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# Dissipative Dirac Operator on Time Scales

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**Abstract.** In this paper, we study symmetric Dirac operator acting on time scales. We give maximal dissipative, maximal accumulative and self-adjoint extensions of such operator via the boundary conditions. Further, we construct a self-adjoint dilation of the maximal dissipative operator and determine the scattering matrix of dilation. Later, we construct a functional model of this operator and define its characteristic function. Finally, we prove that all root vectors of such operator are complete in the convenient Hilbert space .

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## 1 Introduction

Dissipative operators are one of the important class of the operator theory. Using the theory of dilations with applications of functional models, spectral properties of such operators were investigated in [4-6], [12-14]. The functional model of dissipative operators plays an important role within both the abstract operator theory and its more specialized applications in other disciplines.

On the other hand, time scale calculus is a fairly new theory in mathematics. It unites the two approaches of dynamic modelling: differential and difference equations. It was introduced by Stefan Hilger in [22]. The time scales calculus has applications in some mathematical models of real processes, population dynamics, chemical technology, biotechnology and neural networks, economics and social sciences. We refer the reader to consult the references [10, 17, 18, 19, 20, 21, 23].

In this paper, we shall be interested in the one-dimensional Dirac systems on time scales of the form

$$\begin{aligned} -\Delta y_2^{\rho} + p(t)\, y_1 &= \lambda y_1, \\ \Delta y_1 + r(t)\, y_2 &= \lambda y_2, \end{aligned} \tag{1}$$

where $\lambda$ is a complex eigenvalue parameter, $p(.)$ and $r(.)$ are real-valued functions defined on $\mathbb{T}^*$ and $p, r \in L^1_\Delta(\mathbb{T}^*)$.

This system is the time scale generalization of the one-dimensional Dirac system

$$-y_2' + p(x)\, y_1 = \lambda y_1, \tag{2}$$


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$$y_1' + r(x) y_2 = \lambda y_2.$$

As is known, the system (2) describes a relativistic electron in the electrostatic field (see [1]). Spectral properties of this system were investigated in [5]. However, the corresponding results on time scales have not been developed. In [25], the authors studied an eigenvalue problem for the Dirac system with separated boundary conditions on an arbitrary time scale. They improved the results about the spectral theory of the classical Dirac system, such as the orthogonality of eigenfunctions and the simplicity of the eigenvalues. In [26], the author gave some sufficient conditions for the disconjugacy of Dirac systems and obtained a formula about the number of the eigenvalues of the problem. Recently, in [27, 29], the authors studied one-dimensional Dirac operators on time scales. In [28], the authors investigated the dissipative Dirac operator on bounded time scales with general boundary conditions. In this work, motivated by the discussion above, we provide spectral properties of this system on time scales.

In the present article, we consider the Dirac operator acting in the Hilbert space $H := L^2_\Delta(\mathbb{T}^*; E)$ $(E := \mathbb{C}^2)$. In Section 2, we construct a space of boundary value for minimal symmetric Dirac operator and describe all the maximal dissipative, maximal accumulative and self-adjoint extensions of such operator. In Section 3, we construct a self-adjoint dilation and its incoming and outgoing spectral representations. Thus, we determine the scattering matrix of the dilation according to the Lax and Phillips scheme [2, 3]. In Section 4, using incoming spectral representations, we construct a functional model of the maximal dissipative Dirac operator. Furthermore, we determine characteristic function of this operator. Finally, we prove that all root vectors of the maximal dissipative Dirac operator are complete in the space $H$. Hence, our study could fill an important gap in the spectral theory of the Dirac system on time scales.

Now, we recall some necessary concepts of time scale calculus for convenience.

Let $\mathbb{T}$ be a time scale, i.e, a non-empty closed subset of real numbers $\mathbb{R}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \text{ where } t \in \mathbb{T}$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \text{ where } t \in \mathbb{T}.$$

It is convenient to have graininess operators $\mu_\sigma : \mathbb{T} \to [0, \infty)$ and $\mu_\rho : \mathbb{T} \to (-\infty, 0]$ defined by

$$\mu_\sigma(t) = \sigma(t) - t$$

and

$$\mu_\rho(t) = \rho(t) - t,$$

respectively.

**Definition 1.** A point $t \in \mathbb{T}$ is left scattered if $\mu_\rho(t) \neq 0$ and left dense if $\mu_\rho(t) = 0$. A point $t \in \mathbb{T}$ is right scattered if $\mu_\sigma(t) \neq 0$ and right dense if $\mu_\sigma(t) = 0$.

Now, we introduce the sets $\mathbb{T}^k$, $\mathbb{T}_k$, $\mathbb{T}^*$ which are derived from the time scale $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left scattered maximum $t_1$, then $\mathbb{T}^k = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum $t_2$, then $\mathbb{T}_k = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. Finally, $\mathbb{T}^* = \mathbb{T}^k \cap \mathbb{T}_k$.

**Definition 2.** A function $f$ on $\mathbb{T}$ is said to be $\Delta$-differentiable at some point $t \in \mathbb{T}$ if there is a number $f^{\Delta}(t)$ such that for every $\varepsilon > 0$ there is a neighborhood $U \subset \mathbb{T}$ of $t$ such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \text{ where } s \in U.$$

Analogously one may define the notion of $\nabla$-differentiability of some function using the backward jump $\rho$. One can show (see [21])

$$f^{\Delta}(t) = f^{\nabla}(\sigma(t)),\ f^{\nabla}(t) = f^{\Delta}(\rho(t))$$

for continuously differentiable functions.

If $\mathbb{T} = \mathbb{R}$, then

$$f^{\Delta}(t) = f'(t).$$

If $\mathbb{T} = h\mathbb{Z}$ $(h > 0)$, then

$$f^{\Delta}(t) = \frac{f(t+h) - f(t)}{h}.$$

If $\mathbb{T} = q^{\mathbb{N}_0}$ $(q > 1)$, then

$$f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t}.$$

The product and quotient rules on time scales have the following form: If $f, g : \mathbb{T} \to \mathbb{R}$, then

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t),\ (fg)^{\nabla}(t) = f^{\nabla}(t)g(t) + f(\rho(t))g^{\nabla}(t),$$

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)\, g(\sigma(t))},\ \left(\frac{f}{g}\right)^{\nabla}(t) = \frac{f^{\nabla}(t)g(t) - f(t))g^{\nabla}(t)}{g(t)\, g(\rho(t))}.$$

Let $f : \mathbb{T} \to \mathbb{R}$ be a function, and $a, b \in \mathbb{T}$. If there exists a function $F : \mathbb{T} \to \mathbb{R}$ such that $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^k$, then $F$ is a $\Delta$-antiderivative of $f$. In this case the integral is given by the formula

$$\int_a^b f(t)\, \Delta t = F(b) - F(a) \text{ for } a, b \in \mathbb{T}.$$

Analogously one may define the notion of $\nabla$-antiderivative of some function.

