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Global Attractors of Non-autonomous Lattice Dynamical Systems

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Abstract. The aim of this paper is studying the compact global attractors for non-autonomous lattice dynamical systems of the form $u_i' = \nu(u_{i-1} - 2u_i + u_{i+1}) - \lambda u_i + f(u_i) + f_i(t)$ $(i \in \mathbb{Z},\ \lambda > 0)$. We prove their dissipativness, asymptotic compactness and then the existence of compact global attractors.

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1 Introduction

Denote by $\mathbb{R} := (-\infty, \infty)$, $\mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$ and ℓ_2 the Hilbert space of all two-sided sequences $\xi = (\xi_i)_{i\in\mathbb{Z}}$ $(\xi_i \in \mathbb{R})$ with

$$\sum_{i\in\mathbb{Z}} |\xi_i|^2 < +\infty$$

and equipped with the scalar product

$$\langle \xi, \eta \rangle := \sum_{i\in\mathbb{Z}} \xi_i \eta_i.$$

Let $(\mathfrak{B}, |\cdot|)$ be a Banach space with the norm $|\cdot|$, $C(\mathbb{R}, \mathfrak{B})$ be the space of all continuous functions $f : \mathbb{R} \to \mathfrak{B}$ equipped with the distance

$$d(f_1, f_2) := \sup_{L>0} \min\{\max_{|t|\le L} |f_1(t) - f_2(t)|, L^{-1}\}. \tag{1}$$

The metric space $(C(\mathbb{R}, \mathfrak{B}), d)$ is complete and the distance d, defined by (1), generates on the space $C(\mathbb{R}, \mathfrak{B})$ the compact-open topology.

Let $h \in \mathbb{R}$, $f \in C(\mathbb{R}, \mathfrak{B})$, $f^h(t) := f(t + h)$ for any $t \in \mathbb{R}$ and $\sigma : \mathbb{R} \times C(\mathbb{R}, \mathfrak{B}) \to C(\mathbb{R}, \mathfrak{B})$ be a mapping defined by $\sigma(h, f) := f^h$ for any $(h, f) \in \mathbb{R} \times C(\mathbb{R}, \mathfrak{B})$. Then [4, Ch.I] the triplet $(C(\mathbb{R}, \mathfrak{B}), \mathbb{R}, \sigma)$ is a shift dynamical system (or Bebutov's dynamical system) on the space $C(\mathbb{R}, \mathfrak{B})$. By $H(f)$ the closure in the space $C(\mathbb{R}, \mathfrak{B})$ of $\{f^h |\ h \in \mathbb{R}\}$ is denoted.

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In this paper we study the compact global attractors of the systems

$$u_i' = \nu(u_{i-1} - 2u_i + u_{i+1}) - \lambda u_i + F(u_i) + f_i(t) \ (i \in \mathbb{Z}), \tag{2}$$

where $\lambda > 0$, $F \in C(\mathbb{R}, \mathbb{R})$ and $f \in C(\mathbb{R}, \ell_2)$ ($f(t) := (f_i(t))_{i\in\mathbb{Z}}$ for any $t \in \mathbb{R}$).

The system (2) can be considered as a discrete (see, for example,[1, 7] and the bibliography therein) analogue of a reaction-diffusion equation in $\mathbb{R}$:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial^2 x} - \lambda u + F(u) + f(t, x),$$

where grid points are spaced h distance apart and $\nu = D/h^2$.

This study continues the first author's works devoted to the study of compact global attractors of non-autonomous dynamical systems [4] and compact attractors of lattice dynamical systems [1] (autonomous systems) and compact pullback attractors [7] (for non-autonomous systems).

The paper is organized as follows. In the second section we show that under some conditions the equation (2) generates a cocycle which plays a very important role in the study of the asymptotic properties of the equation (2). In the third section we prove under some conditions the existence of an absorbing set for the equation (2). The fourth section is dedicated to the study of the asymptotical compactness of the cocycle generated by the equation (2). In the fifth section we study the problem of existence of a compact global attractor for the equation (2).

2 Cocycles

Consider a non-autonomous system

$$u_i' = \nu(u_{i-1} - 2u_i + u_{i+1}) - \lambda u_i + F(u_i) + f_i(t) \ (i \in \mathbb{Z}). \tag{3}$$

Below we use the following conditions.

Condition (**C1**). The function $f \in C(\mathbb{R}, \mathfrak{B})$ and it is translation-compact, i.e., the set $\{f^h |\ h \in \mathbb{R}\}$ is pre-compact in the space $C(\mathbb{R}, \mathfrak{B})$.

Lemma 1. *[2, Ch.IV, p.236],[9, Ch.III],[10, Ch.IV] The following statements are equivalent:*

1. *the function $f \in C(\mathbb{R}, \mathfrak{B})$ is translation-compact;*
2. *the set $Q := \overline{f(\mathbb{R})}$ is compact in $\mathfrak{B}$ and the function $f \in C(\mathbb{R}, \mathfrak{B})$ is uniformly continuous.*

Condition (**C2**). The function $F \in C(\mathbb{R}, \mathbb{R})$ is Lipschitz continuous on bounded sets and $F(0) = 0$.

Denote by $\widetilde{F} : \ell_2 \to \ell_2$ the Nemytskii operator generated by F, i.e., $\widetilde{F}(\xi)_i := F(\xi_i)$ for any $i \in \mathfrak{N}$.

Condition (**C3**). $sF(s) \leq -\alpha s^2$ for any $s \in \mathbb{R}$.

