BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Numbers 2(108)–3(109), 2025, Pages 21–32
ISSN 1024–7696, E-ISSN 2587–4322

On the Asymptotic Behavior of Solutions of the Cauchy Problem for a Hyperbolic Equation with Periodic Coefficients

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Abstract. The paper establishes the asymptotic behavior as $t \to \infty$ and the principle of limiting amplitude of solutions to the Cauchy problem for a second-order hyperbolic equation with periodic coefficients for large values of the time parameter t, with zero initial data, where the right side of the equation is the function $f(x)\,\exp\{-i\,\omega\,t\}$, $\omega > 0$. To obtain an asymptotic expansion as $t \to \infty$, the basic methods of the spectral theory of differential operators are used, as well as the properties of the spectrum of the Hill operator with periodic coefficients.

Mathematics subject classification: 35B10, 35B40, 35C20, 35L10, 35Q41.
Keywords and phrases: asymptotic behavior of solutions, hyperbolic equation, periodic coefficients, Cauchy problem, Hill operator.

1 Introduction

As is known, for some areas of theoretical physics, such as wave mechanics, the theory of oscillations, etc., the solution of problems is reduced to the problem of eigenvalues. In addition, the question of the unambiguous definition of a mechanical system, i.e., the Hamilton function, through the spectrum of eigenvalues of the linear differential equation associated with it is important.

In the case when the string is vibrating and the boundary conditions are natural, it was shown in [1] that the spectrum of eigenvalues uniquely determines the differential equation which in Schrödinger's theory is called the "amplitude equation".

Paper [2] deals with the problem of determining the Hill equation (or the one-dimensional Schrödinger equation) from its spectrum, as well as deriving the Hill equation from specific properties of its discriminant. A great deal is known about the analytic structure of the discriminant (see, for example, [3, 4]) .

We study the behavior as $x \in [-b, b]$ and $t \to \infty$ of the solution to the following Cauchy problem

$$u_{tt}(x,t) - (p(x)\,u_x(x,t))_x + q(x)\,u(x,t) = f(x)e^{-i\,\omega\,t}, \quad (x,t) \in \mathbb{R} \times \{t > 0\}, \quad (1)$$

$$u(x,t)|_{t=0} = 0, \quad u_t(x,t)|_{t=0} = 0, \quad x \in \mathbb{R}, \quad (2)$$

where $p(x)$ and $q(x)$ are 1-periodic functions,

$$p(x+1) = p(x) \geq \text{const} > 0, \quad q(x+1) = q(x) \geq 0.$$

DOI: https://doi.org/10.56415/basm.y2025.i2-3.p21

Assume that the functions $p(x)$ and $q(x)$ are continuous or have a finite number of discontinuities of the first kind on the period, $f \in C_0^\infty(\mathbb{R})$, $\operatorname{supp} f \subset [0,1]$, $\omega \geq 0$ is a real number, b is an arbitrary fixed constant.

The behavior (as $t \to \infty$) of solutions to problems similar to the problem (1), (2) with $p(x) = 1$, and of the corresponding multidimensional problems under the condition that the potential differs from a constant by a finite function tends to a constant sufficiently fast at infinity, has been studied in many papers, see, for example, [5], and the bibliography there, as well as other papers.

In this regard, we note the paper [6], in which was received the asymptotic expansion (for $t \to \infty$ and $|x| < a < \infty$) of the solution $u(x,t)$ to the following Cauchy problem

$$u_{tt}(x,t) - u_{xx}(x,t) + (\alpha_0 + q_0(x))u = 0, \quad (x,t) \in \mathbb{R} \times \{t > 0\},$$

$$u(x,t)|_{t=0} = \varphi(x), \quad u_t(x,t)|_{t=0} = \psi(x), \quad x \in \mathbb{R},$$

where the initial functions are finite, $\varphi(x) \in C^2(\mathbb{R})$, $\psi(x) \in C^1(\mathbb{R})$, and under weaker restrictions on the potential $q(x) = \alpha_0 + q_0(x)$, where $\alpha_0 = \text{const}$ and $q_0(x)$ is a real-valued continuous function for all $x \in \mathbb{R}$, and for some $k \geq 1$,

$$\int_{-\infty}^{+\infty} |x|^k |q_0(x)| < \infty$$

is satisfied the condition.

