injective, we have $f(x)=f\big(\sigma(x)\big)$. This means that $\operatorname{dis}f\ge d(\rho,\sigma)$. Consequently, $d(\rho,\sigma)\le 2d^c_{GH}(S^n,Y)$. The left inequality follows from the arbitrariness of the involution under consideration.

The right inequality is Property (4) of Proposition 6. $\square$

Remark 1. From Proposition 13 it follows that for the standard sphere of unit radius $S^n\subset\mathbb{R}^n$, $n\ge 1$, endowed with a geodesic or induced Euclidean distance, for every $m>n$, the equality $2d^c_{GH}(S^n,S^m)=\operatorname{diam}S^n$ holds. Note that according to [5, Theorem A, p. 5], we get $2d_{GH}(S^n,S^m)<\operatorname{diam}S^n$.

Corollary 4. *For an arbitrary metric ρ on the sphere S^n and an arbitrary metric space Y, the following conditions are equivalent*:

(1) $d^c_{GH}(S^n,Y)=0$, *and*

(2) *the space Y is isometric to sphere (S^n,ρ).*

Proof. (1) $\Rightarrow$ (2). Since $d_{GH}(S^n, Y) \le d^c_{GH}(S^n, Y) = 0$, then according to Theorem 2, one can assume that the space Y is contained in (S^n, ρ). If Y is a proper subset of the sphere, then it is topologically embedded in the space $\mathbb{R}^n$ and $2d^c_{GH}(S^n, Y) \ge d(\rho) > 0$ according to Proposition 13.

The implication (2) $\Rightarrow$ (1) is obvious. $\square$

We say that a metric space X *has the uniqueness property in the class* $\mathcal{GH}$, $\mathcal{M}$, $\mathcal{GH}^c$, *or* $\mathcal{M}^c$ if for every metric space Y in the class under consideration, the equality $d_{GH}(X, Y) = 0$ in the first two classes and the equality $d^c_{GH}(X, Y) = 0$ in the second two classes implies that X and Y are isometric. Clearly, Theorem 1 and Corollary 4 describe some classes of spaces with the uniqueness property in the classes $\mathcal{M}^c$, $\mathcal{M}$, and $\mathcal{GH}^c$, respectively.

Proposition 14. *If a metric space X has the property of uniqueness in the class $\mathcal{GH}$, then it is complete and discrete.*

Proof. Let X be a incomplete space. Consider its completion Y. Since X can be isometrically identified with a dense subset of Y, we have $d_{GH}(X, Y) \le d_H(X, Y) = 0$. The spaces X and Y themselves are not isometric, since the former is incomplete and the latter is complete.

Let the complete space X be non-discrete. Let us take a non-isolated point $x_0 \in X$ in it. Then $d_{GH}(X, X \setminus x_0) = 0$, but the spaces X and $X \setminus x_0$ are not isometric, because the first is complete and the second is not. $\square$

Example 1. In [9, Example 5.11] an example of a countable complete bounded metric space Y and its clopen subset X with the following properties is constructed.

(1) There is exactly one non-isolated point in the space Y.

(2) The space X is discrete. Therefore, the space Y cannot be topologically (and therefore isometrically) embedded in the space X.

(3) $d_{GH}(X, Y) = 0$, and therefore $d^c_{GH}(X, Y) = 0$ by Corollary 3. However, for these spaces, the corresponding mappings $f_\varepsilon\colon X \to Y$ and $g_\varepsilon\colon Y \to X$ can easily be presented constructively.

(4)
- $0 = s(Y) = s(X) < d(X) = d(Y) = 3.5$,
- $3 = R(Y) = R(X) < \operatorname{diam} X = \operatorname{diam} Y = 4$,
- $d_H(X, Y \setminus X) = 3$,
- $\big|X(Y \setminus X)\big| = 2$.

(5) For each isometric embedding $h\colon X \to Y$, it holds $d_H\big(h(X), Y \setminus h(X)\big) = 3$ and $2 \le \big|h(X)\big(Y \setminus h(X)\big)\big| \le 3$.

Spaces X and Y show that Proposition 14 cannot be conversed.

It is clear that $s(X) > 0$ implies that the space X is complete and discrete. The converse, generally speaking, does not hold.

Example 2. On the line $\mathbb{R}$, we consider the subset $X = \bigl\{n \pm \frac{1}{6n} : n \in \mathbb{N}\bigr\}$. The space X is complete, discrete, and $s(X) = 0$.

For a metric space X, consider the set of all distances in it:

$$\operatorname{dist} X = \bigl\{|xx'| : x, x' \in X\bigr\}.$$

It is easy to check that

- $s(X) = \Bigl|0\,\bigl(\operatorname{dist} X \setminus \{0\}\bigr)\Bigr|$ and $\operatorname{diam} X = d_H\bigl(\{0\}, \operatorname{dist} X\bigr) = \operatorname{diam}\operatorname{dist} X$;
- if the spaces X and Y are isometric, then $\operatorname{dist} X = \operatorname{dist} Y$;
- if $d_{GH}(X,Y) = 0$, then $d_H\bigl(\operatorname{dist} X, \operatorname{dist} Y\bigr) = 0$, i.e., $\overline{\operatorname{dist} X} = \overline{\operatorname{dist} Y}$ (see also [14, Lemma 5.1]).

Proposition 15. *For any $X, Y \in \mathcal{GH}$, the following hold*:

(1) $d_H\bigl(\operatorname{dist} X, \operatorname{dist} Y\bigr) \le 2d_{GH}(X,Y)$;

(2) $s(X) \le 2d_{GH}(X,Y)$, *or* $s(Y) \le s(X) + 2d_{GH}(X,Y)$;

(3) $2s(X) < s(Y)$, *or* $s(Y) - s(X) \le d_H(\operatorname{dist} X, \operatorname{dist} Y)$.

Proof. If $d_{GH}(X,Y) = \infty$, then Items (1) and (2) are obvious (in Item (3) the distance $d_{GH}(X,Y)$ is not contained). Therefore, we will assume that $d_{GH}(X,Y) < \infty$.