Definition 1. A function $F \in C(Y \times \mathfrak{B}, \mathfrak{B})$ is said to be globally Lipschitzian (respectively locally Lipschitzian) with respect to variable $u \in \mathfrak{B}$ uniformly with respect to $y \in Y$ if there exists a positive constant L (for any bounded set $B \subset \mathfrak{B}$ there exists a constant L_B) such that

$$|F(y, u_1) - F(y, u_2)| \leq L|u_1 - u_2|$$

(respectively,

$$|F(y, v_1) - F(y, v_2)| \leq L_B|v_1 - v_2|)$$

for any $u_1, u_2 \in \mathfrak{B}$ and $y \in Y$ (respectively $v_1, v_2 \in B \subset \mathfrak{B}$ and $y \in Y$).

Definition 2. The smallest constant L (respectively L_B) with the property (4) is called Lipshchitz constant of function F (notation $Lip(F)$, respectively $Lip_B(F)$).

Let $B \subset \mathfrak{B}$, denoted by $CL(Y \times B, \mathfrak{B})$ is the Banach space of any Lipschitzian functions $F \in C(Y \times B, \mathfrak{B})$ equipped with the norm

$$||F||_{CL} := \max_{y \in Y} |F(y, 0)| + Lip_B(F).$$

Lemma 2. *[1] Under the Condition (**C2**) is well defined the mapping $\widetilde{F} : \ell_2 \to \ell_2$ and*

$$\|\widetilde{F}(\xi) - \widetilde{F}(\eta)\| \leq Lip_B(F)\|\xi - \eta\|$$

for any $\xi, \eta \in \ell_2$, where $\| \cdot \|^2 := \langle \cdot, \cdot \rangle$ and $\| \cdot \|$ is the norm on the space ℓ_2.

For any $u = (u_i)_{i \in \mathbb{Z}}$, the discrete Laplace operator Λ is defined [7, Ch.III] from ℓ_2 to ℓ_2 component–wise by $\Lambda(u)_i = u_{i-1} - 2u_i + u_{i+1}$ $(i \in \mathbb{Z})$. Define the bounded linear operators D^+ and D^- from ℓ_2 to ℓ_2 by $(D^+u)_i = u_{i+1} - u_i$, $(D^-u)_i = u_{i-1} - u_i$ $(i \in \mathbb{Z})$.

Note that $\Lambda = D^+D^- = D^-D^+$ and $\langle D^-u, v \rangle = \langle u, D^+v \rangle$ for any $u, v \in \ell_2$ and, consequently, $\langle \Lambda u, u \rangle = -|D^+u|^2 \leq 0$. Since Λ is a bounded linear operator acting on the space ℓ_2, it generates a uniformly continuous semi-group on ℓ_2.

Under the Conditions (**C1**) and (**C2**) the system of differential equations (3) can be written in the form of an ordinary differential equation

$$u' = \nu \Lambda u + \Phi(u) + f(t) \tag{4}$$

in the Banach space $\mathfrak{B} = \ell_2$, where $\Phi(u) := -\lambda u + \widetilde{F}(u)$ and $\Lambda(u)_i := u_{i-1} - 2u_i + u_{i+1}$ for any $u = (u_i)_{i \in \mathbb{Z}} \in \ell_2$. Along with equation (4) we consider also its H-class, i.e., the family of equations

$$u' = \nu \Lambda u + \Phi(u) + g(t), \tag{5}$$

where $g \in H(f)$.

The family of equations (5) can be rewritten as follows

$$u' = F(\sigma(t, g), u) \quad (g \in H(f)), \tag{6}$$

where $F : H(f) \times \ell_2 \to \ell_2$ is defined by $F(g,u) := \nu\Lambda u + \Phi(u) + g(0)$. It is easy to see that $F(\sigma(t,g),u) = \nu\Lambda u + \Phi(u) + g(t)$ for any $(t,u,g) \in \mathbb{R} \times \mathfrak{B} \times H(f)$.

Let Y be a complete metric space, $(Y,\mathbb{R},\sigma)$ be a dynamical system on Y and Λ be some complete metric space of linear closed operators acting into Banach space $\mathfrak{B}$. Consider the following linear differential equation

$$x' = A(\sigma(t,y))x, \quad (y \in Y) \tag{7}$$

where $A \in C(Y,\Lambda)$. We assume that the following conditions are fulfilled for equation (7):

a. for any $u \in \mathfrak{B}$ and $y \in Y$ equation (7) has exactly one solution that is defined on $\mathbb{R}_+$ and satisfies the condition $\varphi(0,u,y) = u$;

b. the mapping $\varphi : (t,u,y) \to \varphi(t,u,y)$ is continuous in the topology of $\mathbb{R}_+ \times \mathfrak{B} \times Y$.

Denote by $U(t,y) := \varphi(t,\cdot,y)$ for any $(t,y) \in \mathbb{R}_+ \times Y$.

Consider an evolutionary differential equation

$$u' = A(\sigma(t,y))u + F(\sigma(t,y),u) \quad (y \in Y) \tag{8}$$

in the Banach space $\mathfrak{B}$, where F is a nonlinear continuous mapping ("small" perturbation) acting from $Y \times \mathfrak{B}$ into $\mathfrak{B}$.

Definition 3. A function $u : [0,a) \mapsto \mathfrak{B}$ is said to be a weak (mild) solution of equation (8) passing through the point $x \in \mathfrak{B}$ at the initial moment $t = 0$ (notation $\varphi(t,x,y)$) if $u \in C([0,T],\mathfrak{B})$ and satisfies the integral equation

$$u(t) = U(t,y)x + \int_0^t U(t-s,\sigma(s,y))F(\sigma(s,y),u(s))ds$$

for any $t \in [0,T]$ and $0 < T < a$.