If $\mathbb{T} = \mathbb{R}$ and $f$ is continuous, then

$$\int_a^b f(t)\, \Delta t = \int_a^b f(t)\, dt.$$

If $\mathbb{T}=h\mathbb{Z}$ $(h>0)$ and $a=hx,\ b=hy,\ x<y$, then

$$\int_a^b f(t)\,\Delta t=h\sum_{k=x}^{y-1} f(hk).$$

If $\mathbb{T}=q^{\mathbb{N}_0}$ $(q>1)$ and $a=q^x, b=q^y,\ x<y$, then

$$\int_a^b f(t)\,\Delta t=(q-1)\sum_{k=x}^{y-1} q^k f\left(q^k\right).$$

Let $L^2_\Delta(\mathbb{T}^*)$ be the space of all functions defined on $\mathbb{T}^*$ such that

$$\|f\|:=\left(\int_a^b |f(t)|^2\,\Delta t\right)^{1/2}<\infty.$$

The space $L^2_\Delta\,(\mathbb{T}^*)$ is a Hilbert space with the inner product (see [24])

$$(f,g):=\int_a^b f(t)\,\overline{g(t)}\Delta t,\quad f,g\in L^2_\Delta(\mathbb{T}^*)\ .$$

Let $a\le b$ be fixed points in $\mathbb{T}$ and $a\in\mathbb{T}_k, b\in\mathbb{T}^k$.

Now, we introduce convenient Hilbert space $H:=L^2_\Delta(\mathbb{T}^*;E)$ $(E:=\mathbb{C}^2)$ of vector-valued functions using the inner product

$$(f,g)_H:=\int_a^b (f(x),g(x))_E\Delta t.$$

## 2 Extensions of symmetric Dirac Operator

In this section, we describe all extensions (maximal dissipative, accumulative, self-adjoint and other) of symmetric Dirac operator on time scales. We consider the one-dimensional Dirac systems

$$\Gamma y:=\begin{cases} -\Delta y_2^\rho+p(t)\,y_1 \\ \Delta y_1+r(t)\,y_2\end{cases}=\lambda y=\begin{pmatrix}\lambda y_1\\ \lambda y_2\end{pmatrix},$$

where $f^\rho(t):=f(\rho(t))$, $\Delta f(t):=f^\Delta(t)$, $\lambda$ is a complex eigenvalue parameter, $p(.)$ and $r(.)$ are real-valued functions defined on $\mathbb{T}^*$ and $p,r\in L^1_\Delta(\mathbb{T}^*)$.

Let us consider the set $D$ consisting of all vector-valued functions

$$y(.)=\begin{pmatrix}y_1(.)\\ y_2(.)\end{pmatrix}\in H$$

in which $y_1$ and $y_2$ are $\Delta$ absolutely continuous functions on $\mathbb{T}^*$ and $\Gamma y\in H$. We define the maximal operator $\Upsilon_{\max}$ on the set $D$ by the equality $\Upsilon_{\max}y:=\Gamma y$. Now we have a

**Lemma 1** (Green's formula)**.** *Let*

$$y=\begin{pmatrix} y_1 \\ y_2 \end{pmatrix},\ z=\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\in D.$$

*Then, we have*

$$(\Gamma y,z)_H-(y,\Gamma z)_H=[y,z]_b-[y,z]_a\,,$$

*where* $[y,z]_t:=y_1(t)\,\overline{z_2^\rho(t)}-\overline{z_1(t)}y_2^\rho(t)\,.$

*Proof.* Let $y,z\in D$. Then, we obtain

$$\begin{aligned}
&(\Gamma y,z)_H-(y,\Gamma z)_H\\
&=\int_a^b\left(-\Delta y_2^\rho+p(t)\,y_1\right)\overline{z_1}\Delta t+\int_a^b\left(\Delta y_1+r(t)\,y_2\right)\overline{z_2}\Delta t\\
&-\int_a^b y_1\overline{(-\Delta z_2^\rho+p(t)\,z_1)}\Delta t-\int_a^b y_2\overline{(\Delta z_1+r(t)\,z_2)}\Delta t\\
&=-\int_a^b\left[(\Delta y_2^\rho)\,\overline{z_1}+y_2\overline{(\Delta z_1)}\right]\Delta t+\int_a^b\left[(\Delta y_1)\,\overline{z_2}+y_1\overline{(\Delta z_2^\rho)}\right]\Delta t.
\end{aligned}$$

Since

$$\begin{aligned}
\Delta\left(\overline{z_1(t)}y_2^\rho(t)\right)&=\overline{z_1(t)}\left(\Delta y_2^\rho(t)\right)+\left(y_2^\rho(t)\right)^\sigma\overline{(\Delta z_1(t))}\\
&=\Delta y_2^\rho(t)\,\overline{z_1(t)}+y_2(t)\,\overline{(\Delta z_1(t))}
\end{aligned}$$

and

$$\begin{aligned}
\Delta\left(\overline{z_2^\rho(t)}y_1(t)\right)&=\overline{(\Delta z_2^\rho(t))}y_1(t)+\overline{(z_2^\rho(t))^\sigma}\left(\Delta y_1(t)\right)\\
&=\overline{(\Delta z_2^\rho(t))}y_1(t)+\overline{z_2(t)}\Delta y_1(t)\,.
\end{aligned}$$

Hence we get

$$\begin{aligned}
&(\Gamma y,z)_H-(y,\Gamma z)_H\\
&=-\int_a^b\Delta\left(\overline{z_1(t)}y_2^\rho(t)\right)\Delta t+\int_a^b\Delta\left(y_1(t)\,\overline{z_2^\rho(t)}\right)\Delta t\\
&=\int_a^b\Delta\left[y_1(t)\,\overline{z_2^\rho(t)}-\overline{z_1(t)}y_2^\rho(t)\right]\Delta t=[y,z]_b-[y,z]_a\,.
\end{aligned}$$

□

Let $D_{\min}$ denote the linear set of all vectors $y\in D$ satisfying the conditions

$$y_1(a)=y_2^\rho(a)=y_1(b)=y_2^\rho(b)=0.$$

If we restrict the operator $\Upsilon_{\max}$ to the set $D_{\min}$, then we obtain the minimal operator $\Upsilon_{\min}$. It is clear that $\Upsilon_{\min}^*=\Upsilon_{\max}$, and $\Upsilon_{\min}$ is a closed symmetric operator (see [7]). Now we recall the following.

**Definition 3.** A linear operator $M$ (with dense domain $D(M)$) acting on some Hilbert space $H$ is called dissipative (accumulative) if $\operatorname{Im}(Mf,f) \geq 0$ ($\operatorname{Im}(Mf,f) \leq 0$) for all $f \in D(M)$ and maximal dissipative (maximal accumulative) if it does not have a proper dissipative (accumulative) extension (see [4-6]).

