BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Numbers 2(108)–3(109), 2025, Pages 50–61
ISSN 1024–7696, E-ISSN 2587–4322

The relationships between common multiples of matrices over commutative Bézout domain of stable range 1.5

Andrii Romaniv

Abstract. In the paper, we study common right multiples of matrices over commutative Bézout domains of stable range 1.5. In particular, for nonsingular matrices of arbitrary order, we indicate the relationships between their common right multiples under certain restrictions on invariant factors. We also establish connections between the invariant factors of the given matrices and the invariant factors of their common right multiples. Furthermore, in this paper we study the structure of the least common right multiple of matrices, specifically its Smith normal form and transforming matrices. As a result, we propose a new method for finding the least common right multiple and establish relationships between the Smith forms of the given matrices and the Smith form of their least common right multiple.

Mathematics subject classification: 15A21, 13A05.
Keywords and phrases: commutative Bézout domain, common multiple of matrices, Smith form, stable range 1.5.

1 Introduction

Let R be a commutative elementary divisor domain [2]. Denote by $M_n(R)$ and $\mathrm{GL}_n(R)$ the ring and the complete linear group of $n \times n$ matrices over a ring R, respectively. And let $A \in M_n(R)$ be a nonsingular matrix over R. For matrix A there are invertible matrices P_A and Q_A of appropriate sizes, such that

$$P_A A Q_A = diag(\varphi_1, \ldots, \varphi_n) =: \Phi, \ \varphi_i | \varphi_{i+1}, \ \ i = 1, \ldots, n-1.$$

The matrix $\Phi = diag(\varphi_1, \ldots, \varphi_n)$ is called the Smith normal form, the elements $\varphi_1, \ldots, \varphi_n$ are called invariant factors and the matrices P_A and Q_A are called left and right transforming matrices of A respectively. Denote by $\mathbf{P}_A$ the set of all left transforming matrices for matrix A. According to results [5] we know that $\mathbf{P}_A = \mathbf{G}_\Phi P_A$, where

$$\mathbf{G}_\Phi = \{K \in GL_n(R) \ \ | \ \ K_1 \in GL_n(R) \ \ : \ \ K\Phi = \Phi K_1\}.$$

The set $\mathbf{G}_\Phi$ is the multiplicative group [5] and is called Zelisko group. The structure of elements of this set is investigated in [5, 14] and [13].

DOI: https://doi.org/10.56415/basm.y2025.i2-3.p62

Consider the equality $A = BC$, where $B, C \in M_n(R)$ are nonsingular matrices over R and B has the Smith normal form

$$B \sim diag(\psi_1, \dots, \psi_n) =: \Psi,\ \ \psi_i | \psi_{i+1},\ \ i = 1, \dots, n-1.$$

Note that the notation " $\sim$ " means the equivalence of matrices. Let us recall that two matrices A and B, over a ring R are said to be equivalent if there exist invertible matrices U and V such that $B = UAV$.

We denote by (a, b) and $[a, b]$ the greatest common divisor and the least common multiple of the elements a and b, respectively. The notation $a|b$ means that the element a divides the element b.

Consider the set of matrices

$$\mathbf{L}(\Phi, \Psi) = \{L \in GL_n(R) \ \ | \ \ L\Phi = \Psi L_1\ ,\ \ L_1 \in M_n(R)\},$$

that was introduced in [5] and consists of all invertible matrices of the form

$$\mathbf{L}(\Phi, \Psi) = \left\| \begin{matrix} l_{11} & l_{12} & \dots & l_{1.n-1} & l_{1n} \\ \frac{\psi_2}{(\psi_2, \varphi_1)} l_{21} & l_{22} & \dots & l_{2.n-1} & l_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\psi_n}{(\psi_n, \varphi_1)} l_{n1} & \frac{\psi_n}{(\psi_n, \varphi_2)} l_{n2} & \dots & \frac{\psi_n}{(\psi_n, \varphi_{n-1})} l_{n.n-1} & l_{nn} \end{matrix} \right\|.$$

If $A = BC$ then B is a left divisor of the matrix A and A is a right multiple of B. If $A = DA_1$ and $B = DB_1$, then the matrix D is a common left divisor of the matrices A and B. If $M = AA_1 = BB_1$ then the matrix M is called a common right multiple of matrices A and B. Moreover if, in addition, the matrix M above is a left divisor of any other common right multiple of matrices A and B then we say that M is the least common right multiple of A and B ($[A, B]_r$ in notation).

Common multiples and common divisors of matrices are interconnected and are often used to solve applied problems in algebra. In particular, according to the generalized Bézout theorem, solving a system of one-sided matrix equations over fields can be reduced to finding common left unitary divisors of the first degree of the corresponding matrices. The study of common multiples, particularly the least common multiple, originates from [3], where a method for finding it over a commutative principal ideal domains was proposed. In [4], it was established that the least common right multiple is uniquely determined up to right associativity. In works [6, 10, 11] and [12], relationships between the Smith normal forms of two matrices and the Smith normal forms of their least common right multiple were indicated for certain classes of matrices. Moreover, in [10], a connection between the common right multiples of second-order matrices over a commutative principal ideal domains was established. Naturally, a need arises to study such problems for wider classes of matrices.

