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Temporal Analysis of Hidden and Standard Markov Processes with Final Sequences of States

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Abstract. This research is focused on temporal analysis of hidden and standard Markov processes with final sequences of states and the related games. Initially, the stochastic systems with final sequences of states are considered and enhanced methods for determining the distributions of the evolution time and the game duration are proposed. These approaches are extended later for the hidden Markov models with final sequences of observable states. Additionally, the definition of the total alert time is generalized for the hidden and standard Markov processes and the developed numerical algorithms for the evolution time are adapted to cover also the computation of distribution of the total alert time. For both, hidden and standard Markov processes, and for the associated games, it is shown that the evolution time, the total alert time and the game duration have homogeneous linear recurrent distributions, which allows us to easily evaluate their main probabilistic characteristics, using the previously developed polynomial algorithms.

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1 Introduction

According to [1], a stochastic system with final sequences of states is defined as a Markov process with ω states, enriched with r final sequences of states (of length m), which stops once one of them is reached. Its evolution is driven by the initial and transition probabilities of the Markov process. The evolution time represents the amount of time needed to reach a final sequence of states.

Particularly, for $r = 1$, the stochastic system is named Markov process with final sequence of states. Also, in the case when the probability of each transition from one source state to another target state depends only on target state, and it is equal to initial probability of target state, the stochastic system represents a zero-order Markov process with final sequences of states, deeply studied in [2].

It is known from [1] that, in general case, the evolution time has a homogeneous linear recurrent distribution of order $m^2(r + \omega)$. Even if this order of recurrence is polynomial and it is good enough while studying the stochastic systems with final sequences of states, it is not minimal and this suggests the desire to be decreased.

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Also, the related algorithms, that have been developed until now, are not so extensible. For instance, if we try to use the same approach for hidden Markov models, we will end up with exponential complexity.

The goal of this research is to decrease the order of the recurrence by enhancing the numerical methods that have been developed. These approaches will become extensible to related games and to the hidden Markov models with final sequences of observable states. Also, they will enable us to study the total alert time of hidden and standard Markov processes with final sequences of states, which was previously able to be performed only for zero-order Markov processes with final critical state, due to limitations of used combinatorial method.

So, this study is focused on temporal analysis of hidden and standard Markov processes with final sequences of states and the related games. The existing methods for determining the distribution of the evolution time of stochastic systems with final sequences of states are improved and made more extensible with the help of Knuth–Morris–Pratt and Aho–Corasick string-searching algorithms. The definition of the total alert time is generalized for the hidden and standard Markov processes and the developed numerical algorithms for the evolution time are adapted to cover also the computation of distribution of the total alert time.

2 Evolution Time Analysis

In this section we will be presenting efficient methods for decreasing the order of the homogeneous linear recurrent distribution of the evolution time of stochastic systems with final sequences of states. Additionally, these enhanced methods will be extended to the hidden Markov models with final sequences of observable states.

2.1 Zero-Order Markov Processes with Final Sequence of States

A zero-order Markov process with final sequence of states was defined in [6] as a stochastic system L with finite set of states V, $|V| = \omega$. For each $v \in V$, the system L starts its evolution from v with the probability $p^*(v)$. Also, for every two states $u, v \in V$, the stochastic system L passes from u to v with the same probability $p^*(v)$.

In order to define the stopping rule for the stochastic system L, it was enriched with a sequence of states $X = (x_1, x_2, \dots, x_m) \in V^m$. As result, the system terminates its evolution when the states $x_1, x_2, \dots, x_m$ are consecutively reached.

The evolution time of this system represents the amount of time needed for the system to reach the final sequences of states. So, in the condition that the system starts its evolution at the moment of time $t = 0$ and each transition takes one unit of time, the moment of time T, when the system stops, represents the evolution time of the stochastic system L with given final sequence of states X (of length m).

The distribution $a = (\mathbb{P}(T = n))_{n=0}^{\infty}$ of the evolution time T was studied in [6]. It was theoretically grounded that $a \in Rol[\mathbb{R}][m]$, which means that the sequence a represents a (non-degenerated) homogeneous linear recurrence of order m over $\mathbb{R}$.

Moreover, it was proved that $m = dim[\mathbb{R}](a)$, i.e. the recurrence is m-minimal over the set $\mathbb{R}$. This means that there is no place for improvements by trying to reduce the length of the final sequence of states. Even if it had been possible to handle the zero-order Markov process as a standard Markov process and apply optimizations given by Knuth–Morris–Pratt and Aho–Corasick string-searching algorithms (as it will be shown in the next sections), this would have not led to better results, because the order of the recurrence had already been minimal.

Anyway, we can slightly improve the computations by applying a state merging strategy. Since all the states are independent, the states not belonging to the final sequence of states can be merged together into a single macro-state. In this way, all the states from $V\backslash\{x_1, x_2, \ldots, x_m\}$ form a single macro-state ϖ, for which we have

$$p^*(\varpi) = \sum_{v \in V\backslash\{x_1,x_2,\ldots,x_m\}} p^*(v) = 1 - \sum_{v \in \{x_1,x_2,\ldots,x_m\}} p^*(v).$$

The stochastic system L can be handled as a zero-order Markov process with smaller set of states $\{\varpi, x_1, x_2, \ldots, x_m\}$.

Also, there is an important advantage of this state merging strategy. It allows us to consider also the case when the set of states is infinite. By merging the states not belonging to the final sequence of states, the set of states becomes finite.

2.2 Markov Processes with Final Sequence of States

Next, we extend our study to the standard Markov processes with interdependent states and final sequence of states. These stochastic systems were investigated in [6].

Let L be a such system. Similarly as for zero-order Markov processes, the system L starts its evolution from each state $v \in V$ with the probability $p^*(v)$ and stops when the final sequence of states X is reached. The difference is that, for every two states $u, v \in V$, the stochastic system L passes from state u to the state v with the probability $p(u, v)$, that depends on both states, u and v.