**Definition 4.** A triplet $(\mathbb{H}, \Lambda_1, \Lambda_2)$ is called a space of boundary values of a closed symmetric operator $M$ on a Hilbert space $H$ if $\Lambda_1$ and $\Lambda_2$ are linear maps from $D(M^*)$ to $H$, with equal deficiency numbers and such that:

**i)** For every $f, g \in D(M^*)$ we have

$$(M^* f, g)_H - (f, M^* g)_H = (\Lambda_1 f, \Lambda_2 g)_{\mathbb{H}} - (\Lambda_2 f, \Lambda_1 g)_{\mathbb{H}};$$

**ii)** For any $F_1, F_2 \in H$ there is a vector $f \in D(A^*)$ such that $\Lambda_1 f = F_1$ and $\Lambda_2 f = F_2$ (see [8]).

Let's define by $\Lambda_1,\ \Lambda_2$ the linear maps from $D$ to $\mathbb{C}^2$ by the formula

$$\Lambda_1 y = \begin{pmatrix} -y_1(a) \\ y_1(b) \end{pmatrix},\ \Lambda_2 y = \begin{pmatrix} y_2^{\rho}(a) \\ y_2^{\rho}(b) \end{pmatrix}. \tag{3}$$

Now we will state and prove a theorem.

**Theorem 1.** *The triplet $(\mathbb{C}^2, \Lambda_1, \Lambda_2)$ defined by (3) is a boundary space of the operator $\Upsilon_{\min}$.*

*Proof.* Let $y, z \in D$. Then, we have

$$\begin{aligned}&(\Lambda_1 y, \Lambda_2 z)_{\mathbb{C}^2} - (\Lambda_2 y, \Lambda_1 z)_{\mathbb{C}^2} \\ &= -y_1(a)\,\overline{z}_2^{\rho}(a) + \overline{z_1}(a)\,y_2^{\rho}(a) + y_1(b)\,\overline{z}_2^{\rho}(b) - \overline{z_1}(b)\,y_2^{\rho}(b).\end{aligned}$$

By Green's formula, we obtain

$$(\Lambda_1 y, \Lambda_2 z)_{\mathbb{C}^2} - (\Lambda_2 y, \Lambda_1 z)_{\mathbb{C}^2} = [y,z]_b - [y,z]_a.$$

Hence

$$(\Upsilon_{\max} y, z)_H - (y, \Upsilon_{\max} z)_H = (\Lambda_1 y, \Lambda_2 z)_{\mathbb{C}^2} - (\Lambda_2 y, \Lambda_1 z)_{\mathbb{C}^2}.$$

Thus, we obtain the first condition of the definition of a space of boundary value.

Now, we will prove the second condition. Let

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},\ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{C}^2.$$

Then the vector-valued function

$$y(t) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = u_1\alpha_1(t) + v_1\alpha_2(t) + u_2\beta_1(t) + v_2\beta_2(t),$$

where

$$\alpha_1(.) = \begin{pmatrix} \alpha_{11}(.) \\ \alpha_{12}(.) \end{pmatrix},\ \alpha_2(.) = \begin{pmatrix} \alpha_{21}(.) \\ \alpha_{22}(.) \end{pmatrix},$$

$$\beta_1(.) = \begin{pmatrix} \beta_{11}(.) \\ \beta_{12}(.) \end{pmatrix}, \ \beta_2(.) = \begin{pmatrix} \beta_{21}(.) \\ \beta_{22}(.) \end{pmatrix} \in D$$

satisfy the conditions

$$\begin{aligned} &\alpha_{11}(a) = -1, \ \alpha_{12}^{\rho}(a) = \alpha_{11}(b) = \alpha_{12}^{\rho}(b) = 0, \\ &\alpha_{22}^{\rho}(a) = 1, \ \alpha_{21}(a) = \alpha_{21}(b) = \alpha_{22}^{\rho}(b) = 0, \\ &\beta_{11}(b) = 1, \ \beta_{11}(a) = \beta_{12}^{\rho}(a) = \beta_{12}^{\rho}(b) = 0, \\ &\beta_{22}^{\rho}(b) = 1, \ \beta_{21}(a) = \beta_{21}(b) = \beta_{22}^{\rho}(a) = 0, \end{aligned}$$

belongs to the set $D$ and $\Lambda_1 y = u, \ \Lambda_2 y = v$. This finishes the proof. □

**Corollary 1.** *For any contraction $K$ in $\mathbb{C}^2$ the restriction of the operator $\Upsilon_{\max}$ to the set of functions $y \in D$ satisfying either*

$$(K - I)\Lambda_1 y + i(K + I)\Lambda_2 y = 0 \tag{4}$$

*or*

$$(K - I)\Lambda_1 y - i(K + I)\Lambda_2 y = 0 \tag{5}$$

*is respectively the maximal dissipative and accumulative extension of the operator $\Upsilon_{\min}$. Conversely, every maximal dissipative (accumulative) extension of the operator $\Upsilon_{\min}$ is the restriction of $\Upsilon_{\max}$ to the set of functions $y \in D$ satisfying (4) ((5)), and the extension uniquely determines the contraction $K$. If $K$ is unitary, these conditions define self-adjoint extensions.*

In particular, the boundary conditions

$$y_2^{\rho}(a) - \alpha_1 y_1(a) = 0, \tag{6}$$

$$y_2^{\rho}(b) + \alpha_2 y_1(b) = 0, \tag{7}$$

with $\operatorname{Im}\alpha_1 \geq 0$ or $\alpha_1 = \infty$, $\operatorname{Im}\alpha_2 \geq 0$ or $\alpha_2 = \infty$, ($\operatorname{Im}\alpha_1 = 0$ or $\alpha_1 = \infty$, $\operatorname{Im}\alpha_2 = 0$ or $\alpha_2 = \infty$) describe the maximal dissipative (self-adjoint) extensions of $\Upsilon_{\min}$ with separated boundary conditions. Note that if $\alpha_1 = \infty$ ($\alpha_2 = \infty$), then the boundary condition (6) ((7)) should be replaced by $y_1(a) = 0$ ($y_1(b) = 0$).

From now on, we shall study the maximal dissipative operators $\Upsilon_{\alpha_1\alpha_2}$ generated by expression $\Gamma$ and the boundary conditions (6) and (7) with $\operatorname{Im}\alpha_1 > 0$ and $\operatorname{Im}\alpha_2 = 0$ or $\alpha_2 = \infty$.

## 3 Self-adjoint dilation

While we study the spectral analysis of the maximal dissipative operators, we will use the functional model theory of Sz.Nagy-Foiaş (see [3]). In order to construct the characteristic function of a contraction we will use the abstract scattering function of Lax-Phillips ( see [2]) because it is unitary equivalent to the characteristic function of Sz.-Nagy-Foias (see [3]).

In this section, we construct a self-adjoint dilation and its incoming and outgoing spectral representations. Later, we determine the scattering matrix of the dilation according to the Lax and Phillips scheme [2, 3].