In this paper, for nonsingular matrices under certain restrictions on their Smith normal forms, we establish a connection between the divisibility of common right multiples of matrices over commutative Bézout domains of stable range 1.5. Additionally, we describe the relationship between the invariant factors of the given matrices and the invariant factors of their multiples. The explicit form of the Smith normal form of a least common right multiple for such matrices is also indicated.

Recall that a commutative Bézout domain is a commutative ring without zero divisors in which every finitely generated ideal is principal.

Definition 1. [8]. *A commutative ring R is a ring of stable range 1.5 if, for every triple of relatively prime elements a, b, and c in R, $c \neq 0$, there exists $r \in R$ such that the elements $a + br$ and c are relatively prime.*

Note that the very concept of a stable range of a ring was introduced in [1].

Definition 2. [1]. *The stable range of a ring R is the least natural number n such that for any nonzero relatively prime elements $a_1, \ldots, a_n, a_{n+1} \in R$, there exist elements $r_1, \ldots, r_n \in R$ such that the elements*

$$a_1 + a_{n+1}r_1, a_2 + a_{n+1}r_2, \ldots, a_n + a_{n+1}r_n$$

are relatively prime.

2 Auxiliary results

Let, in the following, R be a commutative Bézout domain of stable range 1.5. According to Theorem 1 of [9], R is an elementary divisor domain. Consider the nonsingular matrices $A, B \in M_n(R)$ which have the Smith normal form

$$A \sim diag(1, \varphi, \ldots, \varphi) =: \Phi,$$

$$B \sim diag(1, \psi, \ldots, \psi) =: \Psi,$$

respectively.

Lemma 1. *Let $A \sim \Phi$, $B \sim \Psi$ and $P_A \in \mathbf{P}_A$, $P_B \in \mathbf{P}_B$, $P_B P_A^{-1} = \|s_{ij}\|_1^n =: S$. Then the element $((\varphi, \psi), s_{21}, s_{31}, \ldots, s_{n1})$ is an invariant with respect to the choice of transforming matrices P_A and P_B.*

Proof. Let $N_A \in \mathbf{P}_A$ and $N_B \in \mathbf{P}_B$ be some other left transforming matrices of A and B. Then matrices $H_A \in \mathbf{G}_\Phi$ and $H_B \in \mathbf{G}_\Psi$ exist such that $N_A = H_A P_A$, $N_B = H_B P_B$. Consider the following product of the matrices:

$$N_B N_A^{-1} = H_B P_B (H_A P_A)^{-1} = H_B P_B P_A^{-1} H_A^{-1} = H_B S H_A^{-1},$$

where $S = P_B P_A^{-1}$. Let's denote $S H_A^{-1} = \|t_{ij}\|_1^n$. Since $H_A^{-1} \in \mathbf{G}_\Phi$ then according to Corollary 6 of [5] the matrix H_A^{-1} has the form

$$H_A^{-1} = \begin{Vmatrix} v_{11} & v_{12} & \ldots & v_{1n} \\ \varphi v_{21} & v_{22} & \ldots & v_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \varphi v_{n-1.1} & v_{n-1.2} & \ldots & v_{n-1.n} \\ \varphi v_{n1} & v_{n2} & \ldots & v_{nn} \end{Vmatrix}.$$

Hence,

$$t_{i1} = \left\| \begin{matrix} s_{i1} & s_{i2} & \dots & s_{in} \end{matrix} \right\| \left\| \begin{matrix} v_{11} \\ \varphi v_{21} \\ \vdots \\ \varphi v_{n1} \end{matrix} \right\| =$$

$$= s_{i1}v_{11} + s_{i2}\varphi v_{21} + \dots + s_{in}\varphi v_{n1},$$

where $i = 2, \dots, n$. Consider the following greatest common divisor:

$$((\varphi,\ \psi), t_{21}, t_{31}, \dots, t_{n1}) =$$

$$= ((\varphi,\ \psi),\ (s_{21}v_{11}+s_{22}\varphi v_{21}+\dots+s_{2n}\varphi v_{n1}), \dots, (s_{n1}v_{11}+s_{n2}\varphi v_{21}+\dots+s_{nn}\varphi v_{n1})) =$$

$$= ((\varphi,\ \psi),\ (s_{21}v_{11}+\varphi(s_{22}v_{21}+\dots+s_{2n}v_{n1})), \dots, (s_{n1}v_{11}+\varphi(s_{n2}v_{21}+\dots+s_{nn}v_{n1}))).$$

Since $(\varphi, \psi)|\varphi$, then

$$((\varphi,\ \psi), t_{21}, t_{31}, \dots, t_{n1}) = ((\varphi,\ \psi), s_{21}v_{11}, s_{31}v_{11}, \dots, s_{n1}v_{11}) = ((\varphi,\ \psi), v_{11}(s_{21}, s_{31}, \dots, s_{n1})).$$

The invertibility of H_A^{-1} implies that $(\varphi, v_{11}) = 1$. Therefore, $((\varphi, \psi), v_{11}) = 1$ and

$$((\varphi,\ \psi), t_{21}, t_{31}, \dots, t_{n1}) = ((\varphi,\ \psi), s_{21}, s_{31}, \dots, s_{n1}).$$