The distribution $a = (\mathbb{P}(T = n))_{n=0}^{\infty}$ of the evolution time T was studied in [6]. It was theoretically grounded that $a \in Rol^*[\mathbb{R}][m\omega]$, which means that the sequence a represents a homogeneous linear recurrence of order $m\omega$ over the set $\mathbb{R}$.

The issue here is that the order $m\omega$ may be non-minimal. We will explore the Knuth–Morris–Pratt algorithm for decreasing this order of recurrence.

As presented in [9], the idea of this string-searching algorithm is to track the current longest found prefix while searching for the occurrence of a word within another string. At each position in the string, if the current char is the next expected one from the word, it is concatenated to the longest found prefix, otherwise the prefix is reset or a fallback to the longest prefix that ends with the current char is applied.

So, let apply this idea to our stochastic system L. We enrich each state $v \in V$ with an additional number k, $0 \leq k \leq m$, that represents the length of the longest found prefix from the final sequence of states. In other words, the stochastic system L is in the state (v, k) at the moment of time t if at the last k moments of time $t-k+1, t-k+2, \ldots, t$ it has been in the states $x_1, x_2, \ldots, x_k$ and this number k is the largest one that satisfies this property. The number k is equal to 0 if any such sequence does not exist (i.e. the current state is not present in the final sequence of states or it is not critical).

In order to minimize the number of joint states, we consider only the states that are possible. So, for each state $v \neq x_1$, we have the possible state $(v, 0)$. The state $(x_1, 0)$ is not possible because, in the worst scenario, the state x_1 acts as fallback indicating the starting of a new matching of the final sequence of states, i.e. the state $(x_1, 1)$. Additionally, for each k, $2 \leq k \leq m$, we have the possible state (x_k, k). In this way, we have built the set of all possible states $Z = \bigcup_{k=0}^{m} Z^{(k)}$, where

$$Z^{(0)} = \{(v, 0) \mid v \in V \backslash \{x_1\}\}, \; Z^{(k)} = \{(x_k, k)\}, \; k = \overline{1, m}.$$

This set contains $|Z| = \omega - 1 + m$ states.

Now, let compute the initial probabilities and transition probability matrix for the new states. It is easy to observe that

$$p^*((v, 0)) = p^*(v), \; \forall v \in V \backslash \{x_1\},$$

$$p^*((x_1, 1)) = p^*(x_1), \; p^*((x_k, k)) = 0, \; k = \overline{2, m}.$$

Regarding transition probabilities, we have the following relations:

$$p((u, 0), (v, 0)) = p(u, v), \; \forall u, v \in V \backslash \{x_1\},$$

$$p((u, 0), (x_1, 1)) = p(u, x_1), \; \forall u \in V \backslash \{x_1\},$$

$$p((x_k, k), (v, 0)) = p(x_k, v), \; \forall v \in V \backslash \{x_1, \; x_2, \ldots, x_m\}, \; k = \overline{1, m},$$

$$p((x_k, k), (x_{k+1}, k + 1)) = p(x_k, x_{k+1}), \; k = \overline{1, m - 1},$$

$$p((x_k, k), (v, f_k(v))) = p(x_k, v), \; \forall v \in \{x_1, \; x_2, \ldots, x_m\} \backslash \{x_{k+1}\}, \; k = \overline{1, m - 1},$$

where $f_k(v)$ is the fallback position of the next state v when the current joint state is (x_k, k). For the rest of transitions, the probability is equal to 0.

In order to clarify more accurately what the fallback position means in this context, let look again at its definition. The position $f_k(v)$ is said to be the fallback position at transition from joint state (x_k, k) to the state $v \in \{x_1, \; x_2, \ldots, x_m\} \backslash \{x_{k+1}\}$ if $0 \leq f_k(v) \leq k$ and $f_k(v)$ is the maximal integer that satisfies the equality

$$(x_1, x_2, \ldots, x_{f_k(v)}) = (x_{k - f_k(v) + 2}, x_{k - f_k(v) + 3}, \ldots, x_k, v).$$

In other words, when the state v occurs instead of x_{k+1} while being in the joint state (x_k, k), a shortcut happens and the critical occurrence $(x_1, x_2, \ldots, x_k)$ is replaced by the shorter one $(x_1, x_2, \ldots, x_{f_k(v)})$.

Now, having all the transition probabilities well defined, we can easily note that they do not depend on the moment of time when the transition is performed. So, the new stochastic system is also a Markov process. Even if the number of the states was increased, from ω to $\omega_{new} = \omega - 1 + m$, the improvement can be seen in the final sequence of states. Indeed, the new stochastic system has only one final state, i.e. $m_{new} = 1$, because the final sequence of states is $X_{new} = ((x_m, m))$. Instead, the evolution time remains unchanged. As result, the order of the homogeneous linear recurrence, which describes the distribution of the evolution time, is decreased from $m\omega$ to $m_{new} \cdot \omega_{new} = 1 \cdot (\omega - 1 + m) = \omega - 1 + m$.

2.3 Markov Processes with Multiple Final Sequences of States

In this section we analyze the complexity reduction for the Markov processes with multiple final sequences of states. These stochastic systems are defined similarly as the Markov processes with final sequence of states investigated in Section 2.2, the main difference being the adjustment of stopping rule.