Now, let us define the *main Hilbert space of the dilation* $\mathcal{H} = L^2(-\infty, 0) \oplus H \oplus L^2(0, \infty)$. In the space $\mathcal{H}$, we consider the operator $\mathbf{\Gamma}$ on the set $D(\mathbf{\Gamma})$, its elements consisting of vectors $w = \langle \varphi_-, y, \varphi_+ \rangle$, generated by the expression

$$\mathbf{\Gamma}\langle \varphi_-, y, \varphi_+ \rangle = \langle i\frac{d\varphi_-}{d\xi}, \Gamma y, i\frac{d\varphi_+}{d\zeta} \rangle \tag{8}$$

satisfying the conditions: $\varphi_- \in W_2^1(-\infty, 0)$, $\varphi_+ \in W_2^1(0, \infty)$, $y \in H$,

$$y_2^\rho(a) - \alpha_1 y_1(a) = \gamma\varphi_-(0), \ y_2^\rho(a) - \overline{\alpha_1} y_1(a) = \gamma\varphi_+(0),$$

$$y_2^\rho(b) - \alpha_2 y_1(b) = 0,$$

where $W_2^1$ is Sobolev space and $\gamma^2 := 2\operatorname{Im}\alpha_1$, $\gamma > 0$.

**Theorem 2.** *The operator* $\mathbf{\Gamma}$ *is self-adjoint in* $\mathcal{H}$ *and it is a self-adjoint dilation of the operator* $\Upsilon_{\alpha_1\alpha_2}$.

*Proof.* Let $f, g \in D(\mathbf{\Gamma})$, $f = \langle \varphi_-, y, \varphi_+ \rangle$ and $g = \langle \psi_-, z, \psi_+ \rangle$. Then we have

$$\begin{aligned}
&(\mathbf{\Gamma}f, g)_{\mathcal{H}} - (f, \mathbf{\Gamma}g)_{\mathcal{H}} \\
&= (\mathbf{\Gamma}\langle \varphi_-, y, \varphi_+ \rangle, \langle \psi_-, z, \psi_+ \rangle) - (\langle \varphi_-, y, \varphi_+ \rangle, \mathbf{\Gamma}\langle \psi_-, z, \psi_+ \rangle) \\
&= i\int_{-\infty}^{0} \frac{d\varphi_-}{d\xi}\overline{\psi}_- d\xi + (\Gamma y, z)_H + i\int_{0}^{\infty} \frac{d\varphi_+}{d\xi}\overline{\psi}_+ d\xi \\
&- i\int_{-\infty}^{0} \varphi_- \overline{\frac{d\psi_-}{d\xi}} d\xi - (y, \Gamma z)_H - i\int_{0}^{\infty} \varphi_+ \overline{\frac{d\psi_+}{d\xi}} d\xi \\
&= i\int_{-\infty}^{0} \frac{d\varphi_-}{d\xi}\overline{\psi}_- d\xi + [y, z]_b + i\int_{0}^{\infty} \frac{d\varphi_+}{d\xi}\overline{\psi}_+ d\xi \\
&- i\int_{-\infty}^{0} \varphi_- \overline{\frac{d\psi_-}{d\xi}} d\xi - [y, z]_a - i\int_{0}^{\infty} \varphi_+ \overline{\frac{d\psi_+}{d\xi}} d\xi \\
&= i\psi_-(0)\overline{\varphi}_-(0) - i\varphi_+(0)\overline{\psi}_+(0) + [y, z]_b - [y, z]_a.
\end{aligned}$$

By direct computation, we get

$$i\psi_-(0)\overline{\varphi}_-(0) - i\varphi_+(0)\overline{\psi}_+(0) + [y, z]_b - [y, z]_a = 0.$$

Thus, $\mathbf{\Gamma}$ is a symmetric operator.

Now, we will prove that $\mathbf{\Gamma}$ is self-adjoint, i.e., $\mathbf{\Gamma}^* \subseteq \mathbf{\Gamma}$. Let $g = \langle \psi_-, z, \psi_+ \rangle \in D(\mathbf{\Gamma}^*)$ and $\mathbf{\Gamma}^* g = g^* = \langle \psi_-^*, z^*, \psi_+^* \rangle \in \mathcal{H}$, such that

$$(\mathbf{\Gamma} f, g)_{\mathcal{H}} = (f, \mathbf{\Gamma}^* g)_{\mathcal{H}} = (f, g^*)_{\mathcal{H}}. \tag{9}$$

Then, it is not difficult to show that $\psi_- \in W_2^1(-\infty, 0)$, $\psi_+ \in W_2^1(0, \infty)$, $g \in D(\mathbf{\Gamma})$ and $g^* = \mathbf{\Gamma} g$. Using (9), we obtain

$$(\mathbf{\Gamma} f, g)_{\mathcal{H}} = (f, \mathbf{\Gamma} g)_{\mathcal{H}}, \ f \in D(\mathbf{\Gamma}^*).$$

Furthermore, $g \in D(\mathbf{\Gamma}^*)$ satisfies the conditions

$$y_2^{\rho}(a) - \alpha_1 y_1(a) = \gamma \varphi_-(0), \ y_2^{\rho}(a) - \overline{\alpha_1} y_1(a) = \gamma \varphi_+(0),$$

$$y_2^{\rho}(b) - \alpha_2 y_1(b) = 0.$$

Consequently, $D(\mathbf{\Gamma}^*) \subseteq D(\mathbf{\Gamma})$, i.e., $\mathbf{\Gamma}$ is self-adjoint.

On the other hand, we know that the self-adjoint operator $\mathbf{\Gamma}$ generates on $\mathcal{H}$ a unitary group $\mathcal{U}_t = \exp(i\mathbf{\Gamma} t)$ $(t \in \mathbb{R})$. Let denote by $\mathcal{P} : \mathcal{H} \to H$ and $\mathcal{P}_1 : H \to \mathcal{H}$ the mappings acting according to the formulae $\mathcal{P} : \langle \varphi_-, y, \varphi_+ \rangle \to y$ and $\mathcal{P}_1 : y \to \langle 0, y, 0 \rangle$. Let $Z_t := \mathcal{P} \mathcal{U}_t \mathcal{P}_1$, $t \geq 0$. Then, the family $\{Z_t\}$ $(t \geq 0)$ of operators is a strongly continuous semigroup of completely non-unitary contraction on $H$. The generator of this semigroup is defined by the formula

$$By = \lim_{t \to +0} \frac{1}{it}(Z_t y - y).$$

The domain of $B$ consists of all the vectors for which the limit exists. The operator $B$ is maximal dissipative. The operator $\mathbf{\Gamma}$ is called the self-adjoint dilation of $B$ (see [3, 11]). We next show that $\Upsilon_{\alpha_1 \alpha_2} = B$ and therefore $\mathbf{\Gamma}$ is self-adjoint dilation of $B$. For this purpose, it is sufficient to verify the equality (see [3, 11])

$$\mathcal{P}(\mathbf{\Gamma} - \lambda I)^{-1} \mathcal{P}_1 y = (\Upsilon_{\alpha_1 \alpha_2} - \lambda I)^{-1} y, \ y \in H, \ \operatorname{Im} \alpha_1 < 0. \tag{10}$$