Let's denote $H_B S = \|k_{ij}\|_1^n$. In view of Corollary 6 of [5] H_B is of the form

$$H_B = \left\| \begin{matrix} u_{11} & u_{12} & \dots & u_{1n} \\ \psi u_{21} & u_{22} & \dots & u_{2n} \\ \dots & \dots & \dots & \dots \\ \psi u_{n-1.1} & u_{n-1.2} & \dots & u_{n-1.n} \\ \psi u_{n1} & u_{n2} & \dots & u_{nn} \end{matrix} \right\|.$$

Hence,

$$k_{i1} = \left\| \begin{matrix} \psi u_{i1} & u_{i2} & \dots & u_{in} \end{matrix} \right\| \left\| \begin{matrix} s_{11} \\ s_{21} \\ \vdots \\ s_{n1} \end{matrix} \right\| =$$

$$= \psi u_{i1}s_{11} + u_{i2}s_{21} + \dots + u_{in}s_{n1} = \psi l_i + u_{i2}s_{21} + \dots + u_{in}s_{n1},$$

where $l_i = u_{i1}s_{11}$, $i = 2, \dots, n$. Consider the following greatest common divisor:

$$((\varphi,\ \psi), k_{21}, k_{31}, \dots, k_{n1}) =$$

$$= ((\varphi,\ \psi),\ (\psi l_2 + u_{22}s_{21} + \dots + u_{2n}s_{n1}), \dots, (\psi l_n + u_{n2}s_{21} + \dots + u_{nn}s_{n1})) =: d.$$

Since $(\varphi, \psi)|\psi$, then

$$d = ((\varphi,\ \psi),\ (u_{22}s_{21} + \dots + u_{2n}s_{n1}), \dots, (u_{n2}s_{21} + \dots + u_{nn}s_{n1})).$$

It is obvious that the element $((\varphi,\ \psi), s_{21}, s_{31}, \dots, s_{n1})$ is a divisor of all the terms above. Thus, we have $((\varphi,\ \psi), s_{21}, s_{31}, \dots, s_{n1})|d$, i.e.

$$((\varphi,\ \psi), s_{21}, s_{31}, \dots, s_{n1})|((\varphi,\ \psi), k_{21}, k_{31}, \dots, k_{n1}).$$

On the other hand $S = H_B^{-1}\,\|k_{ij}\|_1^n$, where $H_B^{-1} \in \mathbf{G}_\Psi$. Based on similar considerations, we obtain that $((\varphi,\ \psi), k_{21}, k_{31}, \dots, k_{n1})|((\varphi,\ \psi), s_{21}, s_{31}, \dots, s_{n1})$, i.e.

$$((\varphi,\ \psi), k_{21}, k_{31}, \dots, k_{n1}) = ((\varphi,\ \psi), s_{21}, s_{31}, \dots, s_{n1}).$$

Appying the associativity of $M_n(R)$ completes the proof.

□

Lemma 2. *Let* $\Omega = diag(\omega_1, \omega_2, \dots, \omega_n)$, *where* $\omega_i \mid \omega_{i+1}$, $i = 1, 2, \dots, n-1$, *and* $\Phi \mid \Omega$, $\Psi \mid \Omega$. *And let* $S = \|s_{ij}\|_1^n \in \mathrm{GL}_n(R)$. *In order* $SL_A = L_B$, *where* $L_A \in \mathbf{L}(\Omega, \Phi)$, $L_B \in \mathbf{L}(\Omega, \Psi)$, *it is necessary and sufficient that*

$$\frac{(\varphi, \psi)}{(\varphi, \psi, \omega_1)} \mid (s_{21}, \dots, s_{n1}).$$

Proof. Necessity. Since $\Phi \mid \Omega$ and $\Psi \mid \Omega$ then according to Corollary 5 of [5] matrices L_A and L_B are of forms:

$$L_A = \left\| \begin{matrix} p_{11} & p_{12} & \cdots & p_{1n} \\ \frac{\varphi}{(\varphi,\omega_1)}p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \frac{\varphi}{(\varphi,\omega_1)}p_{n1} & p_{n2} & \cdots & p_{nn} \end{matrix} \right\|, \quad L_B = \left\| \begin{matrix} q_{11} & q_{12} & \cdots & q_{1n} \\ \frac{\psi}{(\psi,\omega_1)}q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \frac{\psi}{(\psi,\omega_1)}q_{n1} & q_{n2} & \cdots & q_{nn} \end{matrix} \right\|,$$

respectively. Using the Property 4.8 [7], in this case, the sets $\mathbf{L}(\Omega, \Phi)$ and $\mathbf{L}(\Omega, \Psi)$ are groups. Then $S = L_B L_A^{-1}$, where $L_A^{-1} \in \mathbf{L}(\Omega, \Phi)$, i.e.

$$S = \left\| \begin{matrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \dots & \dots & \dots & \dots \\ s_{n1} & s_{n2} & \dots & s_{nn} \end{matrix} \right\| =$$

$$= \left\| \begin{matrix} p'_{11} & p'_{12} & \cdots & p'_{1n} \\ (\frac{\varphi}{(\varphi,\omega_1)}, \frac{\psi}{(\psi,\omega_1)})p'_{21} & p'_{22} & \cdots & p'_{2n} \\ \dots & \dots & \dots & \dots \\ (\frac{\varphi}{(\varphi,\omega_1)}, \frac{\psi}{(\psi,\omega_1)})p'_{n1} & p'_{n2} & \cdots & p'_{nn} \end{matrix} \right\| = L_B L_A^{-1}.$$