Theorem 1. *[5, Ch.VI] Suppose that the function $F \in C(Y \times \mathfrak{B},\mathfrak{B})$ is locally Lipschitzian. Let $x_0 \in \mathfrak{B}$, $r > 0$ and the conditions listed above be fulfilled. Then, there exist positive numbers $\delta = \delta(x_0,r)$ and $T = T(x_0,r)$ such that equation (8) admits a unique solution $\varphi(t,x,y)$ ($x \in B[x_0,\delta] := \{x \in \mathfrak{B} \mid |x - x_0| \le \delta\}$) defined on the interval $[0,T]$ with the conditions: $\varphi(0,x,y) = x$, $|\varphi(t,x,y) - x_0| \le r$ for any $t \in [0,T]$ and the mapping $\varphi : [0,T] \times B[x_0,\delta] \times Y \to \mathfrak{B}$ ($(t,x,y) \mapsto \varphi(t,x,y)$) is continuous.*

Remark 1. Under the conditions of Theorem 1:

1. if ψ is a solution of equation (8) on some interval $[0,h]$, then ψ can be extended over a maximal interval of existence $[0,\alpha)$;

2. if the solution ψ is bounded, then ψ can be extended on the interval $[0,+\infty)$.

This statement can be proved using the same arguments as in the case of ordinary differential equations (see, for example, [3, Ch.IV]).

Theorem 2. *Under the Conditions (**C1**) and (**C2**) there exist positive numbers* $\delta = \delta(u_0, r)$ *and* $T = T(u_0, r)$ *such that equation (8) admits a unique solution* $\varphi(t, g, y)$ $(u \in B[u_0, \delta] = \{u \in \ell_2\ \|u - u_0\| \le \delta\})$ *defined on the interval* $[0, T]$ *with the conditions:* $\varphi(0, u, g) = u$, $\|\varphi(t, u, g) - u_0\| \le r$ *for any* $t \in [0, T]$ *and the mapping* $\varphi : [0, T] \times B[u_0, \delta] \times H(f) \to \ell_2$ $((t, u, g) \mapsto \varphi(t, u, g))$ *is continuous.*

Proof. Assume that the Conditions (**C1**) and (**C2**) are fulfilled. Consider the equation (6), where $F(g, u) := \nu\Lambda u + \Phi(u) + g(0)$ for any $(u, g) \in \ell_2 \times H(f)$. It easy to check that under the conditions of Theorem the mapping F possesses the following properties:

1. F is continuous;
2. the mapping F is locally Lipschitzian in $u \in \ell_2$ uniformly with respect to $g \in H(f)$, i.e., for any bounded subset $B \subset \ell_2$ there exists a positive constant $L_F(B)$ such that
$$\|F(u_1, g) - F(u_2, g)\| \le L_F(B)\|u_1 - u_2\|$$
for any $u_1, u_2 \in B$ and $g \in H(f)$;
3. there exists a positive constant C such that
$$\|F(g, 0)\| \le C$$
for any $g \in H(f)$.

Now to finish the proof of Theorem it suffices to apply Theorem 1. □

Lemma 3. *Assume that the conditions (**C1**)–(**C3**) holds and* $g \in H(f)$*. Then, for every* $T > 0$*, any solution* $v(t)$ *of the problem (5) and* $v(0) = v_0 \in \ell_2$ *satisfies*
$$\|v(t)\| \le M\ , \quad \textit{for all } 0 \le t \le T\ ,$$
where M *is a constant depending only on the data* $(\lambda, C, \|v_0\|)$ *and* T*, where* $C := \sup\{\|f(t)\| \mid t \in \mathbb{R}\}$.

Proof. Let $v(t)$ be a solution of the equation (5) with the initial condition $v(0) = v_0$ defined on the maximal interval $[0, h)$. Denote by $y(t) := |v(t)|^2$ then we have
$$y'(t) = 2(v'(t), v(t)) = 2(\nu\Lambda v(t), v(t)) + 2(\Phi(v(t)), v(t)) + 2(g(t), v(t)) = \\ -2\nu|D^+v(t)|^2 - 2\lambda|v(t)|^2 + 2(\tilde{F}(v(t)), v(t)) + 2(g(t), v(t)) \tag{9}$$
for any $t \in [0, h)$.

Since
$$|(g(t), v(t))| \le \|g(t)\|\|v(t)\| \le \frac{1}{2}\lambda\|v(t)\|^2 + \frac{1}{2\lambda}\|g(t)\|^2\ ,$$

using (**C3**) from (9) we get

$$y'(t) = 2(v'(t), v(t)) \le -(\lambda + 2\alpha)y(t) + \frac{C^2}{\lambda} \tag{10}$$

for any $t \in [0, h)$. By Gronwall's lemma from the inequality (10), taking into account that $y(0) = |v(0)|^2$, we obtain

$$y(t) \le e^{-(\lambda+2\alpha)t}\big((|v(0)|^2 - \frac{C^2}{\lambda(\lambda+2\alpha)}) + \frac{C^2}{\lambda(\lambda+2\alpha)}\big)$$

and, consequently,

$$|v(t)| \le M$$

for any $0 \le t \le T < h$, where

$$M = M(T, |v_0|, C) := \Big(e^{-(\lambda+2\alpha)T}\big((|v_0|^2 - \frac{C^2}{\lambda(\lambda+2\alpha)}) + \frac{C^2}{\lambda(\lambda+2\alpha)}\big)\Big)^{1/2}.$$

Lemma is proved. □

Remark 2. Lemma 3 remains true if we replace the Condition (**C3**) by the weaker condition: $F(s)s \le 0$ for any $s \in \mathbb{R}$.

Theorem 3. *Under the Conditions (**C1**)-(**C3**) the following statements hold:*

1. *for any $(v, g) \in \ell_2 \times H(f)$ there exists a unique solution $\varphi(t, v, g)$ of the equation (5) passing through the point v at the initial moment $t = 0$ and defined on the semi-axis $\mathbb{R}_+ := [0, +\infty)$;*
2. *$\varphi(0, v, g) = v$ for any $(v, g) \in \ell_2 \times H(f)$;*
3. *$\varphi(t + \tau, v, g) = \varphi(t, \varphi(\tau, v, g), g^\tau)$ for any $t, \tau \in \mathbb{R}_+$, $v \in \ell_2$ and $g \in H(f)$;*
4. *the mapping $\varphi : \mathbb{R}_+ \times \ell_2 \times H(f) \to \ell_2$ $((t, v, g) \to \varphi(t, v, g))$ for any $(t, v, g) \in \mathbb{R}_+ \times \ell_2 \times H(f)$ is continuous.*

Proof. The first statement of Theorem follows from Lemma 3, Theorem 1 and Remark 1.