(1) Fix a number $\varepsilon > 0$ and a correspondence $R \in \mathcal{R}(X,Y)$ such that $\operatorname{dis} R < 2d_{GH}(X,Y) + \varepsilon$. Take two arbitrary points $x, x' \in X$, and let $y, y' \in Y$ be points such that (x,y), $(x',y') \in R$. Then $\bigl||xx'| - |yy'|\bigr| \le \operatorname{dis} R < 2d_{GH}(X,Y) + \varepsilon$. The first inequality follows from the obtained inclusion $\operatorname{dist} X \subset B_{2d_{GH}(X,Y)+\varepsilon}(\operatorname{dist} Y)$, the arbitrariness of $\varepsilon > 0$, and the symmetry of the condition under consideration.

(2) Let $s(X) > 2d_{GH}(X,Y)$. Since the second inequality is automatically satisfied for $s(Y) \le s(X)$, we will assume that $s(Y) > s(X)$. Then

$$\bigl|s(X)\,0\bigr| > 2d_{GH}(X,Y) \ge d_H(\operatorname{dist} X, \operatorname{dist} Y) \ge \bigl|s(X)\operatorname{dist} Y\bigr| = \\ = \min\Bigl\{\bigl|s(X)\,0\bigr|, s(Y) - s(X)\Bigr\} = s(Y) - s(X),$$

which is what was required.

(3) If $s(Y) \le s(X)$, then the second inequality is obvious. Now consider the remaining case $s(X) < s(Y) \le 2s(X)$. Then $\bigl|s(X)s(Y)\bigr| = s(Y) - s(X) \le s(X) = \bigl|s(X)\,0\bigr|$, and taking into account that $\bigl|s(X)\operatorname{dist} Y\bigr| \le d_H(\operatorname{dist} X, \operatorname{dist} Y)$, we obtain

$$\bigl|s(X)\operatorname{dist} Y\bigr| = \min\Bigl\{\bigl|s(X)\,0\bigr|,\ \bigl|s(X)s(Y)\bigr|\Bigr\} = \bigl|s(X)s(Y)\bigr| \le d_H(\operatorname{dist} X, \operatorname{dist} Y),$$

which is what was claimed. $\square$

If the diameter of X or Y is finite, then $2d_H(X,Y) \ge \bigl|\operatorname{diam} X - \operatorname{diam} Y\bigr|$, so Inequality (1) of Proposition 15 is a strengthening of Property (5) of Proposition 6. Note also that Property (1) implies

Corollary 5. *For any $X, Y \in \mathcal{GH}$, from $d_{GH}(X,Y) = 0$ it follows that $s(X) = s(Y)$.*

Theorem 3. *If $s(X) > 0$ and $d_{GH}(X,Y) = 0$, then for any $\varepsilon > 0$, there exists a homeomorphism $f_\varepsilon \colon X \to Y$ such that $\bigl||f_\varepsilon(x)f_\varepsilon(x')| - |xx'|\bigr| \le \varepsilon$ for any points $x, x' \in X$.*

Proof. Let $d_{GH}(X,Y) = 0$. By Proposition 8, we have $d^c_{GH}(X,Y) = 0$. Let $0 < \varepsilon < s(X)$ and $f_\varepsilon \colon X \to Y$, $g_\varepsilon \colon Y \to X$ satisfy $\operatorname{dis} R_{f_\varepsilon, g_\varepsilon} \le \varepsilon$.

We show that $g_\varepsilon \circ f_\varepsilon \colon X \to X$ and $g_\varepsilon \circ f_\varepsilon \colon Y \to Y$ are identity mappings. For an arbitrary point $x \in X$ and the point $y = f_\varepsilon(x) \in Y$, we write the codistortion

$$\Bigl|x\, g_\varepsilon\bigl(f_\varepsilon(x)\bigr)\Bigr| = \Bigl||x\, g_\varepsilon(y)| - \bigl|f_\varepsilon(x)\, y\bigr|\Bigr| \le \varepsilon < s(X).$$

Therefore, $x = g_\varepsilon\bigl(f_\varepsilon(x)\bigr)$.

Corollary 5 implies $s(Y) = s(X)$. Therefore, the equality $g_\varepsilon \circ f_\varepsilon = \mathrm{Id}_Y$ can be proved similarly.

The inequality $\bigl||f_\varepsilon(x)f_\varepsilon(x')| - |xx'|\bigr| \le \varepsilon$ is exactly the inequality $\operatorname{dis} f_\varepsilon \le \varepsilon$. □

Proposition 16. *Let $X \in \mathcal{GH}$ have a number $r > 0$ such that $s(\operatorname{dist} X) \ge r$ (the distance between distinct points of the set $\operatorname{dist} X$ is at least r). Then $s(X) \ge r$, and for $\varepsilon < r$, every mapping f_ε from Theorem 3 is an isometry. In particular, this space X has the uniqueness property in the class $\mathcal{GH}$.*

Proof. It is clear that the set $\operatorname{dist} X \subset \mathbb{R}$ is closed and discrete. Let $d_{GH}(X,Y) = 0$. Then from Item (1) of Proposition 15 it follows $d_H\bigl(\operatorname{dist} X, \operatorname{dist} Y\bigr) = 0$, whence, since $\operatorname{dist} X$ is closed, we obtain $\operatorname{dist} X = \overline{\operatorname{dist} X} = \overline{\operatorname{dist} Y}$, therefore $\operatorname{dist} Y \subset \operatorname{dist} X$. Since $\operatorname{dist} X$ is discrete, the closure of a proper subset cannot be equal to $\operatorname{dist} X$, whence $\operatorname{dist} X = \operatorname{dist} Y$. According to the choice of the mapping $f_\varepsilon \colon X \to Y$, for any points $x, x' \in X$, the inequality $\bigl||f_\varepsilon(x)f_\varepsilon(x')| - |xx'|\bigr| \le \varepsilon < r$ holds, and if $\bigl|f_\varepsilon(x)f_\varepsilon(x')\bigr| \ne |xx'|$, then the quantities $\bigl|f_\varepsilon(x)f_\varepsilon(x')\bigr|$ and $|xx'|$ differ from each other by at least r. The latter, together with the previous inequality, implies $\bigl|f_\varepsilon(x)f_\varepsilon(x')\bigr| = |xx'|$. □

Corollary 6. *Every metric space X with a finite set $\operatorname{dist} X$ has the property of uniqueness in the class $\mathcal{GH}$.*

Remark 2. The Corollary 6 can also be immediately obtained from [14, Lemmas 5.1 and 6.1].