It follows that

$$\left(\frac{\varphi}{(\varphi,\omega_1)}, \frac{\psi}{(\psi,\omega_1)}\right) \mid s_{21}, \dots, \left(\frac{\varphi}{(\varphi,\omega_1)}, \frac{\psi}{(\psi,\omega_1)}\right) \mid s_{n1},$$

i.e.

$$\left(\frac{\varphi}{(\varphi,\omega_1)},\frac{\psi}{(\psi,\omega_1)}\right)\mid(s_{21},\dots,s_{n1}).$$

It follows from Lemma 1 (equation 1) [10] that

$$\left(\frac{\varphi}{(\varphi,\omega_1)},\frac{\psi}{(\psi,\omega_1)}\right)=\frac{(\varphi,\psi)}{(\varphi,\psi,\omega_1)},$$

then we will get that $\frac{(\varphi,\psi)}{(\varphi,\psi,\omega_1)}\mid(s_{21},\dots,s_{n1})$.

Sufficiency. Denote by $\frac{\varphi}{(\varphi,\omega_1)}=:a$, $\frac{\psi}{(\psi,\omega_1)}=:b$, then $(a,b)\mid(s_{21},\dots,s_{n1})$. Let $s_{i1}=(a,b)t_i$, $i=2,\dots,n$. By Theorem 2.13 [7] there exist some matrices $H_1\in\mathbf{G}_\Psi$ and $U_1\in\mathbf{G}_\Phi$ such that

$$H_1SU_1=\left\|\begin{array}{c|cccc} 1 & 0 & \dots & 0 & 0\\ \hline (a,b)k_{21} & 1 & \dots & 0 & 0\\ \vdots & \dots & \ddots & \vdots & \vdots\\ (a,b)k_{n-1.1} & k_{n-1.2} & \dots & 1 & 0\\ (a,b)k_{n1} & k_{n2} & \dots & k_{n.n-1} & 1 \end{array}\right\|=:\left\|\begin{matrix} 1 & \mathbf{0}\\ K_{21} & K_{22}\end{matrix}\right\|.$$

Obviously, K_{22} is invertible. Hence, there exists some matrix $U_2=:\left\|\begin{matrix} 1 & \mathbf{0}\\ \mathbf{0} & K_{22}^{-1}\end{matrix}\right\|\in\mathbf{G}_\Phi$ such that

$$H_1SU_1U_2=\left\|\begin{matrix} 1 & 0 & \dots & 0 & 0\\ (a,b)k_{21} & 1 & \dots & 0 & 0\\ \vdots & \dots & \ddots & \vdots & \vdots\\ (a,b)k_{n-1.1} & 0 & \dots & 1 & 0\\ (a,b)k_{n1} & 0 & \dots & 0 & 1 \end{matrix}\right\|=:K.$$

Since $U_1,U_2\in\mathbf{G}_\Phi$ then $U_1U_2=:U_3\in\mathbf{G}_\Phi$. Therefore, $K=H_1SU_3$. Moreover, there exist $u_i,v_i\in R$, $i=2,\dots,n$, such that

$$(a,b)k_{i1}=(au_i+bv_i)k_{i1}=au_ik_{i1}+bv_ik_{i1},\quad i=2,\dots,n.$$

If we consider the matrices

$$\left\|\begin{matrix} 1 & 0 & \dots & 0 & 0\\ bv_2k_{21} & 1 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots\\ bv_{n-1}k_{n-1.1} & 0 & \dots & 1 & 0\\ bv_nk_{n1} & 0 & \dots & 0 & 1 \end{matrix}\right\|=:H_2\in\mathbf{L}(\Omega,\Psi)$$

and

$$\left\|\begin{matrix} 1 & 0 & \dots & 0 & 0\\ -au_1k_{21} & 1 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots\\ -au_{n-1}k_{n-1.1} & 0 & \dots & 1 & 0\\ -av_nk_{n1} & 0 & \dots & 0 & 1 \end{matrix}\right\|=:U_4\in\mathbf{L}(\Omega,\Phi).$$

we obtain that $H_1SU_3U_4 =: H_2$. Then $SU_3U_4 = H_1^{-1}H_2$. Using Properties 2 and 3 [5] we will have $H_1^{-1}H_2 =: L_B \in \mathbf{L}(\Omega, \Psi)$, $U_3U_4 =: L_A \in \mathbf{L}(\Omega, \Phi)$, and so $SL_A = L_B$, which had to be proved. □

In [5], necessary and sufficient conditions are given for a matrix to be a left divisor of a matrix over the commutative elementary divisor domain. For the case of nonsingular matrices over a commutative Bézout domain of stable range 1.5 these conditions can be written as follows.

Theorem 1. [5] *Let R be a commutative Bézout domain of stable range 1.5 and let*

$$P_AAQ_A = diag(\varphi_1, \varphi_2, \dots, \varphi_n) =: \Phi,\ \ \varphi_i|\varphi_{i+1},\ \ i = 1, \dots, n-1,$$

$$P_BBQ_B = diag(\psi_1, \psi_2, \dots, \psi_n) =: \Psi,\ \ \psi_i|\psi_{i+1},\ \ i = 1, \dots, n-1.$$

Then, $A = BC$ if and only if the following conditions hold:
1) $\psi_i|\varphi_i,\ \ i = 1, \dots, n$.
2) $P_B = LP_A$, where $P_B \in \mathbf{P}_B$, $P_A \in \mathbf{P}_A$, $L \in \mathbf{L}(\Phi, \Psi)$.