The second and third statements are evident. The fourth statement follows from Theorem 1. □

Let Y be a complete metric space and $(Y, \mathbb{R}, \sigma)$ be a dynamical system on Y.

Definition 4. Recall [4, Ch.I] that $\langle \mathbb{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ is said to be a cocycle over $(Y, \mathbb{R}, \sigma)$ with the fiber $\mathfrak{B}$ if φ is a continuous mapping acting from $\mathbb{R}_+ \times \mathfrak{B} \times Y \to \mathfrak{B}$ satisfying the following conditions:

1. $\varphi(0, u, y) = v$ for any $(v, y) \in \mathfrak{B} \times Y$;
2. $\varphi(t + \tau, u, y) = \varphi(t, \varphi(\tau, u, t), \sigma(\tau, y))$ for any $(t, \tau \in \mathbb{R}_+$ and $(u, y) \in \mathfrak{B} \times Y$.

Corollary 1. *Under the conditions of Theorem 2 the equation (4) (respectively, the family of equations (5)) generates a cocycle $\langle \ell_2, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$ over the shift dynamical system $(H(f), \mathbb{R}, \sigma)$ with the fiber ℓ_2.*

Proof. This statement directly follows from Theorem 2 and Definition 4. □

3 Existence of an absorbing set

Theorem 4. *Under the Conditions (**C1**)-(**C3**) there exists a closed ball* $B[0,r] := \{\xi \in \ell_2 | \ |\xi| \le r\}$ *such that for any bounded subset* $B \subset \ell_2$ *there exists a positive number* $L = L(B)$ *such that* $\varphi(t, B, Y) \subseteq B[0,r]$ *for any* $t \ge L(B)$*, where* $\varphi(t, M, Y) := \{\varphi(t, u, y) | \ u \in M, \ y \in Y\}$.

Proof. Let $B = B[0,r] := \{x \in \ell_2 | \ ||x|| \le r\}$ be a bounded subset, $v \in B$ and $g \in H(f)$ then we have

$$\begin{aligned}\tfrac{d}{dt}\|\varphi(t,v,g)\|^2 = 2\nu\left\langle \Lambda\varphi(t,v,g), \varphi(t,v,g)\right\rangle + \\ 2\left\langle \Phi(\varphi(t,v,g)), \varphi(t,v,g)\right\rangle + 2\left\langle g(t), \varphi(t,v,g)\right\rangle \\ \le -2\lambda||\varphi(t,v,g)||^2 + 2\textstyle\sum_{i\in\mathbb{Z}}\varphi_i(t,v,g)f(\varphi_i(t,v,g)) + 2\sum_{i\in\mathbb{Z}} g_i(t)\varphi_i(t,v,g) \\ \le -2\lambda||\varphi(t,v,g)||^2 - 2\alpha\|\varphi(t,v,g)\|^2 + \lambda||\varphi(t,v,g)||^2 + \tfrac{\|g(t)\|^2}{\lambda} \\ \le -(2\alpha+\lambda)\|\varphi(t,v,g)\|^2 + \tfrac{C^2}{\lambda}\end{aligned}$$

where the penultimate step follows from Young's inequality because $||g(t)|| \le C := \sup\{||f(t)|| : \ t \in \mathbb{R}\}$ for any $g \in H(f)$). Hence, Gronwall's lemma implies that

$$\|\varphi(t,v,g)\|^2 \le \|v\|^2 e^{-(2\alpha+\lambda)t} + \frac{C^2}{\lambda(\lambda+2\alpha)}\left(1 - e^{-(2\alpha+\lambda)t}\right), \quad t \ge 0.$$

Define the closed ball Q in ℓ^2 by

$$Q := \left\{u \in \ell^2 : \|u\|^2 \le R^2 := 1 + \frac{C^2}{\lambda(\lambda+2\alpha)}\right\}$$

then we have

$$||\varphi(t,v,g)||^2 \le R^2$$

for any $r > R$ and

$$t \ge L(B) := \frac{1}{\lambda(\lambda+2\alpha)}\ln(r^2 - R^2 + 1).$$

Theorem is proved. □

4 Asymptotical compactness of the cocycle generated by the equation (4)

Let $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ (or shortly φ) be a cocycle over dynamical system $(Y, \mathbb{R}, \sigma)$ with the compact phase space Y.

Let A and B be two bounded subsets from $\mathfrak{B}$. Denote by $\rho(a,b) := |a-b|$ $(a, b \in \mathfrak{B})$, $\rho(a, B) := \inf\limits_{b\in B}\rho(a,b)$ and

$$\beta(A,B) := \sup_{a\in A}\rho(a,B).$$

Definition 5. A dynamical system $(X, \mathbb{R}_+, \pi)$ is called asymptotically compact [4, Ch.I] if for any bounded, closed and positively invariant subset $B \subset X$ there exists a nonempty compact subset $K = K(B)$ such that

$$\lim_{t \to +\infty} \beta(\pi(t, B), K) = 0.$$

Definition 6. A cocycle $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ is said to be asymptotically compact if the skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ generated by cocycle φ ($X := W \times Y$, $\pi := (\varphi, \sigma)$) is asymptotically compact.

Definition 7. A cocycle $\langle \ell_2, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ satisfy an asymptotic tails property on the bounded set $K \subset \ell_2$, if for arbitrary positive number ε there exist positive numbers $L(\varepsilon)$ and $k_\varepsilon \in \mathbb{N}$ such that

$$\sum_{|k| \geq k_\varepsilon} |\varphi_k(t, v, y)|^2 < \varepsilon \tag{11}$$

for any $t \geq L(\varepsilon)$ and $(v, y) \in K \times Y$.