Conjecture. Every metric space X with a closed and discrete set $\operatorname{dist} X$ has the property of uniqueness in the class $\mathcal{GH}$.

Example 3. On $X = \{0\} \sqcup \mathbb{N} \sqcup \{\infty\}$ we define the following metric:

$$|xx'| = \begin{cases} 2 & \text{at } 0 \neq x \neq x' \neq 0; \\ 1 + \frac{1}{2n} & \text{for } x = 0,\ x' \in \mathbb{N} \sqcup \{\infty\}. \end{cases}$$

The metric space X and its clopen subset $Y = \{0\} \sqcup \mathbb{N} \subset X$ have the following properties.

(1) $\operatorname{diam} X = 2 = \operatorname{diam} Y$ and $s(X) = s(Y) = 1$, therefore the spaces X and Y are complete and discrete.

(2) $\operatorname{dist} X = \{0, 1, 1 + \frac{1}{2n}, 2 : n \in \mathbb{N}\} \neq \{0, 1 + \frac{1}{2n}, 2 : n \in \mathbb{N}\} = \operatorname{dist} Y$, therefore the spaces X and Y are not isometric.

(3) $d^c_{GH}(X, Y) = 0$. The mapping $h_m \colon X \to Y$, defined by the formula

$$h_m(x) = \begin{cases} 0 & \text{when } x = 0; \\ x & \text{for } 1 \le x < m; \\ x + 1 & \text{for } m \le x < \infty; \\ m & \text{when } x = \infty, \end{cases}$$

is a $\frac{1}{2m}$- isometry .

(4) The isometries of the spaces X and Y are only their identity mappings.

(5) $\big|Y\,(X \setminus Y)\big| = 1$ and $d_H(X, Y) = 1$.

6 Comparison of topologies

Let $p_c \colon \mathcal{GH}^c \to \mathcal{GH}$ be the identity mapping. By Property (1) of Proposition 6, p_c is non-expanding and hence continuous. Recall also that a continuous bijective mapping is called a *condensation*. This concept also carries over to topological classes, so p_c is a condensation. Recall (Proposition 12) that in this case, two metric spaces at positive distance d^c_{GH} can be sometimes mapped to spaces with zero distance d_{GH}.

Proposition 17. *The mapping p_c^{-1} is discontinuous at every point $X \in \mathcal{GH}$ such that* $\operatorname{Ind} X \neq 0$.

Proof. Let us choose an arbitrary non-empty metric space X such that $\operatorname{Ind} X \neq 0$. By Proposition 11, there exists $r > 0$ such that for every non-empty metric space Y with $\operatorname{Ind} Y = 0$, we have $d^c_{GH}(X, Y) \ge r$. For every discrete space Z, it holds $\operatorname{Ind} Z = 0$. On the other hand, it is easy to show that for every $\delta > 0$, there exists a discrete $Y \subset X$ such that $d_{GH}(X, Y) \le d_H(X, Y) < \delta$. The latter means that there is no neighborhood $U_\delta(X)$ with $\delta < r$ and $p_c^{-1}\big(U_\delta(X)\big) \subset U_r(X)$. The latter means the discontinuity of p_c^{-1} at the point X. $\square$

Remark 3. In the proof of Proposition 17 we actually showed more, namely that p_c^{-1} is discontinuous at X, $\operatorname{Ind} X \neq 0$ on various smaller sets, for example,

- on the union of $\{X\}$ with the class of all spaces with zero large inductive dimension;
- on the union of $\{X\}$ with the class of all discrete spaces;
- for compact X, on the union of $\{X\}$ with the set of all finite metric spaces.

Recall that the mapping p_c is an isometry on the subclass of all zero-dimensional spaces in the sense of Lebesgue (Corollary 3). However, the results obtained above show that adding even a single point X can lead to discontinuity in p_c^{-1} and, hence, to violation of isometry.

Below we will show that the mapping p_c^{-1} can be discontinuous at points corresponding to discrete spaces, and the discontinuity manifests itself on a sufficiently thin subset of $\mathcal{GH}$. Nevertheless, on completely discrete spaces X, i.e., when $s(X) > 0$, the mapping p_c^{-1} is continuous. The following result follows immediately from Proposition 8.

Corollary 7. *If $s(X) > 0$, then the mapping p_c^{-1} is continuous at the point $X \in \mathcal{GH}$.*

For compact metric spaces, there is a natural criterion for the continuity of the mapping p_c^{-1}.

Theorem 4. *For a compact metric space $X \in \mathcal{GH}$, the following conditions are equivalent*:

(1) $\operatorname{Ind} X = 0$;

(2) *the mapping p_c^{-1} is continuous at X.*

Proof. (2) $\Rightarrow$ (1). Assume the contrary, i.e., that $\operatorname{Ind} X \neq 0$. By Proposition 11, there exists $r > 0$, depending only on X, such that $2d^c_{GH}(X,Y) \geq r$. But the latter contradicts the continuity of p_c^{-1} at X.

(1) $\Rightarrow$ (2). We need to prove that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d_{GH}(X,Y) < \delta$ implies $d^c_{GH}(X,Y) < \varepsilon$. For $0 < \varepsilon' < \varepsilon/4$, consider the covering $\lambda = \big\{U_{\varepsilon'}(x)\big\}_{x\in X}$. Since $\dim X = 0$, there exists a clopen partition μ inscribed in λ. Since X is compact, the partition μ is finite. Let $\mu = \{U_1, \dots, U_m\}$, then $r := \min\big\{|U_iU_j| : i \neq j\big\} > 0$ and $\operatorname{diam} U_i \leq 2\varepsilon' < \varepsilon/2$. We set $\delta = \min\{r/2, \varepsilon'\} < \varepsilon/4$ and show that δ is what we are looking for.