So, for defining the stopping rule, r different sequences of states

$$X^{(\ell)} = (x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_m^{(\ell)}) \in V^m, \ell = \overline{1, r},$$

are given and the stochastic system stops as soon as the states $x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_m^{(\ell)}$ are reached consecutively in given order for an arbitrary $\ell \in \{1, 2, \ldots, r\}$. Let recall several sets and sequences used in [1] and additional notations for $k = \overline{1, m},\ \ell = \overline{1, r}$:

$$X_k = \{x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(r)}\},\ X = \{X^{(1)}, X^{(2)}, \ldots, X^{(r)}\},$$

$$Y_k^{(\ell)} = (x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_k^{(\ell)}),\ Y_k = \{Y_k^{(1)}, Y_k^{(2)}, \ldots, Y_k^{(r)}\},$$

$$Z_k^{(\ell)} = \{y \in X_k \mid (x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_{k-1}^{(\ell)}, y) \in Y_k\}.$$

From [1] we know that the distribution $a = (\mathbb{P}(T = n))_{n=0}^{\infty}$ of the evolution time T is a homogeneous linear recurrence of order $m^2(r + \omega)$ over the set $\mathbb{R}$. The issue here is that the order $m^2(r + \omega)$ may be non-minimal. Next, we explore the idea of Aho–Corasick algorithm for decreasing this order of the recurrence.

Based on [10], the Aho–Corasick algorithm simultaneously matches multiple words in an input text. This string-searching algorithm tracks the current longest found prefix for each word searched within another string. The common prefixes are reused, in order to avoid repetitions and guarantee the uniqueness of the states. At each position in the string, if the current char is the next expected one from a word, it is concatenated to the longest found prefix for given word (or another word that shares that prefix with given word), otherwise the prefix is reset or a fallback to the longest prefix that ends with the current char is applied.

So, let apply this idea to our stochastic system L. We enrich each state $v \in V$ with two additional characteristics: the number k, $0 \leq k \leq m$, which represents the length of the longest found prefix from the final sequences of states and the corresponding prefix $Y_k^{(\ell)} = (x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_k^{(\ell)})$, $\ell \in \{0, 1, \ldots, r\}$. In other words, the stochastic system L is in the state $(v, Y_k^{(\ell)}, k)$ at the moment of time t if at the last k moments of time $t - k + 1, t - k + 2, \ldots, t$ it has been in the states $x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_k^{(\ell)}$ and this number k is the largest one that satisfies this property along all final sequences of states (i.e. there is no any larger k even for another ℓ, $1 \le \ell \le r$). The number k is equal to 0 if any such sequence does not exist (i.e. the current state is not present in the final sequences of states or it is not critical) and the prefix is considered $Y_0^{(0)} = ()$.

In order to minimize the number of joint states, we consider only the states that are possible and we merge the equivalent states (the joint states that share the same prefix). So, for each state $v \notin X_1 = \{x_1^{(1)}, x_1^{(2)}, \dots, x_1^{(r)}\}$, we have the possible state $(v, Y_0^{(0)}, 0)$. The states of the form $(x_1^{(\ell)}, Y_0^{(0)}, 0)$ are not possible because, in the worst scenario, the state $x_1^{(\ell)}$ acts as fallback indicating the starting of a new matching of the final sequence of states $X^{(\ell)}$, i.e. the state $(x_1^{(\ell)}, Y_1^{(\ell)}, 1)$. Additionally, for each k and ℓ, $2 \le k \le m$, $1 \le \ell \le r$, we have the possible state $(x_k^{(\ell)}, Y_k^{(\ell)}, k)$. In this way, we have built the set of all possible states $Z = \bigcup\limits_{k=0}^{m} Z^{(k)}$, where

$$Z^{(0)} = \{(v, Y_0^{(0)}, 0) \mid v \in V \backslash X_1\},\ Z^{(k)} = \{(x_k^{(\ell)}, Y_k^{(\ell)}, k) \mid 1 \le \ell \le r\},\ k = \overline{1, m}.$$

This set contains $|Z| \le \omega + r(m-1)$ states (the sign "$\le$" is due to the possible existence of the shared prefixes).

Now, let compute the initial probabilities and transition probability matrix for the new states. It is easy to observe that

$$p^*((v, Y_0^{(0)}, 0)) = p^*(v),\ \forall v \in V \backslash X_1,$$

$$p^*((x_1^{(\ell)}, Y_1^{(\ell)}, 1)) = p^*(x_1^{(\ell)}),\ \ell = \overline{1, r},$$

$$p^*((x_k^{(\ell)}, Y_k^{(\ell)}, k)) = 0,\ k = \overline{2, m},\ \ell = \overline{1, r}.$$

Regarding transition probabilities, we have the following relations:

$$p((u, Y_0^{(0)}, 0), (v, Y_0^{(0)}, 0)) = p(u, v),\ \forall u, v \in V \backslash X_1,$$

$$p((u, Y_0^{(0)}, 0), (x_1^{(\ell)}, Y_1^{(\ell)}, 1)) = p(u, x_1^{(\ell)}),\ \forall u \in V \backslash X_1,\ \ell = \overline{1, r},$$

$$p((x_k^{(\ell)}, Y_k^{(\ell)}, k), (v, Y_0^{(0)}, 0)) = p(x_k^{(\ell)}, v),\ \forall v \in V \backslash \bigcup_{j=1}^{m} X_j,\ k = \overline{1, m},\ \ell = \overline{1, r},$$

$$p((x_k^{(\ell)}, Y_k^{(\ell)}, k), (x_{k+1}^{(\ell)}, Y_{k+1}^{(\ell)}, k+1)) = p(x_k^{(\ell)}, x_{k+1}^{(\ell)}),\ k = \overline{1, m-1},\ \ell = \overline{1, r},$$

$$p((x_k^{(\ell)}, Y_k^{(\ell)}, k), (v, Y_\kappa^{(\ell_\kappa)}, \kappa)) = p(x_k^{(\ell)}, v),\ \forall v \in \bigcup_{j=1}^{m} X_j \backslash Z_{k+1}^{(\ell)},\ k = \overline{1, m-1},\ \ell = \overline{1, r},$$

where $\kappa = f_k^{(\ell)}(v)$ is the fallback position of the next state v when the current joint state is $(x_k^{(\ell)}, Y_k^{(\ell)}, k)$ and $\ell_\kappa \in \{1, 2, \dots, r\}$ is the index of the related final sequence of states for which the fallback happens. For other transitions, the probability is 0.