3 Main result

Theorem 2. *Let R be a commutative Bézout domain of stable range 1.5 and let $A, B \in M_n(R)$ be nonsingular matirces which have the Smith normal forms*

$$P_AAQ_A = diag(1, \varphi, \dots, \varphi) =: \Phi,$$

$$P_BBQ_B = diag(1, \psi, \dots, \psi) =: \Psi,$$

respectively. Let also $M, T \in M_n(R)$ be the common right multiples of matrices A and B, i.e. $M = AA_1 = BB_1$ and $T = AA_2 = BB_2$ and have the Smith normal forms

$$P_MMQ_M = diag(\omega_1, \omega_2, \dots, \omega_n) =: \Omega,\ \ \omega_i|\omega_{i+1},\ \ i = 1, \dots, n-1,$$

$$P_TTQ_T = diag(\gamma_1, \gamma_2, \dots, \gamma_n) =: \Gamma,\ \ \gamma_i|\gamma_{i+1},\ \ i = 1, \dots, n-1,$$

respectively. If the following conditions are performed
1) $\omega_1|\gamma_1$,
2) $[\varphi, \psi] =: \omega_i$ and $\omega_i|\gamma_i$, $i = 2, \dots, n$,
then, M is a left divisor of the matrix T, i.e. $T = MN$, where $N \in M_n(R)$.

Proof. The equality $M = AA_1 = BB_1$ means that the matrix M is a common right multiple of the matrices A and B. Taking into account Theorem 1, we get $P_A = L_M^AP_M$, where $P_A \in \mathbf{P}_A$, $P_M \in \mathbf{P}_M$,

$$L_M^A = \left\| \begin{array}{cccc} l_{11} & l_{12} & \dots & l_{1n} \\ \frac{\varphi}{(\varphi,\omega_1)}l_{21} & l_{22} & \dots & l_{2n} \\ \dots & \dots & \dots & \dots \\ \frac{\varphi}{(\varphi,\omega_1)}l_{n1} & l_{n2} & \dots & l_{nn} \end{array} \right\| \in \mathbf{L}(\Omega, \Phi).$$

So, $P_M = (L_M^A)^{-1} P_A$. Similarly, from equality $M = BB_1$ we get that $P_B = L_M^B P_M$, where $P_B \in \mathbf{P}_B$,

$$L_M^B = \left\| \begin{matrix} b_{11} & b_{12} & \dots & b_{1n} \\ \frac{\psi}{(\psi,\omega_1)} b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ \frac{\psi}{(\psi,\omega_1)} b_{n1} & b_{n2} & \dots & b_{nn} \end{matrix} \right\| \in \mathbf{L}(\Omega, \Psi),$$

i.e. $P_M = (L_M^B)^{-1} P_B$.

Since the matrix T is also a common right multiple of the matrices A and B, then by similar considerations we obtain that $P_A = L_T^A P_T$, i.e. $P_T = (L_T^A)^{-1} P_A$, where $P_T \in \mathbf{P}_T$,

$$L_T^A = \left\| \begin{matrix} t_{11} & t_{12} & \dots & t_{1n} \\ \frac{\varphi}{(\varphi,\gamma_1)} t_{21} & t_{22} & \dots & t_{2n} \\ \dots & \dots & \dots & \dots \\ \frac{\varphi}{(\varphi,\gamma_1)} t_{n1} & t_{n2} & \dots & t_{nn} \end{matrix} \right\| \in \mathbf{L}(\Gamma, \Phi).$$

Similarly, $P_B = L_T^B P_T$, where

$$L_T^B = \left\| \begin{matrix} f_{11} & f_{12} & \dots & f_{1n} \\ \frac{\psi}{(\psi,\gamma_1)} f_{21} & f_{22} & \dots & f_{2n} \\ \dots & \dots & \dots & \dots \\ \frac{\psi}{(\psi,\gamma_1)} f_{n1} & f_{n2} & \dots & f_{nn} \end{matrix} \right\| \in \mathbf{L}(\Gamma, \Psi).$$

So, $P_T = (L_T^B)^{-1} P_B$. According to Theorem 1, the matrix M will be the left divisor of the matrix T if $P_M = L_T P_T$, that is $L_T = P_M (P_T)^{-1} \in \mathbf{L}(\Gamma, \Omega)$. Consider the product of matrices

$$P_M (P_T)^{-1} = (L_M^A)^{-1} P_A ((L_T^A)^{-1} P_A)^{-1} = (L_M^A)^{-1} P_A (P_A)^{-1} L_T^A = (L_M^A)^{-1} L_T^A =$$

$$= \underbrace{\left\| \begin{matrix} l'_{11} & l'_{12} & \dots & l'_{1n} \\ \frac{\varphi}{(\varphi,\omega_1)} l'_{21} & l'_{22} & \dots & l'_{2n} \\ \dots & \dots & \dots & \dots \\ \frac{\varphi}{(\varphi,\omega_1)} l'_{n1} & l'_{n2} & \dots & l'_{nn} \end{matrix} \right\|}_{(L_M^A)^{-1}} \cdot \underbrace{\left\| \begin{matrix} t_{11} & t_{12} & \dots & t_{1n} \\ \frac{\varphi}{(\varphi,\gamma_1)} t_{21} & t_{22} & \dots & t_{2n} \\ \dots & \dots & \dots & \dots \\ \frac{\varphi}{(\varphi,\gamma_1)} t_{n1} & t_{n2} & \dots & t_{nn} \end{matrix} \right\|}_{L_T^A} =$$