Let $d_{GH}(X,Y) < \delta$, then there exists $\alpha > 0$ such that $d_{GH}(X,Y) < \delta - \alpha$ and, therefore, we can choose a correspondence $R \in \mathcal{R}(X,Y)$ for which $\operatorname{dis} R < 2\delta - 2\alpha \leq r - 2\alpha$. Let $V_k = R(U_k)$, then for $i \neq j$, we have $V_i \cap V_j = \emptyset$, since for any $y_k \in V_k$ and $x_k \in U_k$ such that $(x_k, y_k) \in R$, we have

$$|y_iy_j| \geq |x_ix_j| - \operatorname{dis} R > r - (r - 2\alpha) = 2\alpha.$$

Thus, $V_k = \cup_{y\in V_k} U_\alpha(y)$ is an open set for every k, so $\{V_k\}_{k=1}^m$ is a partition of Y by clopen sets. Note that $\operatorname{diam} V_k \le \operatorname{diam} U_k + \operatorname{dis} R \le 2\varepsilon' + \varepsilon' < 3\varepsilon/4$.

Let us now choose arbitrary $x_k \in U_k$, $y_k \in V_k$, $(x_k, y_k) \in R$ and define the mappings $f\colon X \to Y$ and $g\colon Y \to X$ as follows: $f(U_k) = y_k$ and $g(V_k) = x_k$, then f and g are continuous. Let us show that $\operatorname{dis} R_{f,g} < 2\varepsilon$, whence $d_{GH}^c(X,Y) \le \frac{1}{2}\operatorname{dis} R_{f,g} < \varepsilon$, which is what we want.

Let us choose arbitrary $x \in U_i$ and $x' \in U_j$, then

$$\begin{aligned}\big||xx'| - |y_iy_j|\big| = \big||xx'| - |x'x_i| + |x'x_i| - |x_ix_j| + |x_ix_j| - |y_iy_j|\big| \le \\ \le \big||xx'| - |x'x_i|\big| + \big||x'x_i| - |x_ix_j|\big| + \big||x_ix_j| - |y_iy_j|\big| \le \\ \le |xx_i| + |x'x_j| + \operatorname{dis} R \le \operatorname{diam} U_i + \operatorname{diam} U_j + \operatorname{dis} R < \varepsilon/2 + \varepsilon/2 + \varepsilon/4 < 2\varepsilon,\end{aligned}$$

therefore $\operatorname{dis} f < 2\varepsilon$.

Let us now choose arbitrary $y \in V_i$ and $y' \in V_j$, then

$$\begin{aligned}\big||x_ix_j| - |yy'|\big| = \big||x_ix_j| - |y_iy_j| + |y_iy_j| - |y_jy| + |y_jy| - |yy'|\big| \le \\ \le \big||x_ix_j| - |y_iy_j|\big| + \big||y_iy_j| - |y_jy|\big| + \big||y_jy| - |yy'|\big| \le \\ \le \operatorname{dis} R + |y_iy| + |y_jy'| \le \operatorname{dis} R + \operatorname{diam} V_i + \operatorname{diam} V_j < \varepsilon/4 + 3\varepsilon/4 + 3\varepsilon/4 < 2\varepsilon,\end{aligned}$$

therefore $\operatorname{dis} g < 2\varepsilon$.

Finally, let us estimate the codistortion $\operatorname{codis}(f,g)$. To do this, we choose arbitrary $x \in U_i$ and $y \in V_j$, then

$$\begin{aligned}\big||x\,x_j| - |y_iy|\big| = \big||x\,x_j| - |x_jx_i| + |x_jx_i| - |y_jy_i| + |y_jy_i| - |y_iy|\big| \le \\ \le \big||x\,x_j| - |x_jx_i|\big| + \big||x_jx_i| - |y_jy_i|\big| + \big||y_jy_i| - |y_iy|\big| \le \\ \le |xx_i| + \operatorname{dis} R + |y_jy| \le \operatorname{diam} U_i + \operatorname{dis} R + \operatorname{diam} U_j < \varepsilon/2 + \varepsilon/4 + \varepsilon/2 < 2\varepsilon,\end{aligned}$$

from which $\operatorname{codis}(f,g) < 2\varepsilon$ and, therefore, $\operatorname{dis} R_{f,g} < 2\varepsilon$, which completes the proof. □

Example 4. Theorem 4 states that at a point X, which is a zero-dimensional compact (complete and totally bounded) space, the mapping p_c^{-1} is continuous. It turns out that if we abandon the condition of total boundedness, then even for countable discrete spaces, continuity is not guaranteed. Below we give an example of a countable (and therefore zero-dimensional) complete discrete space X such that the mapping p_c^{-1} is discontinuous at the point $X \in \mathcal{GH}$. By Corollary 7, $s(X) = 0$ for this X.

Let us take a countable number $Z = I \times \mathbb{N} = \sqcup_{n\in\mathbb{N}} I_n$ of standard unit intervals $I_n = [0,1]$. We define the distance between points of one interval to be standard, and set the distance between points of different intervals to be equal to 1. In the interval I_n, we consider the subset $S_n = \big\{i/2^n : i = 0, \dots, 2^n\big\}$. We put

$$X_\infty = \sqcup_{k\in\mathbb{N}} S_k \subset Z \ \text{ i } \ X_n = \big(\sqcup_{k<n} S_k\big) \sqcup \big(\sqcup_{k\ge n} I_k\big) \subset Z.$$

Note that X_∞ is a countable discrete space, so it is complete. It is easy to see that $d_H(X_\infty, X_n) = 2^{-(n+1)}$, so $X_n \xrightarrow{d_{GH}} X_\infty$. Furthermore, since X_∞ is a totally disconnected space, and the maximum diameter of connected components in X_n is 1, by Proposition 12 we get $2d^c_{GH}(X_\infty, X_n) \ge 1$, so X_n does not converge to X_∞ w.r.t. d^c_{GH}, and hence the mapping p_c^{-1} is discontinuous at X_∞.

7 Incomparable Spaces

We say that a topological space X is *incomparable* with Y if every continuous mapping $f\colon X \to Y$ is trivial, i.e., there is a point $y_f \in Y$ such that $f(X) = y_f$. For example, every connected space is incomparable with every totally disconnected space. Note that the incomparability relation is not symmetric; for example, the interval $X = [0,1]$ is incomparable with the set Y of its rational points, but Y is comparable with X (the inclusion of Y in X is a nontrivial continuous mapping).