Lemma 4. *Assume that the skew product dynamical system $\langle \ell_2, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ satisfies the following conditions:*

1. *the metric space Y is compact;*

2. *the cocycle φ admits a bounded absorbing set $K \subset \ell_2$, i.e., for any bounded subset $B \subset \ell_2$ there exists a positive number $L = L(B)$ such that $\varphi(t, B, Y) \subset K$ for any $t \geq L$ (or equivalently: $\mathcal{K} := K \times Y$ is an absorbing set for the skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ generated by cocycle φ);*

3. *the cocycle φ satisfies an asymptotic tails property on the absorbing set $\mathcal{K} \subset X$.*

Then the skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ generated by the cocycle φ is asymptotically compact.

Proof. Let B be a bounded, closed and positively invariant subset of X. Since the space Y is compact then there exists a positive number $r > 0$ such that $B \subseteq B[0, r] \times Y$. Consider the sequences $\{x_n\} \subset B$ and $t_n \to +\infty$ as $n \to \infty$. We will show that the sequence $\{\pi(t_n, x_n)\}$ is precompact in the space $X := \ell_2 \times Y$ or equivalently the sequence $\{\varphi(t_n, u_n, y_n)\}$ is precompact in ℓ_2 ($x_n = (u_n, y_n)$ and $\pi(t_n, x_n) = (\varphi(t_n, u_n, y_n), \sigma(t_n, y_n))$ for any $n \in \mathbb{N}$) because $\{y_n\} \subset Y$ and the space Y is compact.

Since the set B is positively invariant then $\pi(t, B) \subseteq B \subseteq B[0, r] \times Y$ for any $t \geq 0$ and, consequently,

$$\varphi(t, u, y) \in B[0, r]$$

for any $(u, y) \in B$ and $t \geq 0$. In particular, we have

$$|\varphi(t_n, u_n, y_n)| \leq r$$

for any $n \in \mathbb{N}$). Thus the sequence $\{\varphi(t_n, u_n, y_n)\}$ is weakly precompact in ℓ_2 and, consequently, we can assume that $\{\varphi(t_n, u_n, y_n\}$ is weakly convergent, i.e,

$$\varphi(t_n, u_n, y_n) \rightharpoonup u$$

weakly in ℓ_2.

Let ε be an arbitrary positive number. Since $K \subset \ell_2$ is an absorbing set for the cocycle φ then there exists a positive number $L_1 = L_1(B[0, r])$ such that

$$\varphi(t, u, y) \in K \tag{12}$$

for any $t \geq L_1(B[0, r])$ and $(u, y) \in B[0, r] \times Y$. Let $n_1 \in \mathbb{N}$ be a number such that $t_n \geq L_1$ for any $n \geq n_1$ and, consequently, from (12) we obtain

$$\varphi(t_n, u_n, y_n) \in K \tag{13}$$

for any $n \geq n_1$.

We will show that the sequence $\{\varphi(t_n, u_n, y_n)\}$ converges in the space ℓ_2, that is, for any $\varepsilon > 0$ there exists a number $n(\varepsilon) \in \mathbb{N}$ such that

$$\|\varphi(t_n, u_n, y_n) - u\| < \varepsilon$$

for any $n \geq n(\varepsilon)$.

By (13) we have

$$\varphi(L_1, u_n, y_n) \in K \tag{14}$$

for any $n \in \mathbb{N}$. Since the cocycle φ satisfies an asymptotic tails on the set K, for given $\varepsilon > 0$, there exist $k_1(\varepsilon) \in \mathbb{N}$ and $L_1(\varepsilon) > 0$ such that

$$\sum_{|i| \geq k_1(\varepsilon)} |\varphi_i(t, \varphi(L_1, u_n, y_n), \sigma(L_1, y_n))|^2 < \varepsilon^2/8 \tag{15}$$

for any $t \geq L_1(\varepsilon)$.

Since $t_n \to +\infty$ as $n \to \infty$, there exists $n_2(\varepsilon) \in \mathbb{N}$ such that if $n \geq n_2(\varepsilon)$, then $t_n - L_1 \geq L_1(\varepsilon)$, and hence from (15), we have

$$\sum_{|i| \geq k_1(\varepsilon)} |\varphi_i(t_n, u_n, y_n)|^2 = \tag{16}$$
$$\sum_{|i| \geq k_1(\varepsilon)} |\varphi_i(t_n - L_1, \varphi(L_1 u_n, y_n), \sigma(L_1, y_n))|^2 \leq \varepsilon^2/8.$$

Since $u \in \ell_2$ then there exists $k_2(\varepsilon)$ such that

$$\sum_{|i| \geq k_2(\varepsilon)} |u_i|^2 \leq \varepsilon^2/8. \tag{17}$$

Let $k(\varepsilon) = \max\{k_1(\varepsilon), k_2(\varepsilon)\}$. By the weak convergence of $\{\varphi(t_n, u_n, y_n)\}$ we have that

$$\varphi_i(t_n, u_n, y_n) \to u_i$$

as $n \to \infty$ (for any $|i| \le k(\varepsilon)$), which implies that there exists $n_3(\varepsilon)$ such that

$$\sum_{|i|\le k(\varepsilon)} |\varphi_i(t_n,u_n,y_n)-u_i|^2 \le \varepsilon^2/2 \tag{18}$$

for any $n \ge n_3(\varepsilon)$.