$$= \left\| \begin{matrix} h_{11} & h_{12} & \dots & h_{1n} \\ \left(\frac{\varphi}{(\varphi,\omega_1)}, \frac{\varphi}{(\varphi,\gamma_1)}\right) h_{21} & h_{22} & \dots & h_{2n} \\ \dots & \dots & \dots & \dots \\ \left(\frac{\varphi}{(\varphi,\omega_1)}, \frac{\varphi}{(\varphi,\gamma_1)}\right) h_{n1} & h_{n2} & \dots & h_{nn} \end{matrix} \right\| =: K.$$

It follows from Lemma 1 (equation 2) [10] that

$$\left(\frac{\varphi}{(\varphi, \omega_1)}, \frac{\varphi}{(\varphi, \gamma_1)} \right) = \frac{\varphi}{(\varphi, [\omega_1, \gamma_1])} = \frac{\varphi}{(\varphi, \gamma_1)}.$$

Then

$$K=\left\|\begin{array}{cccc} h_{11} & h_{12} & \dots & h_{1n} \\ \frac{\varphi}{(\varphi,\gamma_1)}h_{21} & h_{22} & \dots & h_{2n} \\ \dots & \dots & \dots & \dots \\ \frac{\varphi}{(\varphi,\gamma_1)}h_{n1} & h_{n2} & \dots & h_{nn} \end{array}\right\| =: \left\|\begin{array}{cccc} k_{11} & k_{12} & \dots & k_{1n} \\ k_{21} & k_{22} & \dots & k_{2n} \\ \dots & \dots & \dots & \dots \\ k_{n1} & k_{n2} & \dots & k_{nn} \end{array}\right\|.$$

On the other hand

$$P_M(P_T)^{-1}=(L_M^B)^{-1}P_B((L_T^B)^{-1}P_B)^{-1}=(L_M^B)^{-1}P_B(P_B)^{-1}L_T^B=(L_M^B)^{-1}L_T^B=$$

$$=\underbrace{\left\|\begin{array}{cccc} b'_{11} & b'_{12} & \dots & b'_{1n} \\ \frac{\psi}{(\psi,\omega_1)}b'_{21} & b'_{22} & \dots & b'_{2n} \\ \dots & \dots & \dots & \dots \\ \frac{\psi}{(\psi,\omega_1)}b'_{n1} & b'_{n2} & \dots & b'_{nn} \end{array}\right\|}_{(L_M^B)^{-1}} \cdot \underbrace{\left\|\begin{array}{cccc} f_{11} & f_{12} & \dots & f_{1n} \\ \frac{\psi}{(\psi,\gamma_1)}f_{21} & f_{22} & \dots & f_{2n} \\ \dots & \dots & \dots & \dots \\ \frac{\psi}{(\psi,\gamma_1)}f_{n1} & f_{n2} & \dots & f_{nn} \end{array}\right\|}_{L_T^B} =$$

$$=\left\|\begin{array}{cccc} h_{11} & h_{12} & \dots & h_{1n} \\ \frac{\psi}{(\psi,\gamma_1)}h'_{21} & h_{22} & \dots & h_{2n} \\ \dots & \dots & \dots & \dots \\ \frac{\psi}{(\psi,\gamma_1)}h'_{n1} & h_{n2} & \dots & h_{nn} \end{array}\right\| =: \left\|\begin{array}{cccc} k_{11} & k_{12} & \dots & k_{1n} \\ k_{21} & k_{22} & \dots & k_{2n} \\ \dots & \dots & \dots & \dots \\ k_{n1} & k_{n2} & \dots & k_{nn} \end{array}\right\|.$$

Since $\frac{\varphi}{(\varphi,\gamma_1)}|k_{21}$ and $\frac{\psi}{(\psi,\gamma_1)}|k_{21}$, then

$$\left[\frac{\varphi}{(\varphi,\gamma_1)},\frac{\psi}{(\psi,\gamma_1)}\right]|k_{21}.$$

It follows from Lemma 1 (equation 4) [10] that

$$\left[\frac{\varphi}{(\varphi,\gamma_1)},\frac{\psi}{(\psi,\gamma_1)}\right]=\frac{[\varphi,\psi]}{([\varphi,\psi],\gamma_1)}.$$

Noting that according to the conditions of the theorem, $[\varphi,\psi]=\omega_i$, $i=2,\dots,n$, then

$$\frac{[\varphi,\psi]}{([\varphi,\psi],\gamma_1)}=\frac{\omega_i}{(\omega_i,\gamma_1)}|k_{i1},\quad i=2,\dots,n.$$

We get that $K\in\mathbf{L}(\Gamma,\Omega)$. And this means that $T=MN$, where $N\in M_n(R)$.