Proposition 18. *Let the metric space X be incomparable with the metric space Y. Then*

(1) *for any mapping $f\colon X \to Y$, it holds* $\operatorname{dis} f = \operatorname{diam} X$;

(2) *for any mappings $f\colon X \to Y$ and $g\colon Y \to X$, it holds*

$$\operatorname{codis}(f,g) \ge \max\bigl\{R(X), R(Y)\bigr\}.$$

Proof. (1) We have

$$\operatorname{dis} f = \sup_{x,x' \in X} \bigl||xx'| - |f(x)f(x')|\bigr| = \sup_{x,x'\in X} |xx'| = \operatorname{diam}(X).$$

(2) We have

$$\operatorname{codis}(f,g) = \sup_{x\in X,\, y\in Y} \bigl||x\,g(y)| - |f(x)y|\bigr| \ge \sup_{x\in X}\bigl||x\,g(y_f)| - |f(x)y_f|\bigr| = \\ = \sup_{x\in X} |x\,g(y_f)| = R_{g(y_f)}(X) \ge R(X).$$

Similarly,

$$\operatorname{codis}(f,g) = \sup_{x\in X, y\in Y} \Bigl|\bigl|x\,g(y)\bigr| - \bigl|f(x)y\bigr|\Bigr| \ge \sup_{y\in Y} \Bigl|\bigl|g(y)g(y)\bigr| - \bigl|f\bigl(g(y)\bigr)y\bigr|\Bigr| = \\ \sup_{y\in Y} \Bigl|\bigl|g(y)g(y)\bigr| - |y_f y|\Bigr| = \sup_{y\in Y} |y_f y| = R_{y_f}(Y) \ge R(Y).$$

To complete the proof, it remains to gather together the two resulting inequalities. ◻

Corollary 8. *Let a metric space X be incomparable with a metric space Y. Then* $2d^c_{GH}(X,Y) \ge \operatorname{diam}(X)$.

Corollary 9. *Let X and Y be mutually incomparable metric spaces. Then*

$$2d^c_{GH}(X,Y)=\max\bigl\{\operatorname{diam}(X),\operatorname{diam}(Y)\bigr\}.$$

Proof. The inequality $2d^c_{GH}(X,Y)\ge\max\bigl\{\operatorname{diam}(X),\operatorname{diam}(Y)\bigr\}$ follows from Corollary 8. The inverse inequality is Item (4) of Proposition 6. □

8 Hyperspace of Cook's continuum

Recall that a *continuum* is every connected compact Hausdorff topological space. We will restrict ourselves to continua that are metric spaces. Thus, in what follows, by *continuum* we mean a metrizable continuum.

We will also be interested in various *hyperspaces.* If X is a metric space, then by $\mathcal{H}(X)$ we denote the family of all non-empty closed bounded subsets of X, endowed with the Hausdorff distance d_H. It is well known [3] that d_H is a metric on $\mathcal{H}(X)$. This $\mathcal{H}(X)$ will be in our case the most general example of hyperspaces (a richer list of hyperspaces can be found in [19]). An important subspace of $\mathcal{H}(X)$ is the family $\mathcal{K}(X)$ of all non-empty compact subsets of X. If X is compact, then $\mathcal{K}(X)=\mathcal{H}(X)$. Another "restriction" of hyperspace is obtained if we restrict ourselves to subcontinua of X. The family of all subcontinua in X is denoted by $\mathcal{CK}(X)$. Since every point in X is a degenerate subcontinuum and the Hausdorff distance d_H between one-point subspaces of X coincides with the distance in X between the corresponding points, there is an isometric embedding $x\mapsto\{x\}$ from X into $\mathcal{CK}(X)\subset\mathcal{K}(X)\subset\mathcal{H}(X)$. In what follows, we will informally write $X\subset\mathcal{CK}(X)$ to refer to this embedding.

Theorem 5. *The spaces X and $\mathcal{CK}(X)$ are closed subsets of $\mathcal{K}(X)$.*

The following result is well known [3].

Theorem 6. *A space $\mathcal{H}(X)$ is complete* (*totally bounded, compact, boundedly compact*) *if and only if X is of the same type.*

If X is a continuum, then, by Theorem 6, the space $\mathcal{K}(X)=\mathcal{H}(X)$ is also compact. By Theorem 5, $\mathcal{CK}(X)$ is a closed subset of the compact set $\mathcal{K}(X)$, and therefore is itself compact. A less trivial fact is that the connectedness of X is also inherited. And even more interesting: $\mathcal{CK}(X)$ turns out to be path-connected, and for a non-degenerate X, one can estimate the dimension of $\mathcal{CK}(X)$.

Theorem 7. *For any continuum X, the space $\mathcal{CK}(X)$ is linearly connected continuum* [19, Theorem 14.9]*, has trivial shape* [19, Theorem 19.10] *and, in particular, is acyclical in each dimension* [19, Theorem 19.3].

For a hereditarily indecomposable or locally connected continuum X, the space $\mathcal{CK}(X)$ is contractible [19, Theorem 20.3, 20.14].

If the continuum X is non-degenerate, then $\dim\mathcal{CK}(X)\ge 2$ [19, Theorem 22.18].

Example 5. (1) For the standard interval I and the circle S^1, the spaces $\mathcal{CK}(I)$ and $\mathcal{CK}(S^1)$ are homeomorphic to two-dimensional disk B^2 [19, sections 5.1 and 5.2].

(2) The spaces $\mathcal{CK}^c_{GH}(I)$, $\mathcal{CK}_{GH}(I^1)$, and $\mathcal{CK}_{GH}(S^1)$ are homeomorphic to an interval.

(3) The space $\mathcal{CK}^c_{GH}(S^1)$ is homeomorphic to the union of a half-interval and an isolated point. Note that the point $\{S^1\}$ corresponding to the entire circle is at distance 1 in the d^c_{GH} metric from every proper subset of the circle and is therefore isolated in $\mathcal{CK}^c_{GH}(S^1)$.