Let $(\mathbf{\Gamma} - \lambda I)^{-1} \mathcal{P}_1 y = g = \langle \psi_-, z, \psi_+ \rangle$. Then, we have $(\mathbf{\Gamma} - \lambda I) g = \mathcal{P}_1 y$. Consequently, $\Gamma z - \lambda z = y$, $\psi_-(\xi) = \psi_-(0) e^{-i\lambda\xi}$ and $\psi_+(\xi) = \psi_+(0) e^{-i\lambda\xi}$. Since $g \in D(\mathbf{\Gamma})$, then $\psi_- \in W_2^1(-\infty, 0)$, it follows that $\psi_-(0) = 0$, and consequently $z$ satisfies the boundary condition $y_2^{\rho}(a) - \alpha_1 y_1(a) = 0$. Therefore $z \in D(\Upsilon_{\alpha_1 \alpha_2})$, and since point $\lambda$ with $\operatorname{Im} \lambda < 0$ cannot be an eigenvalue of dissipative operator, then $z = (\Upsilon_{\alpha_1 \alpha_2} - \lambda I)^{-1} y$. Thus

$$(\mathbf{\Gamma} - \lambda I)^{-1} \mathcal{P}_1 y = \langle 0, \ (\Upsilon_{\alpha_1 \alpha_2} - \lambda I)^{-1} y, \ \gamma^{-1}(y_2^{\rho}(a) - \overline{\alpha}_1 y_1(a)) e^{-i\lambda\xi} \rangle$$

for $y \in H$ and $\operatorname{Im} \lambda < 0$. By applying the mapping $\mathcal{P}$, we obtain (10). Furthermore, using (10), we get

$$(\Upsilon_{\alpha_1 \alpha_2} - \lambda I)^{-1} = \mathcal{P}(\mathbf{\Gamma} - \lambda I)^{-1} \mathcal{P}_1 = -i\mathcal{P} \int_0^{\infty} \mathcal{U}_t e^{-i\lambda t} dt \mathcal{P}_1$$

$$= -i\int_0^\infty Z_t e^{-i\lambda t}dt = (B-\lambda I)^{-1},\ \operatorname{Im}\lambda < 0,$$

i.e., $\Upsilon_{\alpha_1\alpha_2} = B$. □

On the other hand, the unitary group $\{\mathcal{U}_t\}$ has an important property which makes it possible to apply it to the Lax-Phillips (see [2]). In the following theorem, we will give its properties.

**Theorem 3.** *Let $M_- = \langle L^2(-\infty,0),0,0\rangle$ and $M_+ = \langle 0,0,L^2(0,\infty)\rangle$ be orthogonal incoming and outgoing subspaces of the unitary group $\{\mathcal{U}_t\}$, $t\in\mathbb{R}$. Then they have the following properties:*

$(i)$ $\mathcal{U}_t M_- \subset M_-$, $t\le 0$ and $\mathcal{U}_t M_+ \subset M_+$, $t\ge 0$;
$(ii)$ $\bigcap_{t\le 0}\mathcal{U}_t M_- = \bigcap_{t\ge 0}\mathcal{U}_t M_+ = \{0\}$;
$(iii)$ $M_- \perp M_+$.

*Proof.* $(i)$ For all $\lambda$, with $\operatorname{Im}\lambda<0$, we have

$$\mathcal{R}_\lambda f = (\mathbf{\Gamma}-\lambda I)^{-1} f = \langle 0,0,-ie^{-i\lambda\xi}\int_0^\xi e^{i\lambda s}\varphi_+(s)\,ds\rangle,\ f=\langle 0,0,\varphi_+\rangle\in M_+,$$

i.e., $\mathcal{R}_\lambda f\in M_+$. Furthermore, if $g\perp M_+$, then

$$0 = (\mathcal{R}_\lambda f,g)_{\mathcal{H}} = -i\int_0^\infty e^{-i\lambda t}(\mathcal{U}_t f,g)_{\mathcal{H}}\,dt,\ \operatorname{Im}\lambda<0.$$

which implies that $(\mathcal{U}_t f,g)_{\mathcal{H}}=0$ for all $t\ge 0$. Hence, for $t\ge 0$, $\mathcal{U}_t M_+\subset M_+$, the proof for $M_-$ is similar. □

$(ii)$ Let us define the mappings $\mathcal{P}^+:\mathcal{H}\to L^2(0,\infty)$ and $\mathcal{P}_1^+ : L^2(0,\infty)\to M_+$ as follows $\mathcal{P}^+:\langle\varphi_-,y,\varphi_+\rangle\to\varphi_+$ and $\mathcal{P}_1^+:\varphi\to\langle 0,0,\varphi\rangle$, respectively. We take into consideration that the semigroup of isometries $\mathcal{U}_t := \mathcal{P}^+\mathcal{U}_t\mathcal{P}_1^+$ $(t\ge 0)$ is a one-sided shift in $L^2(0,\infty)$. Indeed, the generator of the semigroup of the one-sided shift $V_t$ in $L^2(0,\infty)$ is the differential operator $i\frac{d}{d\xi}$ with the boundary condition $\varphi(0)=0$. On the other hand, the generator $S$ of the semigroup of isometries $\mathcal{U}_t$ $(t\ge 0)$ is the operator

$$S\varphi = \mathcal{P}^+\mathbf{\Gamma}\mathcal{P}_1^+\varphi = \mathcal{P}^+\mathbf{\Gamma}\langle 0,0,\varphi\rangle = \mathcal{P}^+\langle 0,0,i\frac{d\varphi}{d\xi}\rangle = i\frac{d\varphi}{d\xi},$$

where $\varphi\in W_2^1(0,\infty)$ and $\varphi(0)=0$. Since a semigroup is uniquely determined by its generator, it follows that $\mathcal{U}_t = V_t$, and, hence,

$$\bigcap_{t\ge 0}\mathcal{U}_t M_+ = \langle 0,0,\bigcap_{t\le 0} V_t L^2(0,\infty)\rangle = \{0\}.$$

$(iii)$ The proof is clear.

Now, we will give a definition and three lemmas to prove another property of incoming and outgoing subspaces of the unitary group $\{\mathcal{U}_t\}$, $t\in\mathbb{R}$.

**Definition 5.** [4] In the Hilbert space $H$, the linear operator $A$ (with domain $D(A)$) is called *simple (or completely non-self-adjoint)* if there is no invariant subspace $N \subseteq D(A)$ $(N \neq \{0\})$ of the operator $A$ on which the restriction $A$ to $N$ is self-adjoint.