In this context, the position $f_k^{(\ell)}(v)$ represents the fallback position at transition from joint state $(x_k^{(\ell)}, Y_k^{(\ell)}, k)$ to the state $v \in \bigcup_{j=1}^{m} X_j \backslash Z_{k+1}^{(\ell)}$ because $0 \leq f_k^{(\ell)}(v) \leq k$ and $\kappa = f_k^{(\ell)}(v)$ is the maximal integer that satisfies the equality

$$(x_1^{(\ell_\kappa)}, x_2^{(\ell_\kappa)}, \ldots, x_\kappa^{(\ell_\kappa)}) = (x_{k-\kappa+2}^{(\ell)}, x_{k-\kappa+3}^{(\ell)}, \ldots, x_k^{(\ell)}, v)$$

for an index $\ell_\kappa \in \{1, 2, \ldots, r\}$. This means that, when the state v occurs instead of $x_{k+1}^{(\ell)}$ while being in the joint state $(x_k^{(\ell)}, Y_k^{(\ell)}, k)$, a shortcut happens and the critical occurrence $(x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_k^{(\ell)})$ is replaced by the shorter one $(x_1^{(\ell_\kappa)}, x_2^{(\ell_\kappa)}, \ldots, x_\kappa^{(\ell_\kappa)})$.

As result, we can easily observe that the new stochastic system is also a Markov process with multiple final sequences of states. Even if the number of the states was eventually increased, from ω to $\omega_{new} \leq \omega + r(m-1)$, the improvement can be seen in the final sequences of states. The new stochastic system has r final sequences of states of length $m_{new} = 1$, because the final sequences of states are

$$X_{new} = \{((x_m^{(1)}, X^{(1)}, m)), ((x_m^{(2)}, X^{(2)}, m)), \ldots, ((x_m^{(r)}, X^{(r)}, m))\}.$$

The evolution time remains unchanged and the order of the homogeneous linear recurrent distribution of the evolution time is decreased from $m^2(r+\omega)$ to

$$m_{new}^2 \cdot (r + \omega_{new}) \leq 1^2 \cdot (r + \omega + r(m-1)) = \omega + mr.$$

2.4 Zero-Order Markov Processes with Multiple Final Sequences

The zero-order Markov processes with multiple final sequences of states were studied in [2]. They represent the particular case of the Markov processes with multiple final sequences of states, when the states of the system are independent.

From [2] we know that the evolution time distribution $a = (\mathbb{P}(T = n))_{n=0}^{\infty}$ is a homogeneous linear recurrence of order $m^2 r$ over $\mathbb{R}$. By applying the state merging technique, exposed in Section 2.1, we can reduce the number of states from ω to

$$\omega^* = \left| \bigcup_{k=1}^{m} X_k \right| + 1 \leq mr + 1,$$

where $X_k = \{x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(r)}\}$. Next, considering the obtained zero-order Markov process as a standard Markov process and applying the enhancement developed in Section 2.3, we obtain a new Markov process with r final sequences of states of length $m_{new} = 1$ and the number of states

$$\omega_{new} \leq \omega^* + r(m-1) \leq 2mr + 1 - r.$$

The evolution time remains unchanged and the order of the homogeneous linear recurrent distribution of the evolution time is decreased, in the end, from $m^2 r$ to

$$\omega^* + mr \leq 2mr + 1.$$

2.5 HMMs with Final Sequence of Observable States

Let consider a hidden Markov model (HMM), represented by underlying Markov process L with finite sets of hidden states $H = \{h_1, h_2, \ldots, h_\varpi\}$ and observable states $V = \{v_1, v_2, \ldots, v_\omega\}$. The system L starts from each state $h \in H$ with initial probability $p^*(h)$. At each moment of time $t = 0, 1, 2, \ldots$, the system L passes from a state $g \in H$ to the next state $h \in H$ with the probability $p(g, h)$ and emits each observable state $v \in V$ with the probability $e(h, v)$. Additionally, a final sequence of observable states $U = (u_1, u_2, \ldots, u_m) \in V^m$ is given. The amount of time T, needed to observe U, represents the evolution time.

In order to improve the complexity of determining the evolution time of hidden Markov models, a joint Markov process J over the set of states $H \times V$ is considered. The process J is in the state $(h, v) \in H \times V$ at the moment of time t if the hidden state of L is $h \in H$ and the observed state is $v \in V$ at the same moment of time t. The final sequence of observable states $U = (u_1, u_2, \ldots, u_m) \in V^m$ is mapped to a set of multiple final sequences of states of the form $((x_1, u_1), (x_2, u_2), \ldots (x_m, u_m))$ for each tuple $(x_1, x_2, \ldots, x_m) \in H^m$.

So, J is a Markov process with multiple final sequences of states. We have

- The number of the states: $\omega_J = \omega \cdot \varpi$;
- The number of final sequences of states: $r_J = \varpi^m$;
- The length of the final sequences of states: $m_J = m$;
- The order of the recurrence: $m_J^2 \cdot (r_J + \omega_J) = m^2 \cdot (\varpi^m + \omega \cdot \varpi)$.

The bad news here, for the moment, are the exponential parameters. We will explore the Knuth–Morris–Pratt algorithm for decreasing this order of recurrence.

Similarly as done in Section 2.2, we enrich each state $(h, v) \in H \times V$ with an additional number k, $0 \leq k \leq m$, that represents the length of the longest observed prefix from the final sequence of states. The stochastic system J is in the state (h, v, k) at the moment of time t if at the last k moments of time $t-k+1, t-k+2, \ldots, t$ the states $u_1, u_2, \ldots, u_k$ have been observed and this number k is the largest one that satisfies this property. The number k is considered equal to 0 if any such sequence does not exist. The set of all possible states is $Z = \bigcup_{k=0}^{m} Z^{(k)}$, where

$$Z^{(0)} = \{(h, v, 0) \mid h \in H,\ v \in V \backslash \{u_1\}\},\ Z^{(k)} = \{(h, u_k, k) \mid h \in H\},\ k = \overline{1, m}.$$

This set contains $|Z| = \varpi \cdot (\omega - 1 + m)$ states.