□

Theorem 3. *Let R be a commutative Bézout domain of stable range 1.5 and let*

$$A\sim diag(1,\varphi,\dots,\varphi),$$

$$B\sim diag(1,\psi,\dots,\psi),$$

$P_BP_A^{-1}=\|s_{ij}\|_1^n$, $P_B\in\mathbf{P}_B$, $P_A\in\mathbf{P}_A$. *Then*

$$[A,B]_r=(L_AP_A)^{-1}\Omega=(L_BP_B)^{-1}\Omega,$$

where

$$\Omega = diag\left(\frac{(\varphi,\psi)}{((\varphi,\psi), s_{21}, s_{31}, \ldots, s_{n1})}, [\varphi,\psi], \ldots, [\varphi,\psi], [\varphi,\psi]\right),$$

L_A , L_B belong to sets $\mathbf{L}(\Omega,\Phi)$, $\mathbf{L}(\Omega,\Psi)$ respectively and satisfy the equality:

$$(P_B P_A^{-1})L_A = L_B.$$

Proof. Remark that according to Lemma 1, the element $((\varphi,\psi), s_{21}, s_{31}, \ldots, s_{n1})$, and hence the matrix Ω, do not depend on the choice of transforming matrices P_A and P_B. We will show that the matrix $P_B P_A^{-1}$ can be written in the form

$$P_B P_A^{-1} T = G,$$

where $G \in \mathbf{L}(\Omega,\Psi)$, $T \in \mathbf{L}(\Omega,\Phi)$. By Lemma 1 (equation 1) [10] we will have

$$\left(\frac{\varphi}{(\varphi,\omega_1)}, \frac{\psi}{(\psi,\omega_1)}\right) = \frac{(\varphi,\psi)}{((\varphi,\psi),\omega_1)} =: \mu.$$

Since $\frac{(\varphi,\psi)}{((\varphi,\psi),(s_{21},\ldots,s_{n1}))} =: \omega_1$, then

$$\mu = \frac{(\varphi,\psi)}{\left((\varphi,\psi), \frac{(\varphi,\psi)}{((\varphi,\psi),(s_{21},\ldots,s_{n1}))}\right)} =$$

$$= \frac{(\varphi,\psi)((\varphi,\psi),(s_{21},\ldots,s_{n1}))}{((\varphi,\psi)((\varphi,\psi),(s_{21},\ldots,s_{n1})),(\varphi,\psi))} = ((\varphi,\psi),(s_{21},\ldots,s_{n1}))\,.$$

This means that $\mu \mid (s_{21},\ldots,s_{n1})$. According to Lemma 2 there exist matrices $T \in \mathbf{L}(\Omega,\Phi)$, $R \in \mathbf{L}(\Omega,\Psi)$ such that $P_B P_A^{-1} T = G$. So

$$G^{-1} P_B = T^{-1} P_A.$$

Denote by $G^{-1} =: L_B$ and $T^{-1} =: L_A$. Then

$$(L_A P_A)^{-1}\Omega = (L_B P_B)^{-1}\Omega = M.$$

Since $\Phi \mid \Omega$ and $\Psi \mid \Omega$, then using Theorem 1, the matrix M is the common right multiple of A and B.

Let $C = P_C \Lambda Q_C$ be the least common right multiple of matrices A and B. Hence, $\Phi \mid \Lambda$ and $\Psi \mid \Lambda$, i.e. $\varphi \mid \lambda_i$ and $\psi \mid \lambda_i$, $i = 2,\ldots,n$. This means that $[\varphi,\psi] \mid \lambda_i$, where $[\varphi,\psi] =: \omega_i$, $i = 2,\ldots,n$, i.e. $\omega_i \mid \lambda_i$. Since, $C = AA_3$, $C = BB_3$ then $P_A = K_A P_C$, where $K_A \in \mathbf{L}(\Lambda,\Phi)$ and $P_B = K_B P_C$, where $K_B \in \mathbf{L}(\Lambda,\Psi)$. In addition $P_C = K_A^{-1} P_A$ and $P_C = K_B^{-1} P_B$, hence $K_B K_A^{-1} = P_B P_A^{-1}$. The matrix $K_B K_A^{-1}$ has the form

$$K_B K_A^{-1} = \begin{Vmatrix} u_{11} & u_{12} & \ldots & u_{1n} \\ \frac{\psi}{(\psi,\lambda_1)} u_{21} & u_{22} & \ldots & u_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\psi}{(\psi,\lambda_1)} u_{n1} & u_{n2} & \ldots & u_{nn} \end{Vmatrix} \cdot \begin{Vmatrix} v_{11} & v_{12} & \ldots & v_{1n} \\ \frac{\varphi}{(\varphi,\lambda_1)} v_{21} & v_{22} & \ldots & v_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\varphi}{(\varphi,\lambda_1)} v_{n1} & v_{n2} & \ldots & v_{nn} \end{Vmatrix} =$$

$$=\left\|\begin{matrix} z_{11} & z_{12} & \dots & z_{1n} \\ \frac{(\varphi,\psi)}{((\varphi,\psi),\lambda_1)}z_{21} & z_{22} & \dots & z_{2n} \\ \dots & \dots & \dots & \dots \\ \frac{(\varphi,\psi)}{((\varphi,\psi),\lambda_1)}z_{n1} & z_{n2} & \dots & z_{nn} \end{matrix}\right\| = P_B P_A^{-1} =: \left\|\begin{matrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \dots & \dots & \dots & \dots \\ s_{n1} & s_{n2} & \dots & s_{nn} \end{matrix}\right\|.$$