**Lemma 2.** *The operator $\Upsilon_{\alpha_1\alpha_2}$ is simple.*

*Proof.* Suppose the assertion of the lemma is false. Then we could find a nontrivial subspace $H' \subset H$ such that $\Upsilon_{\alpha_1\alpha_2}$ induces a self-adjoint operator $\Upsilon'_{\alpha_1\alpha_2}$ with domain $D(\Upsilon'_{\alpha_1\alpha_2}) = H' \cap D(\Upsilon_{\alpha_1\alpha_2})$. If $y \in D(\Upsilon'_{\alpha_1\alpha_2})$, then $y \in D(\Upsilon'^{*}_{\alpha_1\alpha_2})$ and

$$y_2^{\rho}(a) - \alpha_1 y_1(a) = 0, \ y_2^{\rho}(a) - \overline{\alpha_1} y_1(a) = 0,$$

$$y_2^{\rho}(b) + \alpha_2 y_1(b) = 0.$$

Since the eigenfunctions of the operator $\Upsilon_{\alpha_1\alpha_2}$ lie in $H'$ and are eigenfunctions of the operator $\Upsilon'_{\alpha_1\alpha_2}$, we have $y_2^{\rho}(a) = y_1(a) = 0$. By the uniqueness theorem of the Cauchy problem for the equation $\Gamma y = \lambda y$, we obtain $y(x,\lambda) \equiv 0$. Hence, the resolvent $R_\lambda(\Upsilon_{\alpha_1\alpha_2})$ of the operator $\Upsilon_{\alpha_1\alpha_2}$ is a compact operator, and the spectrum of $\Upsilon_{\alpha_1\alpha_2}$ is purely discrete. Consequently, by the theorem on expansion in the eigenvectors of the self-adjoint operator $\Upsilon'_{\alpha_1\alpha_2}$, we obtain $H' = \{0\}$. This contradicts our assumption. □

Now, let us define $H_- = \overline{\underset{t\geq 0}{\cup} \mathcal{U}_t M_-}$, $H_+ = \overline{\underset{t\leq 0}{\cup} \mathcal{U}_t M_+}$. Then, we have a

**Lemma 3.** *The equality $H_- + H_+ = \mathcal{H}$ holds.*

*Proof.* From Theorem 7, we show that the subspace $\acute{\mathcal{H}} = \mathcal{H} \ominus (H_- + H_+)$ is invariant relative to the group $\{\mathcal{U}_t\}$ and has the form $\acute{\mathcal{H}} = \langle 0, \acute{H}, 0\rangle$, where $\acute{H}$ is a subspace in $H$. Therefore, if the subspace $\acute{\mathcal{H}}$ (and hence also $\acute{H}$) were nontrivial, then the unitary group $\{\mathcal{U}_t\}$ restricted to this subspace would be a unitary part of the group $\{\mathcal{U}_t\}$, and hence, the restriction $\acute{\Upsilon}_{\alpha_1\alpha_2}$ of $\Upsilon_{\alpha_1\alpha_2}$ to $\acute{H}$ would be a self-adjoint operator in $\acute{H}$. Then, it follows that $\acute{\mathcal{H}} = \{0\}$, since the operator $\Upsilon_{\alpha_1\alpha_2}$ is simple. □

Assume that

$$\varphi(x,\lambda) = \begin{pmatrix} \varphi_1(x,\lambda) \\ \varphi_2(x,\lambda) \end{pmatrix}, \ \psi(x,\lambda) = \begin{pmatrix} \psi_1(x,\lambda) \\ \psi_2(x,\lambda) \end{pmatrix}$$

are solutions of $\Gamma y = \lambda y$ satisfying the conditions

$$\varphi_1(a,\lambda) = 0, \ \varphi_2^{\rho}(a,\lambda) = -1, \ \psi_1(a,\lambda) = 1, \ \psi_2^{\rho}(a,\lambda) = 0.$$

The Titchmarsh-Weyl function $m_{\infty,\alpha_2}(\lambda)$ of the self-adjoint operator $\Upsilon_{\infty,\alpha_2}$ generated by the boundary conditions $y_1(a) = 0$, $y_2^{\rho}(b) + \alpha_2 y_1(b) = 0$ is determined by the condition

$$\psi_2^{\rho}(b) + m_{\infty,\alpha_2}(\lambda)\varphi_2^{\rho}(b) - \alpha_2\left[\psi_1(b) + m_{\infty,\alpha_2}(\lambda)\varphi_1(b)\right] = 0.$$

Hence, we have

$$m_{\infty,\alpha_2}(\lambda) = -\frac{\psi_2^{\rho}(b) + \alpha_2\psi_1(b)}{\varphi_2^{\rho}(b) + \alpha_2\varphi_1(b)}. \tag{11}$$

Note that the Weyl-Titchmarsh function $m_{\infty,\alpha_2}(\lambda)$ is a meromorphic function on $\mathbb{C}$, and is a holomorphic function with $\operatorname{Im}\lambda \neq 0$, $\operatorname{Im}\lambda \operatorname{Im} m_{\infty,\alpha_2}(\lambda) > 0$ and $\overline{m_{\infty,\alpha_2}(\lambda)} = m_{\infty,\alpha_2}(\overline{\lambda})$. Then $m_{\infty,\alpha_2}(\lambda)$ has a countable number of isolated poles on the real axis, these poles are the eigenvalues of the self-adjoint operator $\Upsilon_{\infty,\alpha_2}$, and the operator $\Upsilon_{\infty,\alpha_2}$ (also every self-adjoint extension of the symmetric operator $\Upsilon_{\min}$) has a purely discrete spectrum ([7, 15, 16]).

We set

$$\Omega_{\lambda}^{-}(x,\xi,\zeta) = \langle e^{-i\lambda\xi},\ \frac{1}{m_{\infty,\alpha_2}(\lambda) - \alpha_1}\gamma\varkappa(x,\lambda),\ \overline{K_{\alpha_1\alpha_2}}(\lambda)\, e^{-i\lambda\zeta}\rangle, \tag{12}$$

$$\Omega_{\lambda}^{+}(x,\xi,\zeta) = \langle K_{\alpha_1\alpha_2}(\lambda)\, e^{-i\lambda\xi},\ \frac{1}{m_{\infty,\alpha_2}(\lambda) - \overline{\alpha}_1}\gamma\varkappa(x,\lambda),\ e^{-i\lambda\zeta}\rangle, \tag{13}$$

where

$$\varkappa(x,\lambda) = \psi(x,\lambda) + m_{\infty,\alpha_2}(\lambda)\,\varphi(x,\lambda),$$
$$K_{\alpha_1\alpha_2}(\lambda) = \frac{m_{\infty,\alpha_2}(\lambda) - \alpha_1}{m_{\infty,\alpha_2}(\lambda) - \overline{\alpha_1}}. \tag{14}$$

It is clear that the vectors $\Omega_{\lambda}^{\mp}(x,\xi,\zeta)$ for real $\lambda$ do not belong to the space $\mathcal{H}$. However, $\Omega_{\lambda}^{\mp}(x,\xi,\zeta)$ satisfies the equation $\mathbf{\Gamma}U = \lambda U$ and the corresponding boundary conditions for the operator $\mathbf{\Gamma}$.