For each $g, h \in H$, the initial and transition probabilities of the new states are:

$$p^*((h, v, 0)) = p^*(h) \cdot e(h, v),\ \forall v \in V \backslash \{u_1\},$$

$$p^*((h, u_1, 1)) = p^*(h) \cdot e(h, u_1),\ p^*((h, u_k, k)) = 0,\ k = \overline{2, m}.$$

$$p((g, u, 0), (h, v, 0)) = p(g, h) \cdot e(h, v),\ \forall u, v \in V \backslash \{u_1\},$$

$$p((g,u,0),(h,u_1,1)) = p(g,h)\cdot e(h,u_1),\ \forall u \in V\backslash\{u_1\},$$

$$p((g,u_k,k),(h,v,0)) = p(g,h)\cdot e(h,v),\ \forall v \in V\backslash\{u_1,\dots,u_m\},\ k=\overline{1,m},$$

$$p((g,u_k,k),(h,u_{k+1},k+1)) = p(g,h)\cdot e(h,u_{k+1}),\ k=\overline{1,m-1},$$

$$p((g,u_k,k),(h,v,f_k(v))) = p(g,h)\cdot e(h,v),\ \forall v \in \{u_1,\dots,u_m\}\backslash\{u_{k+1}\},\ k=\overline{1,m-1},$$

where $f_k(v)$ is the fallback position of the next state v when the current joint state is (g,u_k,k). For the rest of transitions, the probability is equal to 0.

So, in the end, the parameters of the stochastic system J become polynomial:

- The number of states: $\omega_{new} = \varpi\cdot(\omega-1+m)$;
- The final sequences of states: $\{((h,u_m,m)) \mid h \in H\}$;
- The number of final sequences of states: $r_{new} = \varpi$;
- The length of the final sequence of states: $m_{new} = 1$;
- The order of the recurrence: $m_{new}^2\cdot(r_{new}+\omega_{new}) = \varpi\cdot(\omega+m)$.

2.6 HMMs with Multiple Final Sequences of Observable States

Now, let consider a similar HMM with multiple final sequences of observable states $U^{(\ell)} = (u_1^{(\ell)},u_2^{(\ell)},\dots,u_m^{(\ell)}) \in V^m, \ell=\overline{1,r}$. The underlying system stops as soon as the states $u_1^{(\ell)},u_2^{(\ell)},\dots,u_m^{(\ell)}$ are observed for an arbitrary $\ell \in \{1,2,\dots,r\}$. The evolution time T represents the amount of time needed to observe $U^{(\ell)}$.

Similarly as in Section 2.5, the joint Markov process J over $H\times V$ is considered. The process J is in the state $(h,v)\in H\times V$ at the moment of time t if the hidden state of L is $h\in H$ and the observed state is $v\in V$. Each $U^{(\ell)} = (u_1^{(\ell)},u_2^{(\ell)},\dots,u_m^{(\ell)}) \in V^m$, $\ell \in \{1,2,\dots,r\}$, is mapped to a subset of final sequences of states of the form $((x_1,u_1^{(\ell)}),(x_2,u_2^{(\ell)}),\dots(x_m,u_m^{(\ell)}))$ for each tuple $(x_1,x_2,\dots,x_m)\in H^m$.

So, J is a Markov process with multiple final sequences of states. We have

- The number of the states: $\omega_J = \omega\cdot\varpi$;
- The number of final sequences of states: $r_J = r\cdot\varpi^m$;
- The length of the final sequences of states: $m_J = m$;
- The order of the recurrence: $m_J^2\cdot(r_J+\omega_J) = m^2\cdot(r\cdot\varpi^m+\omega\cdot\varpi)$.

The parameters are exponential also in this case. We will explore the Aho-Corasick algorithm for decreasing this order of recurrence.

We enrich each state $(h,v)\in H\times V$ with two additional characteristics: the number k, $0\le k\le m$, which represents the length of the longest observed prefix from the final sequence of states, and the corresponding prefix $Y_k^{(\ell)} = (u_1^{(\ell)},u_2^{(\ell)},\dots,u_k^{(\ell)})$, $\ell\in\{0,1,\dots,r\}$. The HMM is in the state $(h,v,Y_k^{(\ell)},k)$ if at the last k moments of time the states $u_1^{(\ell)}, u_2^{(\ell)}, \ldots, u_k^{(\ell)} = v$ have been observed and k is the largest one with this property along all $U^{(\ell)}$, $\ell = \overline{1,r}$. We have $k = 0$ and $Y_0^{(0)} = ()$ if any such sequence does not exist. The set of all possible states is $Z = \bigcup_{k=0}^{m} Z^{(k)}$, where

$$Z^{(0)} = \{(h, v, Y_0^{(0)}, 0) \mid h \in H,\ v \in V \backslash U_1\},$$
$$Z^{(k)} = \{(h, u_k^{(\ell)}, Y_k^{(\ell)}, k) \mid h \in H,\ 1 \le \ell \le r\},\ k = \overline{1,m}.$$

This set contains $|Z| \le \varpi \cdot [\omega + r(m-1)]$ states.