A continuum X is called *hereditarily indecomposable* if each of its subcontinuum Y cannot be represented as the union of two proper (non-empty and distinct from Y) subcontinua.

Recall that we call a topological space Y incomparable with a topological space X if the only continuous mappings from X to Y are mappings to a point.

Proposition 19. *Let X be a path-connected space, and Y a hereditarily indecomposable continuum. Then X is incomparable with Y. In particular, $2d^c_{GH}(X,Y) \geq \operatorname{diam} X$.*

Proof. Let $f\colon X \to Y$ be a continuous mapping such that $f(X)$ contains at least two distinct points $y_0 = f(x_0) \neq f(x_1) = y_1$. Since X is path-connected, there exists a continuous mapping $\varphi\colon [0,1] \to X$ such that $\varphi(0) = x_0$ and $\varphi(1) = x_1$. Set $t_0 = \sup\big[(f\circ\varphi)^{-1}(x_0)\big] = \max\big[(f\circ\varphi)^{-1}(x_0)\big]$. Clearly, $\varphi(t_0) = x_0$. We put $t_1 = \inf\big[(f\circ\varphi)^{-1}(x_1)\big] \cap [t_0,1] = \min\big[(f\circ\varphi)^{-1}(x_1)\big] \cap [t_0,1]$. For an arbitrary $t_0 < \tau < t_1$, the continuum $K = (f\circ\varphi)\big([t_0,t_1]\big) \subset Y$ is represented as the union of two proper subcontinua $y_0 \in K_0 = (f\circ\varphi)\big([t_0,\tau]\big) \not\ni y_1$ and $y_0 \notin K_1 = (f\circ\varphi)\big([\tau,t_1]\big) \ni y_1$. The resulting representation contradicts the indecomposability of Y, i.e., it means the triviality of any continuous mapping $f\colon X \to Y$. $\square$

A *chain* in a topological space is a finite sequence of its subsets in which only successive elements intersect (they are called *links*). If for some $\varepsilon > 0$ each link in a chain lying in a metric space has diameter at most ε, then such a chain is called an *ε-chain.* If for every $\varepsilon > 0$ the continuum is covered by an ε-chain, then such a continuum is called *arc-shaped.* A hereditarily indecomposable arc-shaped continuum is called a *pseudoarc.* An example of a non-degenerate pseudoarc that is a subset of the plane was given by Knaster in [20]. Moise [21] showed that every proper subcontinuum of a pseudoarc is homeomorphic to the pseudoarc itself. Bing [22] proved that all pseudo-arcs are homeomorphic to each other. Bing also found in [6] that almost all continua in the Euclidean space $\mathbb{R}^n$, $n \geq 2$, are pseudo-arcs. More formally, a subset Y of a topological space X is called a *G_δ-set* if Y is equal to at most a countable intersection of open subsets of X. We say that *most* *elements of a complete metric space X are elements of* $Y \subset X$ if Y is a dense G_δ-subset of X. Recall that the space $\mathcal{CK}(\mathbb{R}^n)$ of all continua in $\mathbb{R}^n$ is equipped with the Hausdorff metric and the corresponding metric topology.

Theorem 8 (Bing [6]). *Most continua in $\mathbb{R}^n$, $n \geq 2$, i.e., most points from $\mathcal{CK}(\mathbb{R}^n)$ are pseudo-arcs.*

It follows that the Gromov–Hausdorff distance from any subcontinuum in $\mathbb{R}^n$, $n \geq 2$, to the set of all pseudo-arcs is zero, but the continuous Gromov–Hausdorff distance from any path-connected subcontinuum to the set of all pseudo-arcs is equal to the diameter of this subcontinuum.

Let X be a continuum. The distances d^c_{GH} and d_{GH} are pseudometrics on the set $\mathcal{CK}(X)$. The quotient spaces $\mathcal{CK}(X)$ by the pseudometrics d^c_{GH} and d_{GH} are denoted by $\mathcal{CK}^c_{GH}(X)$ and $\mathcal{CK}_{GH}(X)$, respectively. The projections $p^c_{GH}\colon \mathcal{CK}^c_{GH}(X) \to \mathcal{CK}_{GH}(X)$ and $p_H\colon \mathcal{CK}_H(X) \to \mathcal{CK}_{GH}(X)$ are non-expanding mappings, so they are continuous.

Proposition 20. *The space $\mathcal{CK}_{GH}(X)$ is a path-connected continuum.*

Proof. By Theorem 7 the space $\mathcal{CK}(X)$ is a linearly connected continuum. Now the result follows from the fact that a continuous mapping preserves both compactness and path connectivity. □

H. Cook in [23] constructed an example of a hereditarily indecomposable continuum M_1 such that any two of its different non-degenerate subcontinua are incomparable.

Theorem 9 (Cook [23]). *The continuum M_1 is one-dimensional in the sense of Lebesgue dimension, therefore M_1 is embedded in $\mathbb{R}^3$, but no one non-degenerate subcontinuum of it is embeddable in the plane.*

Let us apply Corollary 9.

Corollary 10. *Let K and H be arbitrary distinct subcontinua in the Cook continuum M_1, then $2d^c_{GH}(K,H) = \max\{\operatorname{diam} H, \operatorname{diam} K\}$. In particular, the mapping p_c^{-1} is discontinuous on the entire $\mathcal{CK}_{GH}(M_1)$, except for the one-point space.*

Remark 4. According to Item (4) of Proposition 6, the value $d^c_{GH}(K,H)$ is the maximum possible value of the d^c_{GH}-distance between H and K.

9 Continuous GH-distance is intrinsic

Let X and Y be arbitrary metric spaces such that $2d^c_{GH}(X,Y) < \infty$. Choose an arbitrary $\varepsilon > 0$, then there exist continuous $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ such that $\operatorname{dis} R_{f,g} < 2d^c_{GH}(X,Y) + 2\varepsilon$. For each $t \in (0,1)$, define a metric d_t on $R := R_{f,g}$ as follows:

$$d_t\big((x,y),(x',y')\big) = (1-t)|xx'| + t\,|yy'|,$$

and we denote the resulting metric space by R_t. We extend R_t by setting $R_0 = X$ and $R_1 = Y$.