**Lemma 4.** *Let us define the transformation* $F_{\mp} : f \to \widetilde{f}_{\mp}(\lambda)$ *by*

$$(F_{-}f)(\lambda) := \widetilde{f}_{-}(\lambda) := \frac{1}{\sqrt{2\pi}}(f, \Omega_{\overline{\lambda}}^{-})_{\mathcal{H}},$$
$$(F_{+}f)(\lambda) := \widetilde{f}_{+}(\lambda) := \frac{1}{\sqrt{2\pi}}\left(f, \Omega_{\lambda}^{+}\right)_{\mathcal{H}},\ f = \langle\varphi_{-}, y, \varphi_{+}\rangle$$

*where $\varphi_{-}$, $\varphi_{+}$, $y$ are smooth, compactly supported functions. Then the transformation $F_{\mp}$ isometrically maps $H_{\mp}$ onto $L^2(\mathbb{R})$. For all vectors $f$, $g \in H_{\mp}$, the Parseval equality and the inversion formulae hold:*

$$(f,g)_{\mathcal{H}} = (\widetilde{f}_{-}, \widetilde{g}_{-})_{L^2}$$

$$= \int_{-\infty}^{\infty} \widetilde{f}_{-}(\lambda)\overline{\widetilde{g}_{-}(\lambda)}d\lambda,\ f = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widetilde{f}_{-}(\lambda)\,\Omega_{\overline{\lambda}}^{-}d\lambda, \tag{15}$$

$$(f,g)_{\mathcal{H}} = (\widetilde{f}_{+}, \widetilde{g}_{+})_{L^2}$$

$$= \int_{-\infty}^{\infty} \widetilde{f_+}(\lambda) \overline{\widetilde{g_+}(\lambda)} d\lambda, \; f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widetilde{f_+}(\lambda) \Omega_\lambda^+ d\lambda, \tag{16}$$

*where* $\widetilde{f_-}(\lambda) = (F_- f)(\lambda)$, $\widetilde{g_-}(\lambda) = (F_- g)(\lambda)$, $\widetilde{f_+}(\lambda) = (F_+ f)(\lambda)$ *and* $\widetilde{g_+}(\lambda) = (F_+ g)(\lambda)$.

*Proof.* We will just prove the formula (15) since the proof of (16) is similar. By Paley-Wiener theorem, we have

$$\widetilde{f_-}(\lambda) = \frac{1}{\sqrt{2\pi}} (f, \Omega_{\overline{\lambda}})_{\mathcal{H}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \varphi_-(\xi) e^{-i\lambda\xi} d\xi \in H_-^2,$$

where $f, g \in M_-, f = \langle \varphi_-, 0, 0 \rangle, g = \langle \psi_-, 0, 0 \rangle$. Using Parseval equality for Fourier integrals, we obtain

$$(f,g)_{\mathcal{H}} = \int_{-\infty}^{\infty} \varphi_-(\xi) \overline{\psi_-(\xi)} d\xi = \int_{-\infty}^{\infty} \widetilde{f_-}(\lambda) \overline{\widetilde{g_-}(\lambda)} d\lambda = (F_- f, F_- g)_{L^2},$$

where $H_\pm^2$ denote the Hardy classes in $L^2(\mathbb{R})$ consisting of the functions analytically extendible to the upper and lower half-planes, respectively. Now, we extend the Parseval equality to the whole of $H_-$. We consider in $H_-$ the dense set of $\acute{H_-}$ of the vectors obtained as follows from the smooth, compactly supported functions in $M_- : f \in \acute{H_-}$ if $f = \mathcal{U}_t \; f_0$, $f_0 = \langle \varphi_-, 0, 0 \rangle$, $\varphi_- \in C_0^\infty(-\infty, 0)$, where $T = T_f$ is a nonnegative number depending on $f$. If $f, g \in \acute{H_-}$, then for $T > T_f$ and $T > T_g$ we have $U_{-T} f, U_{-T} g \in M_-$, moreover, the first components of these vectors belong to $C_0^\infty(-\infty, 0)$. Therefore, since the operators $\mathcal{U}_t$ $(t \in \mathbb{R})$ are unitary, by the equality

$$F_- \mathcal{U}_t f = (\mathcal{U}_t f, \Omega_{\overline{\lambda}})_{\mathcal{H}} = e^{i\lambda t} (f, \Omega_{\overline{\lambda}})_{\mathcal{H}} = e^{i\lambda t} F_- f,$$

we have

$$(f,g)_{\mathcal{H}} = (\mathcal{U}_{-T} f, \mathcal{U}_{-T} g)_{\mathcal{H}} = (F_- \mathcal{U}_{-T} f, F_- \mathcal{U}_{-T} g)_{L^2}$$

and

$$(e^{i\lambda T} F_- f, e^{i\lambda T} F_- g)_{L^2} = (\widetilde{f}, \widetilde{g})_{L^2}. \tag{17}$$

By taking the closure (17), we obtain the Parseval equality for the space $H_-$. The inversion formula is obtained from the Parseval equality if all integrals in it are considered as limits in the integrals over finite intervals. Finally

$$F_- H_- = \overline{\bigcup_{t \geq 0} F_- \mathcal{U}_t M_-} = \overline{\bigcup_{t \geq 0} e^{i\lambda t} H_-^2} = L^2(\mathbb{R}),$$

that is $F_-$ maps $H_-$ onto the whole of $L^2(\mathbb{R})$. The lemma is proved. □

It is immediate that the function $K_{\alpha_1\alpha_2}(\lambda)$ is meromorphic in $\mathbb{C}$ and all poles are in the lower half-plane. From (14), $|K_{\alpha_1\alpha_2}(\lambda)| \leq 1$ for all $\lambda \in \mathbb{R}$. Hence, it explicitly follows from the formulae for the vectors $\Omega_\lambda^-$ and $\Omega_\lambda^+$ that

$$\Omega_\lambda^+ = K_{\alpha_1\alpha_2}(\lambda)\,\Omega_\lambda^-. \tag{18}$$

Moreover, $H_- = H_+$. Together with Lemma 10, this shows that $H_- = H_+ = \mathcal{H}$.

Summarizing, we have proved the following theorem for the incoming and outgoing subspaces (i.e., for the spaces $M_-$ and $M_+$).