Similarly as before, we consider the following notations for $k = \overline{1,m}$, $\ell = \overline{1,r}$:

$$U_k = \{u_k^{(1)}, u_k^{(2)}, \ldots, u_k^{(r)}\},\ U = \{U^{(1)}, U^{(2)}, \ldots, U^{(r)}\},$$
$$Y_k^{(\ell)} = (u_1^{(\ell)}, u_2^{(\ell)}, \ldots, u_k^{(\ell)}),\ Y_k = \{Y_k^{(1)}, Y_k^{(2)}, \ldots, Y_k^{(r)}\},$$
$$Z_k^{(\ell)} = \{y \in U_k \mid (u_1^{(\ell)}, u_2^{(\ell)}, \ldots, u_{k-1}^{(\ell)}, y) \in Y_k\}.$$

For each $h \in H$, the initial probabilities for the new states are:

$$p^*((h, v, Y_0^{(0)}, 0)) = p^*(h) \cdot e(h, v),\ \forall v \in V \backslash U_1,$$
$$p^*((h, u_1^{(\ell)}, Y_1^{(\ell)}, 1)) = p^*(h) \cdot e(h, u_1^{(\ell)}),\ \ell = \overline{1,r},$$
$$p^*((h, u_k^{(\ell)}, Y_k^{(\ell)}, k)) = 0,\ k = \overline{2,m},\ \ell = \overline{1,r}.$$

Also, for each $g, h \in H$ and $\ell = \overline{1,r}$, the transition probabilities become:

$$p((g, u, Y_0^{(0)}, 0), (h, v, Y_0^{(0)}, 0)) = p(g,h) \cdot e(h, v),\ \forall u, v \in V \backslash U_1,$$
$$p((g, u, Y_0^{(0)}, 0), (h, u_1^{(\ell)}, Y_1^{(\ell)}, 1)) = p(g,h) \cdot e(h, u_1^{(\ell)}),\ \forall u \in V \backslash U_1,$$
$$p((g, u_k^{(\ell)}, Y_k^{(\ell)}, k), (h, v, Y_0^{(0)}, 0)) = p(g,h) \cdot e(h, v),\ \forall v \in V \backslash \bigcup_{j=1}^{m} U_j,\ k = \overline{1,m},$$
$$p((g, u_k^{(\ell)}, Y_k^{(\ell)}, k), (h, u_{k+1}^{(\ell)}, Y_{k+1}^{(\ell)}, k+1)) = p(g,h) \cdot e(h, u_{k+1}^{(\ell)}),\ k = \overline{1,m-1},$$
$$p((g, u_k^{(\ell)}, Y_k^{(\ell)}, k), (h, v, Y_\kappa^{(\ell_\kappa)}, \kappa)) = p(g,h) \cdot e(h, v),\ \forall v \in \bigcup_{j=1}^{m} U_j \backslash Z_{k+1}^{(\ell)},\ k = \overline{1,m-1},$$

where $\kappa = f_k^{(\ell)}(v)$ is the fallback position of the next state v when the current joint state is $(g, u_k^{(\ell)}, Y_k^{(\ell)}, k)$ and $\ell_\kappa \in \{1, 2, \ldots, r\}$ is the index of the final sequence of states for which the fallback happens. For other transitions the probability is 0.

So, for the stochastic system J we have polynomial parameters:

- The number of states: $\omega_{new} \le \varpi \cdot [\omega + r(m-1)]$;
- The final sequences of states: $\{((h, u_m^{(\ell)}, U^{(\ell)}, m)) \mid h \in H,\ \ell = \overline{1,r}\}$;
- The number of final sequences of states: $r_{new} = r \cdot \varpi$;
- The length of the final sequences of states: $m_{new} = 1$;
- The order of the recurrence: $m_{new}^2 \cdot (r_{new} + \omega_{new}) \le \varpi \cdot (\omega + mr)$.

3 Game Duration Analysis

In this section we will be applying the previously developed techniques for decreasing the order of the homogeneous linear recurrent distribution of the duration of the games defined on Markov processes with final sequence of states.

3.1 Games on Markov Processes with Final Sequence of States

Let L be a Markov process with final sequence of states X, as defined in Section 2.2, with initial distribution of states $(p^*(v))_{v\in V}$. A game defined on stochastic system L was considered in [4].

According to the game rules, each player $\mathcal{P}_\ell$ has an own strategy, which is a transition matrix $(p^{(\ell)}(u,v))_{u,v\in V}$, $\ell=\overline{0,r-1}$. The game is started by the first player $\mathcal{P}_0$. At every moment of time, the stochastic system passes consecutively to the next state according to the strategy of the current player. After the last player $\mathcal{P}_{r-1}$, the first player $\mathcal{P}_0$ acts on the system evolution and the game continues in this way until the given final sequence of states X is reached. The player $\mathcal{P}_{T mod\ r}$, who acts the last on the system evolution, is considered the winner of the game.

It is known from [4] that the distribution $a=(\mathbb{P}(T=n))_{n=0}^{\infty}$ of the game duration T is a homogeneous linear recurrence of order $mr\omega$ over the set $\mathbb{R}$.

Let apply the same optimization from Section 2.2, i.e. the Knuth–Morris–Pratt string-searching algorithm. Each state $v\in V$ is enriched with an additional number k, $0\le k\le m$, that represents the length of the longest prefix from the final sequence of states X. The set of states becomes $Z=\bigcup\limits_{k=0}^{m} Z^{(k)}$, where

$$Z^{(0)}=\{(v,0)\mid v\in V\backslash\{x_1\}\},\ Z^{(k)}=\{(x_k,k)\},\ k=\overline{1,m}.$$

This set contains $|Z|=\omega-1+m$ states.

The initial probabilities and the strategies of the players, $\ell=\overline{0,r-1}$, are:

$$p^*((v,0))=p^*(v),\ \forall v\in V\backslash\{x_1\},$$

$$p^*((x_1,1))=p^*(x_1),\ p^*((x_k,k))=0,\ k=\overline{2,m},$$
$$p^{(\ell)}((u,0),(v,0))=p^{(\ell)}(u,v),\ \forall u,v\in V\backslash\{x_1\},$$
$$p^{(\ell)}((u,0),(x_1,1))=p^{(\ell)}(u,x_1),\ \forall u\in V\backslash\{x_1\},$$
$$p^{(\ell)}((x_k,k),(v,0))=p^{(\ell)}(x_k,v),\ \forall v\in V\backslash\{x_1,\ x_2,\dots,x_m\},\ k=\overline{1,m},$$
$$p^{(\ell)}((x_k,k),(x_{k+1},k+1))=p^{(\ell)}(x_k,x_{k+1}),\ k=\overline{1,m-1},$$
$$p^{(\ell)}((x_k,k),(v,f_k(v)))=p^{(\ell)}(x_k,v),\ \forall v\in\{x_1,\ x_2,\dots,x_m\}\backslash\{x_{k+1}\},\ k=\overline{1,m-1},$$

where $f_k(v)$ is the fallback position of the next state v when the current joint state is (x_k,k). For the rest of transitions, the probability is equal to 0.