Setting $n(\varepsilon) := \max\{n_1, n_2(\varepsilon), n_3(\varepsilon)\}$, from (16)-(18) we get that, for $n \ge n(\varepsilon)$,

$$\|\varphi(t_n,u_n,y_n)-u\|^2 = \sum_{|i|\le k(\varepsilon)} |\varphi_i(t_n,u_n,y_n)-u_i|^2 +$$
$$\sum_{|i|>k(\varepsilon)} |\varphi_i(t_n,u_n,y_n)-u_i|^2 \le \varepsilon^2/2 + 2\sum_{|i|\ge k_\varepsilon} (|\varphi_i(t_n,u_n,y_n)|^2 + |u_i|^2) \le \varepsilon^2.$$

as desired. Hence we obtain $\varphi(t_n,u_n,y_n)$ converges to u in the space ℓ_2. Lemma is completely proved. □

Theorem 5. *Under the Conditions (**C1**)-(**C3**) the cocycle $\langle \ell_2, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$ generated by the equation (4) satisfies an asymptotic tails property on the set Q.*

Proof. We will prove this statement using the ideas and methods elaborated in the work [1] (see also [7, Ch.2.3]). Consider a smooth function $\xi : \mathbb{R}^+ \to [0,1]$ satisfying

$$\xi(s) = \begin{cases} 0, & 0 \le s \le 1, \\ \in [0,1], & 1 \le s \le 2, \\ 1, & s \ge 2 \end{cases}$$

and note that there exists a constant C_0 such that $|\xi'(s)| \le C_0$ for all $s \ge 0$. Then for a fixed $k \in \mathbb{N}$ (its value will be specified later), define

$$\xi_k(s) = \xi\left(\frac{s}{k}\right) \quad \text{for all} \quad s \in \mathbb{R}_+.$$

Given $u \in C^1(\mathbb{R}_+, \ell_2)$, define $v \in C^1(\mathbb{R}_+, \ell_2)$ componentwise as

$$v_i(t) := \xi_k(|i|)u_i(t) \quad \text{for} \quad i \in \mathbb{Z} \quad (\forall\ t \in \mathbb{R}_+).$$

Note that

$$\langle u(t), v(t)\rangle = \sum_{i\in\mathbb{Z}} \xi_k(|i|)|u_i(t)|^2 \tag{19}$$

and

$$\frac{d\langle u(t), v(t)\rangle}{dt} = 2\langle \frac{du(t)}{dt}, v(t)\rangle \tag{20}$$

for any $t \in \mathbb{R}_+$.

Taking the inner product of equation (5) with $\mathbf{v}(t)$ gives

$$\frac{d}{dt}\langle \mathbf{u}(t), \mathbf{v}(t)\rangle + \nu\langle \mathrm{D}^+\mathbf{u}(t), \mathrm{D}^+\mathbf{v}(t)\rangle = \langle \Phi(\mathbf{u}(t)), \mathbf{v}(t)\rangle + \langle \mathbf{g}(\mathbf{t}), \mathbf{v}(t)\rangle,$$

that is

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_i|^2 + 2\nu\langle \mathrm{D}^+\mathbf{u}, \mathrm{D}^+\mathbf{v}\rangle = 2\sum_{i\in\mathbb{Z}}\xi_k(|i|)u_i f(u_i) + 2\sum_{i\in\mathbb{Z}}\xi_k(|i|)g_i(t)u_i \quad (21)$$

Each term in the equation (21) will now be estimated. First,

$$\left\langle \mathrm{D}^+\mathbf{u}, \mathrm{D}^+\mathbf{v}\right\rangle = \sum_{i\in\mathbb{Z}}(u_{i+1}-u_i)(v_{i+1}-v_i)$$

$$= \sum_{i\in\mathbb{Z}}(u_{i+1}-u_i)\left[(\xi_k(|i+1|)-\xi_k(|i|))\,u_{i+1} + \xi_k(|i|)(u_{i+1}-u_i)\right]$$

$$= \sum_{i\in\mathbb{Z}}\left(\xi_k(|i+1|)-\xi_k(|i|)\right)(u_{i+1}-u_i)u_{i+1} + \sum_{i\in\mathbb{Z}}\xi_k(|i|)(u_{i+1}-u_i)^2$$

$$\geq \sum_{i\in\mathbb{Z}}\left(\xi_k(|i+1|)-\xi_k(|i|)\right)(u_{i+1}-u_i)u_{i+1}.$$

Since

$$\left|\sum_{i\in\mathbb{Z}}\left(\xi_k(|i+1|)-\xi_k(|i|)\right)(u_{i+1}-u_i)u_{i+1}\right| \leq \sum_{i\in\mathbb{Z}}\frac{1}{k}|\xi'(s_i)|\cdot|u_{i+1}-u_i|\cdot|u_{i+1}|,$$

for some s_i between $|i|$ and $|i+1|$, and

$$\sum_{i\in\mathbb{Z}}|\xi'(s_i)|\,|u_{i+1}-u_i|\,|u_{i+1}| \leq C_0\sum_{i\in\mathbb{Z}}\left(|u_{i+1}|^2 + |u_i||u_{i+1}|\right) \leq 4C_0\,\|\mathbf{u}\|^2\,.$$

Then it follows that for all $\mathbf{u}\in Q$ and $\mathbf{v}\in\ell^2$ defined componentwise as $v_i := \xi_k(|i|)u_i$, for $i\in\mathbb{Z}$,

$$\left\langle \mathrm{D}^+\mathbf{u}, \mathrm{D}^+\mathbf{v}\right\rangle \geq -\frac{4C_0\|Q\|^2}{k}, \quad (22)$$

where $\|Q\| := \sup_{\mathbf{u}\in Q}\|\mathbf{u}\|$. On the other hand, by Condition (**C3**),

$$2\sum_{i\in\mathbb{Z}}\xi_k(|i|)u_i f(u_i) \leq -2\alpha\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_i|^2$$

and by Young's inequality

$$2\sum_{i\in\mathbb{Z}}\xi_k(|i|)g_i u_i \leq \alpha\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_i|^2 + \frac{1}{\alpha}\sum_{i\in\mathbb{Z}}\xi_k(|i|)|g_i|^2.$$