From the above, it follows that

$$\frac{(\varphi,\psi)}{((\varphi,\psi),\lambda_1)} \mid s_{21},\dots,\frac{(\varphi,\psi)}{((\varphi,\psi),\lambda_1)} \mid s_{n1}$$

i.e., $\frac{(\varphi,\psi)}{((\varphi,\psi),\lambda_1)} \mid (s_{21},\dots,s_{n1})$. Since $\frac{(\varphi,\psi)}{((\varphi,\psi),\lambda_1)} \mid (\varphi,\psi)$ then

$$\frac{(\varphi,\psi)}{((\varphi,\psi),\lambda_1)} \mid ((\varphi,\psi),(s_{21},\dots,s_{n1})).$$

There is the element $m \in R$ such that $\frac{(\varphi,\psi)}{((\varphi,\psi),\lambda_1)} = \frac{((\varphi,\psi),(s_{21},\dots,s_{n1}))}{m}$. It follows that

$$\frac{((\varphi,\psi),\lambda_1)}{m} = \frac{(\varphi,\psi)}{((\varphi,\psi),(s_{21},\dots,s_{n1}))}.$$

Since $\frac{(\varphi,\psi)}{((\varphi,\psi),(s_{21},\dots,s_{n1}))} =: \omega_1$, then $\omega_1 m = ((\varphi,\psi),\lambda_1)$. This means that $\omega_1 \mid \lambda_1$. Since $\omega_i \mid \lambda_i$, $i = 2,\dots,n$ then according to Theorem 2, the matrix $C = MC_1$, i.e. M is the least common right multiple of matrices A and B.

□

4 Conclusion

In this paper, we investigated the relationships between common right multiples of matrices and the structure of the least common right multiple of matrices over commutative Bézout domains of stable range 1.5. Also, we described the relationship between the invariant factors of matrices and the invariant factors of their multiples. Under certain restrictions on the Smith normal forms of matrices, a new method for finding the least common right multiple of matrices is proposed, based on determining its Smith normal form and transforming matrices.

References

[1] Bass H. *K-theory and stable algebra.* Publ. Math. I.H.E.S., 1964, **22** , 5–60.

[2] Kaplansky I. *Elementary divisor and modules.* Trans. Amer. Math. Soc., 1949, **66**, 2, 464–491, DOI: https://doi.org/10.2307/1990591.

[3] MacDuffee C. *Matrices with elements in a principal ring.* Bull. Amer. Math. Soc., 1933, **39**, 8, 570–573.

[4] Stewart B. *A note on least common left multiples.* Bull. Amer. Math. Soc., 1949, **55**, 6, 587–591.

[5] Shchedryk V. *Factorization of matrices over elementary divisor domain.* Algebra Discrete Math., 2009, **8**, 2, 79–98.

[6] Romaniv A. *On the structure of least common multiple matrices from some class of matrices.* Carpathian Math. Publ., 2018, **10**, 179–184, DOI: https://doi.org/10.15330/cmp.10.1.179-184.

[7] Shchedryk V. *Factorization of matrices over elementary divisor rings.* Pidstryhach Inst. Appl. Probl. Mech. and Math. NAS of Ukraine, Lviv, 2017.

[8] Shchedryk V. *Bezout Rings of Stable Range 1.5.* Ukr. Math. J., 2015, **67**, 960–974, DOI: https://doi.org/10.1007/s11253-015-1126-9

[9] Shchedryk V. *Bezout rings of stable range 1.5 and the decomposition of a complete linear group into the product of its subgroups.* Ukr. Math. J., 2017, **69**, 2, 138–147, DOI: https://doi.org/10.1007/s11253-017-1352-4

[10] Romaniv A., Shchedryk V. *The greatest common left divisor and the least common right multiple of second order matrices.* Math. Bull. Shevchenko Sci. Soc., 2012, **9**, 269–284.

[11] Romaniv A. *On the Smith normal form of the least common multiple of matrices from some class of matrices.* Journal of Mathematical Sciences, 2023, **277**, 1, 53–59, DOI: https://doi.org/10.1007/s10958-023-06813-3.

[12] Thompson R. *Left multiples and right divisors of integral matrices.* Linear Multilinear Algebra, 1986, **19**, 3, 287–295, DOI: 10.1080/03081088608817722.

[13] Bovdi V., Shchedryk V. *Generating solutions of a linear equation and structure of elements of the Zelisko group II.* Quaestiones Mathematicae, 2023, **46**, 9, 1789–1798, DOI: https://doi.org/10.2989/16073606.2022.2112629

[14] Bovdi V., Shchedryk V. *Generating solutions of a linear equation and structure of elements of the Zelisko group.* Linear Multilinear Algebra, 2021, **625**, 55–67, Doi: https://doi.org/10.1016/j.laa.2021.04.019

Andrii Romaniv
Pidstryhach Institute for Applied Problems of Mechanics
and Mathematics, National Academy of Sciences of
Ukraine, Lviv, Ukraine
E-mail: *romaniv_a@ukr.net*

Received October 6, 2025