So, the initial game is transformed into an equivalent game with final sequence of states $X_{new}=((x_m,m))$ of length $m_{new}=1$. The number of the players r remains the same and the number of states of the new game becomes $\omega_{new} = \omega - 1 + m$. The game duration remains unchanged and its distribution becomes a homogeneous linear recurrence of smaller order $m_{new} \cdot r \cdot \omega_{new} = 1 \cdot r \cdot (\omega - 1 + m) = r(\omega - 1 + m)$.

3.2 Zero-Order Games with Final Sequence of States

The games defined on zero-order Markov processes with final sequence of states were studied in [5] and represent a particular case of the games from Section 3.1. It is known that the distribution $a = (\mathbb{P}(T = n))_{n=0}^{\infty}$ of the game duration T is a homogeneous linear recurrence of order mr over $\mathbb{R}$. By applying the state merging technique, exposed in Section 2.1, we can reduce the number of states from ω to

$$\omega^* = |\{x_1, x_2, \ldots, x_m\}| + 1 \leq m + 1.$$

We can consider this zero-order game as a general game with final sequence of states and apply the enhancement from Section 3.1, obtaining a new game with final sequence of states of length $m_{new} = 1$ and the number of states

$$\omega_{new} = \omega^* - 1 + m \leq 2m,$$

but this transformation does not give us any improvement in this case. Indeed, the game duration remains unchanged and the order of the homogeneous linear recurrence, which describes its distribution, becomes

$$r(\omega^* - 1 + m) \geq r(2 - 1 + m) > mr.$$

So, only the state merging technique gives us a small improvement for the games defined on zero-order Markov processes with final sequence of states. This state merging strategy allows us to consider also the games defined on zero-order Markov processes with infinite set of states. By merging the states that do not belong to the final sequence of states, the entire set of states becomes finite.

4 Total Alert Time Analysis

In this section we will be studying the total alert time of the stochastic systems with final sequences of states and hidden Markov Models with final sequences of observable states. It will be proved that the total alert time has homogeneous linear recurrent distribution and the order of the recurrence is polynomial.

4.1 Zero-Order Markov Processes with Final Critical State

A zero-order Markov process with final critical state v^* (of degree m) represents a zero-order Markov process L with final sequence of states $X = (x_1, x_2, \ldots, x_m)$, where $x_1 = x_2 = \ldots = x_m = v^*$ and $p = p^*(v^*)$ is the probability of the critical state. The total alert time η was defined in [6] as the cumulative time during which the system remains in its critical state throughout its evolution. Also, the moment of the first alert of order k, $k=\overline{1,m}$, was introduced as the moment of time when, for the first time, the system L consecutively passes k times through its critical state. The number $\gamma=\eta+1$ indicates how many times the critical state is visited.

From [6], it is already known that the evolution time T has the homogeneous linear recurrent distribution $a=(\mathbb{P}(T=n))_{n=0}^{\infty}\in Rol[\mathbb{R}][m]$ with generating vector $q=((1-p)p^k)_{k=0}^{m-1}\in G[\mathbb{R}][m](a)$. The expectation and variance of evolution time T satisfy the following formulas:

$$\mathbb{E}(T)=\overline{m}=m-2+\frac{1+\Delta_1(q)}{w_m},$$

$$\mathbb{V}(T)=(m-1)(m-2)+\overline{m}(\overline{m}-2m+3)+\frac{\Delta_1(q)+\Delta_2(q)}{w_m},$$

where $w_m=\prod\limits_{k=1}^{m}p^*(x_k)\neq 0$ and $\Delta_j(q)=\sum\limits_{k=1}^{m-1}k^j q_k,\ \forall j\in\mathbb{N}^*$.

For determining the total alert time and the moments of the first alert, the related systems L_k, $k=\overline{1,m}$, are considered. The system L_k has the same set of states V, the same critical state v^* and the same probability $p=p^*(v^*)$ of the critical state. The difference is made by the degree of the critical state, which is equal to k for the stochastic system L_k, $k=\overline{1,m}$. It is shown in [6] that μ_k represents the evolution time of the system L_k, $k=\overline{1,m}$. Also, the total alert time η_k of the system L_k and $\mu_{k-1}+1$ have the same distribution for all $k>1$, where $\eta_1=0$.

4.2 Markov Processes with Multiple Final Sequences of States

Now, let consider the general case, represented by Markov processes with multiple final sequences of states, which were analyzed in Section 2.3. We will extend the definition of criticality and total alert time.

A state v is considered critical at one moment of time if it is part of one final sequence of states $X^{(\ell)}$ and the system has just consecutively passed through $x_1^{(\ell)},x_2^{(\ell)},\ldots,x_k^{(\ell)}=v$. It corresponds to a state $(v,Y_k^{(\ell)},k)$ with $k>0$ in the reduced system (the related Markov process with final sequences of states of length 1, obtained by using the Aho-Corasick algorithm, as described in Section 2.3).