Thus

$$2\sum_{i\in\mathbb{Z}}\xi_k(|i|)u_i f(u_i) + 2\sum_{i\in\mathbb{Z}}\xi_k(|i|)g_i(t)u_i \leq -\alpha\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_i|^2 + \frac{1}{\alpha}\sum_{|i|\geq k}|g_i|^2. \quad (23)$$

Using the estimates (22) and (23) in the equation (21) gives

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_i|^2+\alpha\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_i|^2\leq\nu\frac{4C_0\|Q\|^2}{k}+\frac{1}{\alpha}\sum_{|i|\geq k}|g_i|^2 \tag{24}$$

Since $g\in H(f)$ and the set $H(f)$ is a compact subset in the space $C(\mathbb{R},\ell_2)$ then by Lemma 1 the set $\overline{f(\mathbb{R})}$ is a compact subset of ℓ_2. In particular for any $\varepsilon>0$ there exists a natural number $k(\varepsilon)$ such that

$$\sum_{|i|\geq k(\varepsilon)}|v_i|^2<\varepsilon \tag{25}$$

for any $v\in\overline{f(\mathbb{R})}$. Note that $g\in H(f)$ and, consequently,

$$\overline{g(\mathbb{R})}\subseteq\overline{f(\mathbb{R})}. \tag{26}$$

From (25) and (26) we obtain

$$\sum_{|i|\geq k(\varepsilon)}|g_i(t)|^2<\varepsilon$$

for any $g\in H(f)$ and $t\in\mathbb{R}$.

Note that $g(t)\in\overline{g(\mathbb{R})}\subseteq\overline{f(\mathbb{R})}$ and, consequently, for every $\varepsilon>0$, there exists k_ε such that

$$\nu\frac{4C_0\|Q\|^2}{k}+\frac{1}{\alpha}\sum_{|i|\geq k}|g_i(t)|^2\leq\varepsilon,\quad k\geq k(\varepsilon)$$

for any $g\in H(f)$ and $t\in\mathbb{R}$.

The inequality (24) along with the relation above give

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_i|^2+\alpha\sum_{i\in\mathbb{Z}}\xi_k(|i|)|u_i|^2\leq\varepsilon$$

for any $g\in H(f)$ and $t\in\mathbb{R}$.

Then, Gronwall's lemma implies that

$$\sum_{i\in\mathbb{Z}}\xi_k(|i|)|\varphi(t,v,g)_i(t,\mathbf{u}_o)|^2\leq e^{-\alpha t}\sum_{i\in\mathbb{Z}}\xi_k(|i|)|v_i|^2+\frac{\varepsilon}{\alpha}\leq e^{-\alpha t}\|v\|^2+\frac{\varepsilon}{\alpha}.$$

Hence for every $v\in Q$,

$$\sum_{i\in\mathbb{Z}}\xi_k(|i|)|\varphi(t,v,g)_i|^2\leq e^{-\alpha t}\|Q\|^2+\frac{\varepsilon}{\alpha}$$

and therefore

$$\sum_{i\in\mathbb{Z}}\xi_k(|i|)\,|u_i(t,\mathbf{u}_o)|^2\leq\frac{2\varepsilon}{\alpha},\quad\text{for } t\geq T(\varepsilon):=\frac{1}{\alpha}\ln\frac{\alpha\|Q\|^2}{\varepsilon}.$$

This means that the cocycle $\langle\ell_2,\varphi,(Y,\mathbb{R},\sigma)\rangle$ is asymptotic tails on the absorbing set Q. Theorem is proved. □

5 Compact global attractors

Definition 8. A family $\{I_y |\ y \in Y\}$ of compact subsets I_y of $\mathfrak{B}$ is said to be a compact global attractor for the cocycle $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ if the following conditions are fulfilled:

1. the set
$$\mathcal{I} := \bigcup \{I_y |\ y \in Y\}$$
is precompact;

2. the family of subsets $\{I_y |\ y \in Y\}$ is invariant, i.e., $\varphi(t, I_y, y) = I_{\sigma(t,y)}$ for any $(t, y) \in \mathbb{R}_+ \times Y$;

3.
$$\lim_{t\to+\infty} \sup_{y\in Y} \beta(\varphi(t, M, y), \mathcal{I}) = 0$$

for any compact subset M from $\mathfrak{B}$.

Definition 9. A cocycle φ is said to be dissipative if there exists a bounded subset $K \subset \mathfrak{B}$ such that for any bounded subset $B \subset \mathfrak{B}$ there exists a positive number $L = L(B)$ such that $\varphi(t, B, Y) \subseteq K$ for any $t \geq L(B)$, where $\varphi(t, B, Y) := \{\varphi(t, u, y)|\ (u, y) \in B \times Y\}$.

Theorem 6. *[6, Ch.II] Assume that the metric space Y is compact and the cocycle $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ is dissipative and asymptotically compact.*

Then the cocycle φ has a compact global attractor.

Theorem 7. *Under the Conditions (**C1**)-(**C3**) the equation (4) (the cocycle φ generated by the equation (4)) has a compact global attractor $\{I_g |\ g \in H(f)\}$.*

Proof. This statement follows from Theorems 4, 5 and 6. □

Below we give an example which illustrate our general results.

Example 1. Let $\{\omega_i\}_{i\in\mathbb{Z}}$ be a sequence of real numbers. For every $i \in \mathbb{Z}$ we define a function $f_i \in C(\mathbb{R}, \mathbb{R})$ by the equality

$$f_i(t) := \frac{\sin(\omega_i t + \ln(1 + t^2))}{2^{|i|}}$$

for all $t \in \mathbb{R}$.