Theorem 10. *In the notation introduced above, we have* $d^c_{GH}(R_t, R_s) \le |ts| \operatorname{dis} R$. *In particular, the mapping* $\gamma\colon [0,1] \to \mathcal{GH}^c$ *is a continuous curve whose length* $|\gamma|$ *satisfies the inequality* $|\gamma| < d^c_{GH}(X,Y) + \varepsilon$, *and the continuous Gromov–Hausdorff distance is intrinsic.*

Proof. Let $0 < t,\ s < 1$, and $R' \in \mathcal{R}(R_t, R_s)$ be the correspondence generated by the identity mapping. Then

$$\operatorname{dis} R' = \sup_{(x,y),(x',y')\in R} \Big| d_t\big((x,y),(x',y')\big) - d_s\big((x,y),(x',y')\big)\Big| = \\ = \sup_{(x,y),(x',y')\in R} |ts|\, \big||xx'| - |yy'|\big| = |ts| \operatorname{dis} R.$$

Further, for $t = 0$, $X = R_t$ and R_s, $0 < s < 1$, we choose the correspondence $R_{f',g'}$ as $R' \in \mathcal{R}(R_t, R_s)$, where $f'(x) = \big(x, f(x)\big)$ for all $x \in X$, and $g'\big((x,y)\big) = x$. The mapping f' is continuous, since it is the restriction to $X \times f(X)$ of the mapping $x \mapsto \big(x, f(x)\big)$ from X to $X\times Y$ with continuous coordinate mappings. The mapping g' is also continuous as the restriction of the projection $X \times Y \to X$. Further,

$$\operatorname{dis} f' = \sup_{x,x'\in X} \Big| |xx'| - (1-s)|xx'| - s\,\big|f(x)f(x')\big|\Big| = \\ = \sup_{x,x'\in X} s\,\Big||xx'| - \big|f(x)f(x')\big|\Big| = |ts| \operatorname{dis} f \le |ts| \operatorname{dis} R;$$

$$\operatorname{dis} g' = \sup_{(x,y),(x',y')\in R} \big||xx'| - (1-s)|xx'| - s\,|yy'|\big| = \\ = \sup_{(x,y),(x',y')\in R} s\,\big||xx'| - |yy'|\big| = |ts| \operatorname{dis} R;$$

$$\operatorname{codis}(f',g') = \sup_{x\in X,\,(x',y')\in R} \Big|\Big|x\,g'\big((x',y')\big)\Big| - \big|(x',y')\,f'(x)\big|\Big| = \\ = \sup_{x\in X,\,(x',y')\in R} \Big||xx'| - (1-s)|xx'| - s\,\big|y'\,f(x)\big|\Big| = \\ = \sup_{(x,f(x)),\,(x',y')\in R} s\,\Big||xx'| - \big|y'\,f(x)\big|\Big| \le |ts| \operatorname{dis} R.$$

Thus, we have shown that $\operatorname{dis} R' \le |ts| \operatorname{dis} R$ also in the case $t = 0$ and $0 < s < 1$. The case $0 < t < 1$ and $s = 1$ is analyzed similarly. Finally, for $t = 0$ and $s = 1$, we set $R' = R$, so that here $\operatorname{dis} R' = |ts| \operatorname{dis} R$. Thus, for all $0 \le t,\ s \le 1$, we have

$$d^c_{GH}(R_t, R_s) \le \frac{1}{2}|ts| \operatorname{dis} R,$$

from which it immediately follows that the mapping $\gamma\colon t \mapsto R_t$ is continuous with respect to d^c_{GH}, and the length $|\gamma|$ of the curve γ does not exceed $\frac{1}{2}\operatorname{dis} R < d^c_{GH}(X,Y)+\varepsilon$. Since $\varepsilon>0$ can be chosen arbitrarily small, we conclude that the distance d^c_{GH} is intrinsic. $\square$

As in the standard theory, the correspondence $R_{f,g}\in\mathcal{R}(X,Y)$ is called *optimal* if $2d^c(X,Y)=\operatorname{dis} R_{f,g}$.

Corollary 11. *If the correspondence $R=R_{f,g}$ is optimal, then the curve R_t constructed above is the shortest geodesic, the length of which is equal to the distance between its ends.*

10 Incompleteness

Proposition 21. *The continuous Gromov–Hausdorff distance restricted to the space $\mathcal{M}^c$ of metric compacts is not complete.*

Proof. Consider the sequence $X_n=\{i/n\}_{i=0}^n$ of subsets of the interval $[0,1]$ with induced distance. Since the continuous and ordinary GH-distances coincide on finite metric spaces, this sequence is fundamental in $\mathcal{M}^c$. Suppose that $X_n \xrightarrow{d^c_{GH}} X$, then $X_n \xrightarrow{d_{GH}} X$, and since $\mathcal{M}$ is a metric space, the limit is uniquely defined, hence $X=[0,1]$. However, by Proposition 12 we have $d^c(X_n,X)=1/2$, so the sequence X_n does not converge to X in $\mathcal{M}^c$. $\square$

The validity of Proposition 21 is due to the fact that we have restricted the class of metric spaces (compacta) in which we seek the limit of the given fundamental sequence. However, in the class of all metric spaces, this sequence has a limit, namely, it is each zero-dimensional dense subset of the interval. Let us construct a fundamental sequence for which no metric space is its limit w.r.t. the continuous Gromov-Hausdorff metric.

We will need additional technical result.

Lemma 2. *If for a subset $A\subset X$ and a mapping $f\colon X\to Y$ at least one of the sets A or $f(A)$ has a finite diameter, then it holds* $\operatorname{dis} f \ge \bigl|\operatorname{diam} f(A)-\operatorname{diam} A\bigr|$.

Proof. Suppose that only one of the sets A and $f(A)$ has a finite diameter. Then, in the set of finite diameter, each pair of points is at a bounded distance, while in the second, the corresponding points (from the image or preimage of f) can be chosen arbitrarily far apart, which proves $\operatorname{dis} f=\infty$, and the inequality holds.

Now let both A and $f(A)$ have finite diameters. We fix $\varepsilon>0$.