The total alert time η is defined similarly, as the cumulative time during which the state is critical. It represents the visiting time of the set of critical-related states $\bigcup\limits_{k=1}^{m-1}Z^{(k)}$ and its distribution, conditioned by the initial state y, satisfies the following recurrences:

$$P_y(n)=\sum_{z\in Z}p(y,z)P_z(n),\ \forall y\in Z^{(0)},\ \forall n\geq 0,$$

$$P_y(0)=0,\ P_y(n)=\sum_{z\in Z}p(y,z)P_z(n-1),\ \forall y\in\bigcup_{k=1}^{m-1}Z^{(k)},\ \forall n\geq 1,$$

$$P_y(0)=1,\ P_y(n)=0,\ \forall y\in Z^{(m)},\ \forall n\geq 1.$$

As a remark, the subset of states $Z_\infty^{(0)} \subseteq Z^{(0)}$, starting from which any critical-related state is not reachable, can be ignored, since the probability of visiting time is zero. For the remaining states $Z\backslash Z_\infty^{(0)}$ we have the recurrence

$$P(n) = Q \cdot P(n) + R \cdot P(n-1),\ \forall n \geq 1,$$

where

$$P(n) = (P_y(n))_{y \in Z\backslash Z_\infty^{(0)}},\ \forall n \geq 0,$$
$$Q = (q(y,z))_{y,z \in Z\backslash Z_\infty^{(0)}},\ R = (r(y,z))_{y,z \in Z\backslash Z_\infty^{(0)}},$$

and, for $\forall z \in Z\backslash Z_\infty^{(0)}$,

$$q(y,z) = p(y,z),\ r(y,z) = 0,\ \forall y \in Z^{(0)}\backslash Z_\infty^{(0)},$$
$$q(y,z) = 0,\ r(y,z) = p(y,z),\ \forall y \in \bigcup_{k=1}^{m-1} Z^{(k)},$$
$$q(y,z) = 0,\ r(y,z) = 0,\ \forall y \in Z^{(m)}.$$

Q is a weakly chained substochastic matrix [3] due to the following properties:

- $q(y,z) \in \{p(y,z), 0\} \subseteq [0,1],\ \forall y,z \in Z\backslash Z_\infty^{(0)}$;
- $\sum_{z \in Z\backslash Z_\infty^{(0)}} q(y,z) \leq 1,\ \forall y \in Z\backslash Z_\infty^{(0)}$;
- $\sum_{z \in Z\backslash Z_\infty^{(0)}} q(y,z) = 0 < 1,\ \forall y \in \bigcup_{k=1}^{m} Z^{(k)}$;
- for each $y \in Z^{(0)}\backslash Z_\infty^{(0)}$ there exists $z \in \bigcup_{k=1}^{m} Z^{(k)}$ such that the state z is reachable from y (in a finite number of steps).

As consequence, the matrix $I - Q$ is invertible ([3],[7],[8]), which involves

$$P(n) = (I-Q)^{-1} \cdot R \cdot P(n-1),\ \forall n \geq 1,$$

i.e. $(P(n))_{n=0}^\infty \in Rol^*[\mathcal{M}_M(\mathbb{R})][1]$, where M is the size of square matrix Q.

Next, we have

$$d = (P(n))_{n=0}^\infty \in Rol^*[\mathcal{M}_M(\mathbb{R})][1],\ ((I-Q)^{-1}R) \in G^*[\mathcal{M}_M(\mathbb{R})][1](d) \Rightarrow$$
$$d \in Rol^*[\mathbb{R}][M],\ H(z) = |I - (I-Q)^{-1}R \cdot z| \in H^*[\mathbb{R}][M](d) \Rightarrow$$
$$d_y = (P_y(n))_{n=0}^\infty \in Rol^*[\mathbb{R}][M],\ H(z) \in H^*[\mathbb{R}][M](d_y),\ \forall y \in Z\backslash Z_\infty^{(0)} \Rightarrow$$
$$c = (\mathbb{P}(\eta = n))_{n=0}^\infty = \sum_{y \in Z\backslash Z_\infty^{(0)}} p^*(y) d_y \in Rol^*[\mathbb{R}][M],\ H(z) \in H^*[\mathbb{R}][M](c) \Rightarrow$$
$$c \in Rol^*[\mathbb{R}][M],\ H(z) = |I - (I-Q)^{-1}R \cdot z| \in H^*[\mathbb{R}][M](c).$$

So, the total alert time distribution is a homogeneous linear recurrence over $\mathbb{R}$. The order of the recurrence is $M = |Z\backslash Z_\infty^{(0)}| \leq |Z| \leq \omega + r(m-1)$.

4.3 HMMs with Multiple Final Sequences of Observable States

Let extend the previous results to hidden Markov models with multiple final sequences of observable states, defined and deeply studied in 2.6.

An observable state v is considered to be critical at one moment of time if it is part of one final sequence of observable states $U^{(\ell)}$ and the system has just consecutively passed through $u_1^{(\ell)}, u_2^{(\ell)}, \ldots, u_k^{(\ell)} = v$. It corresponds to a state $(h, v, Y_k^{(\ell)}, k)$ with $k > 0$ in the reduced stochastic system J, which represents a Markov process with multiple final sequences of states.

The total alert time η is defined similarly as the cumulative time during which a critical state was observed. It represents the visiting time of the set of critical-related states $\bigcup_{k=1}^{m-1} Z^{(k)}$. So, the distribution of η is a homogeneous linear recurrence over $\mathbb{R}$ and its order is $M \leq |Z| \leq \varpi \cdot [\omega + r(m-1)]$.

5 Conclusions

This research has done a temporal analysis of the following stochastic models:

- Markov processes with final sequences of states;
- Hidden Markov models with final sequences of observable states;
- Games defined on Markov processes with final sequence of state.

The both situations were analyzed: the case when the states are independent (given by zero-order Markov processes) and the case when the states are interdependent (in the context of standard Markov processes).

As part of the temporal analysis that has been done, three important numerical random variables were considered: the evolution time, the total alert time and the game duration. For each of them and for each stochastic model mentioned above, it was proved that the distribution represents a homogeneous linear recurrence.

We saw that the length of the final sequences of states could be reduced to 1. This allowed us to reduce the order of the homogeneous linear recurrent distribution.

As result, the algorithms developed in [6] can be applied for finding the main probabilistic characteristics: the expectation, the variance, the k-order moments. The complexity remains to be polynomial, even for hidden Markov models.

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