**Theorem 4.**

$$\overline{\underset{t\geq 0}{\cup}\mathcal{U}_t M_-} = \overline{\underset{t\leq 0}{\cup}\mathcal{U}_t M_+} = \mathcal{H}.$$

Thus, the transformation $F_-$ isometrically maps $H_-$ onto $L^2(\mathbb{R})$ with the subspace $M_-$ mapped onto $H_-^2$ and the operators $\mathcal{U}_t$ are transformed into the operators of multiplication by $e^{i\lambda t}$. This means that $F_-$ is the incoming spectral representation for the group $\{\mathcal{U}_t\}$. Similarly, $F_+$ is the outgoing spectral representation for the group $\{\mathcal{U}_t\}$. It follows from (18) that the passage from the $F_-$ representation of an element $f \in \mathcal{H}$ to its $F_+$ representation is accomplished as $\widetilde{f}_+(\lambda) = K_{\alpha_1\alpha_2}(\lambda)\,\widetilde{f}_-(\lambda)$. Consequently, according to [2], we have proved the following.

**Theorem 5.** *The scattering function of the group $\{\mathcal{U}_t\}$ is the function $\overline{K_{\alpha_1\alpha_2}}(\lambda)$ i.e., the scattering function of the self-adjoint operator $\mathbf{\Gamma}$ is the function $\overline{K_{\alpha_1\alpha_2}}(\lambda)$.*

## 4 Functional model of the maximal dissipative Dirac operators

In this section, we construct a functional model of the maximal dissipative Dirac operators on time scales with the help of incoming spectral representation. We will determine characteristic function of this operator and prove that all root vectors of the maximal dissipative Dirac operator are complete. The characteristic function carries important information regarding the spectral properties of operators. We know that the absence of the singular factor in the factorization of the characteristic function guarantees the completeness of the system of root vectors of maximal dissipative operators [3].

Now, we will give some definitions.

**Definition 6.** [3] The analytic function $S(\lambda)$ on the upper half-plane $\mathbb{C}_+$ is called *inner function* on $\mathbb{C}_+$ if $|S(\lambda)| \leq 1$ for all $\lambda \in \mathbb{C}_+$ and $|S(\lambda)| = 1$ for almost all $\lambda \in \mathbb{R}$.

**Definition 7.** [3] Let us define $\Psi = H_+^2 \ominus SH_+^2$, where $S(\lambda)$ be an arbitrary nonconstant inner function on the upper half-plane. It is obvious that $\Psi \neq \{0\}$ is a subspace of the Hilbert space $H_+^2$. We consider the semigroup of operators $Z_t$ $(t \geq 0)$ acting in $\Psi$ according to the formula

$$Z_t\varphi = \mathcal{P}\left[e^{i\lambda t}\varphi\right], \varphi = \varphi(\lambda) \in \Psi,$$

where $\mathcal{P}$ is the orthogonal projection from $H_+^2$ onto $\Psi$. The generator of the semigroup $\{Z_t\}$ is denoted by

$$T\varphi = \lim_{t\to+0} (it)^{-1} (Z_t\varphi - \varphi),$$

which $T$ is a maximal dissipative operator acting in $\Psi$ and with the domain $D(T)$ consisting of all functions $\varphi \in \Psi$, such that the limit exists. The operator $T$ is called *a model dissipative operator.*

Recall that this model dissipative operator, which is associated with the names of Lax-Phillips [2], is a special case of a more general model dissipative operator constructed by Nagy and Foiaş [3]. The basic assertion is that $S(\lambda)$ is the *characteristic function* of the operator $T$.

Let $V = \langle 0, H, 0\rangle$, so that $\mathcal{H} = M_- \oplus V \oplus M_+$. It follows from the explicit form of the unitary transformation $F_-$ under the mapping $F_-$

$$\begin{aligned}&\mathcal{H}\to L^2(\mathbb{R}), f \to \widetilde{f}_-(\lambda) = (F_- f)(\lambda), M_- \to H_-^2, M_+ \to K_{\alpha_1\alpha_2} H_+^2,\\ &V \to H_+^2 \ominus K_{\alpha_1\alpha_2} H_+^2,\ \mathcal{U}_t \to (F_- \mathcal{U}_t F_-^{-1} \widetilde{f}_-)(\lambda) = e^{i\lambda t}\widetilde{f}_-(\lambda).\end{aligned} \tag{19}$$

The formulas (19) show that operator $\Upsilon_{\alpha_1\alpha_2}$ is a unitarily equivalent to the model dissipative operator with the characteristic function $K_{\alpha_1\alpha_2}(\lambda)$. We have thus proved following theorem.

**Theorem 6.** *The characteristic function of the maximal dissipative operator $\Upsilon_{\alpha_1\alpha_2}$ coincides with the function $K_{\alpha_1\alpha_2}(\lambda)$ defined by (14).*

**Theorem 7.** *For all the values of $\alpha_1$ with $\operatorname{Im}\alpha_1 > 0$, except possibly for a single value $\alpha_1 = \alpha_1^0$ and for fixed $\alpha_2$ ($\operatorname{Im}\alpha_2 = 0$ or $\alpha_2 = \infty$), the characteristic function $K_{\alpha_1\alpha_2}(\lambda)$ of the maximal dissipative operator $\Upsilon_{\alpha_1\alpha_2}$ is a Blaschke product. The spectrum of $\Upsilon_{\alpha_1\alpha_2}$ is purely discrete and belongs to the open upper half-plane. The operator $\Upsilon_{\alpha_1\alpha_2}$ $(\alpha_1 \neq \alpha_1^0)$ has a countable number of isolated eigenvalues with finite multiplicity and limit points at infinity. The system of all eigenvectors and associated vectors (or all root vectors) of the operator $\Upsilon_{\alpha_1\alpha_2}$ is complete in the space $H$.*

*Proof.* It is obvious that $K_{\alpha_1\alpha_2}(\lambda)$ is an inner function in the upper half-plane, and it is meromorphic in the whole complex $\lambda$-plane. Therefore, we can say

$$K_{\alpha_1\alpha_2}(\lambda) = e^{i\lambda c} B_{\alpha_1\alpha_2}(\lambda),\ c = c(\alpha_1) \geq 0, \tag{20}$$

where $B_{\alpha_1\alpha_2}(\lambda)$ is a Blaschke product. Hence, we get

$$|K_{\alpha_1\alpha_2}(\lambda)| \leq e^{-c(\alpha_1)\operatorname{Im}\lambda},\ \operatorname{Im}\lambda \geq 0. \tag{21}$$

From (14), we obtain

$$m_{\infty,\alpha_2}(\lambda) = \frac{\overline{\alpha_1} K_{\alpha_1\alpha_2}(\lambda) - \alpha_1}{K_{\alpha_1\alpha_2}(\lambda) - 1}. \tag{22}$$

If $c(\alpha_1) > 0, (\operatorname{Im}\alpha_1 > 0)$, then (21) implies that

$$\lim_{x\to+\infty} K_{\alpha_1\alpha_2}(ix) = 0,$$

and then (22) gives us that

$$\lim_{x\to+\infty} m_{\infty,\alpha_2}(ix) = \alpha_1^0.$$

Since $m_{\infty,\alpha_2}(\lambda)$ does not depend on $\alpha_1$, this implies that $c(\alpha_1)$ can be nonzero at not more than a single point $\alpha_1 = \alpha_1^0$. □

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