Note that the functions f_i $(i \in \mathbb{Z})$ possess the following properties:

1.
$$|f_i(t)| \leq \frac{1}{2^{|i|}} \tag{27}$$
for all $t \in \mathbb{R}$ and $i \in \mathbb{Z}$;

2.
$$|f_i'(t)| \le (1 + |\omega_i|) \tag{28}$$
for all $t \in \mathbb{R}$ and $i \in \mathbb{Z}$.

Lemma 5. *For every $i \in \mathbb{Z}$ the function f_i is bounded and uniformly continuous on $\mathbb{R}$.*

Proof. This statement directly follows from (27) and (28). □

Lemma 6. *The following statements hold:*

1. *$f(t) \in \ell_2$, where $f(t) := (f_i(t))_{i\in\mathbb{Z}}$;*
2. *for every $\varepsilon > 0$ there exists a number $n(\varepsilon) \in \mathbb{N}$ such that*
$$\sum_{|i|\ge n(\varepsilon)} |f_i(t)|^2 < \frac{\varepsilon^2}{4} \tag{29}$$
for all $t \in \mathbb{R}$.

Proof. Since
$$\|f(t)\|^2 = \sum_{i\in\mathbb{Z}} |f_i(t)|^2 = \sum_{i\in\mathbb{Z}} \frac{\sin^2(\omega_i t + \ln(1+t^2))}{2^{2|i|}} \le \sum_{i\in\mathbb{Z}} \frac{1}{2^{2|i|}} = \frac{11}{3}$$
then $f(t) \in \ell_2$.

Note that for every $\varepsilon > 0$ there exists a number $n(\varepsilon) \in \mathbb{N}$ such that
$$\sum_{|i|\ge n(\varepsilon)} \frac{1}{4^{|i|}} < \frac{\varepsilon^2}{4}. \tag{30}$$

By (27) and (30) we obtain
$$\sum_{|i|\ge n(\varepsilon)} |f_i(t)|^2 \le \sum_{|i|\ge n(\varepsilon)} \frac{1}{4^{|i|}} < \frac{\varepsilon^2}{4} \tag{31}$$
for all $t \in \mathbb{R}$. □

Consider the function $f : \mathbb{R} \to \ell_2$ defined by $f(t) := (f_i(t))_{i\in\mathbb{Z}}$ for all $t \in \mathbb{R}$.

Lemma 7. *The following statements hold:*

1. *the function $f : \mathbb{R} \to \ell_2$ is uniformly continuous on $\mathbb{R}$;*
2. *the set $f(\mathbb{R})$ is a precompact subset of ℓ_2.*

Proof. For every $\varepsilon > 0$ we choose $n(\varepsilon) \in \mathbb{N}$ such that (31) holds. Since the functions f_i ($|i| < n(\varepsilon)$) are uniformly continuous then for ε there exists a positive number $\delta = \delta(\varepsilon)$ such that $|t_1 - t_2| < \delta$ implies (see Lemma 5)

$$\sum_{|i|<n(\varepsilon)} |f_i(t_1) - f_i(t_2)|^2 < \frac{\varepsilon^2}{2}. \tag{32}$$

On the other hand we have

$$\|f(t_1) - f(t_2)\|^2 = \sum_{|i|<n(\varepsilon)} |f_i(t_1) - f_i(t_2)|^2 + \sum_{|i|\geq n(\varepsilon)} |f_i(t_1) - f_i(t_2)|^2 \tag{33}$$

for any $t_1, t_2 \in \mathbb{R}$. From (33), (32) and (29) we receive

$$\begin{aligned} \|f(t_1) - f(t_2)\|^2 = \sum_{|i|<n(\varepsilon)} |f_i(t_1) - f_i(t_2)|^2 + \sum_{|i|\geq n(\varepsilon)} |f_i(t_1) - f_i(t_2)|^2 \leq \\ \sum_{|i|<n(\varepsilon)} |f_i(t_1) - f_i(t_2)|^2 + \sum_{|i|\geq n(\varepsilon)} 2(|f_i(t_1)|^2 + |f_i(t_2)|^2) \\ < \tfrac{\varepsilon^2}{2} + 2(\tfrac{\varepsilon^2}{4} + \tfrac{\varepsilon^2}{4}) = \varepsilon^2 \end{aligned} \tag{34}$$

and, consequently, $\|f(t_1) - f(t_2)\| < \varepsilon$ for any $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| < \delta$.

Let now v be an arbitrary element of the set $f(\mathbb{R})$, then there exists a number $s \in \mathbb{R}$ such that $v = f(s)$. By Lemma 6 (item (ii)) for every $\varepsilon > 0$ there exists a number $n(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{|i|\geq n(\varepsilon)} |v_i|^2 = \sum_{|i|\geq n(\varepsilon)} |f_i(s)|^2 < \frac{\varepsilon^2}{4}$$

and, consequently, by Theorem 5.25 [8, Ch.V, p.167] the subset $f(\mathbb{R})$ of ℓ_2 is precompact. □

Corollary 2. *The function f is Lagrange stable, i.e., the set $H(f)$ is a compact subset of $C(\mathbb{R}, \ell_2)$.*

Proof. This statement follows from Lemmas 1 and 7. □

Consider the system of differential equations

$$u_i' = \nu(u_{i-1} - 2u_i + u_{i+1}) - \lambda u_i + F(u_i) + \frac{\sin(\omega_i t + \ln(1 + t^2))}{2^{|i|}} \quad (i \in \mathbb{Z}), \tag{35}$$

where $F(u) = -u - u^3$ for all $u \in \mathbb{R}$.

Along with this system of equations (35), consider the (equivalent) equation

$$u' = \Lambda u - \lambda u + \tilde{F}(u) + f(t) \tag{36}$$

in the space ℓ_2.

Taking into account the results above it is easy to show that the Conditions (**C1**)-(**C3**) for the equation (36) are fulfilled. According to our result this equation admits a compact global attractor.

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7 Conflict of Interest

The authors declare that they have not conflict of interest.

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