Let us take points $a,a'\in A$ such that $|aa'|>\operatorname{diam} A-\varepsilon$. Then

$$\operatorname{dis} f \ge |aa'| - \bigl|f(a)f(a')\bigr| > \operatorname{diam} A-\varepsilon-\bigl|f(a)f(a')\bigr| \ge \operatorname{diam} A-\varepsilon-\operatorname{diam} f(A).$$

Due to the arbitrariness of the number $\varepsilon>0$, we obtain $\operatorname{dis} f\ge \operatorname{diam} A-\operatorname{diam} f(A)$.

Next, we take points $a, a' \in A$ such that $\bigl|f(a)f(a')\bigr| > \operatorname{diam} f(A) - \varepsilon$. Then

$$\operatorname{dis} f \geq \bigl|f(a)f(a')\bigr| - |aa'| > \operatorname{diam} f(A) - \varepsilon - |aa'| \geq \operatorname{diam} f(A) - \varepsilon - \operatorname{diam} A.$$

Since the number $\varepsilon > 0$ is arbitrary, we obtain $\operatorname{dis} f \geq \operatorname{diam} f(A) - \operatorname{diam} A$, which completes the proof. $\square$

The resulting inequality is a strengthening of Proposition 6, Property (5).

Lemma 3. *For any mappings $f\colon X \to Y$, $g\colon Y \to X$, and any point $y_0 \in Y$, we have* $\operatorname{codis}(f,g) \geq \bigl|y_0\, f(X)\bigr|$.

Proof. Take a point $x = g(y_0) \in X$. Then

$$\operatorname{codis}(f,g) \geq \bigl|f(x)\, y_0\bigr| - \bigl|x\, g(y_0)\bigr| = \bigl|f(x)\, y_0\bigr| \geq \bigl|y_0\, f(X)\bigr|,$$

which is what was required. $\square$

Example 6. In the plane with coordinates (x,y) we choose the following intervals: $J = \{(0,y) : -1 \leq y \leq 1\}$, $I = \{(x,0) : 0 \leq x \leq 1\}$, and $I_n = \{(x,0) : \frac{1}{n} \leq x \leq 1\}$, $n \in \mathbb{N}$. Consider the triode $X = J \cup I$ and the sequence of spaces $X_n = J \cup I_n$, $n \in \mathbb{N}$. On the triode $X = J \cup I$ we take the intrinsic metric, and on its subsets X_n the metric induced by this intrinsic metric.

Theorem 11. *The sequence of spaces $X_n = J \cup I_n$, $n \in \mathbb{N}$ is fundamental w.r.t. the metric d^c_{GH}, but no metric space is its limit in this metric.*

Proof. It is easy to see that $d^c_{GH}(X_m, X_n) \leq \frac{m-n}{mn} < \frac{1}{n}$ for $m \geq n$. Therefore, the sequence of compact metric spaces X_n is fundamental in the metric d^c_{GH}.

One can easily verify that $d_{GH}(X, X_n) \leq d_H(X, X_n) = \frac{1}{2n}$. Therefore, the triode X is the limit of the sequence X_n in the metric d_{GH}.

Let this sequence have a limit X_∞ in the metric d^c_{GH}.

From the inequality $d_{GH} \leq d^c_{GH}$ it follows that the space X_∞ is also the limit of the sequence X_n in the metric d_{GH}. Therefore, $d_{GH}(X, X_\infty) = 0$.

According to Theorem 2, we can assume that the space X_∞ belongs to the triode X and is dense there.

Let us show that $\bigl\{(0,-1)\bigr\} \cup \bigl\{(0,1)\bigr\} \cup \bigl\{(1,0)\bigr\} \cup X_\infty = X$. The inclusion of the left set into the right set is obvious.

Take an arbitrary $(x_0, y_0) \in X \setminus \Bigl(\bigl\{(0,-1)\bigr\} \cup \bigl\{(0,1)\bigr\} \cup \bigl\{(1,0)\bigr\} \cup X_\infty\Bigr)$.

a. Let $(x_0, y_0) \neq (0,0)$. Without loss of generality we can assume that $(x_0, y_0) = (r, 0)$, $r > 0$. Then one of the connected components K of the space $X \setminus \bigl\{(x_0, y_0)\bigr\}$ has diameter $\operatorname{diam} K = 1 - r < 1$. We fix an index n such that $\operatorname{diam} I_n = 1 - 1/n > \operatorname{diam} K$.

Let $m \geq n$ and the mapping $f\colon X_m \to X_\infty$ be continuous. The point $(r, 0)$ partitions the space X_∞, so any connected set intersecting K lies entirely in K.

If $f(J) \cap K \neq \emptyset$, then $f(J) \subset K$ and $\operatorname{dis} f \geq 2 - (1 - r) = 1 + r$ according to Lemma 2.

If $f(I_m) \cap K \neq \emptyset$, then $f(I_m) \subset K$ and $\operatorname{dis} f \geq 1 - \frac{1}{m} - (1-r) = r - \frac{1}{m} > r - \frac{1}{n}$ according to Lemma 2.

If $f(X_m) \cap K = \emptyset$, then $\operatorname{codis}(f,g) \geq 1-r$ according to Lemma 3 for any mapping $g\colon X_m \to X_\infty$.

b. Let $(x,y) = (0,0)$. Then the space $X \setminus \{(x,y)\} = K_1 \sqcup K_2 \sqcup K_3$ consists of three connected components of diameter 1. Hence, for every continuous mapping $f\colon X_m \to X_\infty \subset X$, there is a connected component K_i such that $f(J) \subset K_i$. Therefore, $\operatorname{dis}(f) \geq \operatorname{diam} J - \operatorname{diam} K_i = 1$ by Lemma 2.

In account, we proved that the space X_∞ is connected. This means that for every continuous mapping $g\colon X_\infty \to X_m$ of a connected space into a space with two connected components, the latter has a connected component K such that $g(X_\infty) \cap K = \emptyset$. By Lemma 3, $\operatorname{codis}(f,g) \geq 1 - \frac{1}{m}$ for any mapping $g\colon X_m \to X_\infty$.

Therefore, for any metric space X_∞, the sequence of distances $d^c_{GH}(X_m, X_\infty)$ cannot tend to zero. $\square$

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