The papers [7] and [8] studied the following Cauchy problem

$$u_{tt}(x,t) - (a(x)\, u_x(x,t))_x = 0, \quad 0 < a_0 \leq a(x) \leq A_0 < +\infty, \quad (x,t) \in \mathbb{R} \times \{t > 0\},$$

$$u(x,t)|_{t=0} = \varphi(x), \quad u_t(x,t)|_{t=0} = 0, \quad x \in \mathbb{R},$$

for which, under certain assumptions on the tension coefficient $a(x)$, such as

$$\frac{1}{a_0} \int_{-\infty}^{+\infty} |a'(x)|\, dx < 1,$$

sufficient conditions for the stabilization of the solution $u(x,t)$ as $t \to +\infty$ uniformly in x on any compact set, as well as necessary and sufficient conditions for the stabilization of the solution $u(x,t)$ in the mean were obtained.

In [9] the asymptotic behavior as $t \to \infty$ of the solution of the initial-boundary value problem for a hyperbolic equation in the following formulation

$$a(x)u_{tt}(x,t) = u_{xx}(x,t), \quad 0 < a_0 \leq a(x) \leq A < +\infty, \quad x > 0,\, t > 0,$$

$$u(x,t)|_{t=0} = f(x), \quad u_t(x,t)|_{t=0} = g(x), \quad x \geq 0,$$

$$u(x,t)|_{x=0} = 0, \quad t \geq 0,$$

is studied, and it follows from the asymptotic expansion obtained that the solution to the problem under study decreases exponentially in t uniformly in x on any compact set as $t \to \infty$.

This article is devoted to the asymptotic behavior as $t \to \infty$ of solutions to the Cauchy problem for a non-homogeneous second-order hyperbolic equation with periodic coefficients, as well as to establishing the principle of the limiting amplitude of this problem, with zero initial data, and with the right side of the equation being the function $f(x)\exp\{-i\,\omega\,t\}, \quad \omega > 0$.

In the papers [10] and [11], the Cauchy problem for a second-order homogeneous hyperbolic equation with periodic coefficients was considered, and an asymptotic expansion was obtained for $t \to \infty$, as in the case of the positive Hill operator $H_0 > 0$, and also in the case when the left end of the spectrum $\sigma(H_0)$ of the Hill operator H_0 is non-positive. Similar problems were considered in [12] with $p(x) = 1$, that is, in the case of a periodic potential $q(x)$.

We also note the papers [13] and [14], in which the asymptotic behavior as $t \to \infty$ of solutions of the initial-boundary value problem for a second-order hyperbolic equation with periodic coefficients on the semi-axis was obtained.

The article [15] studies the spectral properties of the perturbed Hill operator. In particular, it is proven that the perturbed Hill operator with an exponentially decreasing impurity potential in each sufficiently distant lacuna on the "non-physical" sheet has an odd number of resonances.

2 Preliminaries and Auxiliary Statements

2.1 Green's function and spectrum of the Hill operator

Continuing the function $u(x,t)$ by zero in the region $t < 0$ and applying the Fourier transform with respect to the variable t in the Cauchy problem (1) and (2), for the function

$$y(x,k) = \int_0^{\infty} u(x,t)\, e^{ikt} dt$$

we obtain the equation

$$(p(x)\, y'(x,k))' + (k^2 - q(x))\, y(x,k) = -\frac{f(x)}{k-\omega}. \tag{3}$$

If the function $g(x)$ is defined on the entire axis $(-\infty, +\infty)$, then by $\hat{g}(x)$ we denote the restriction of this function on the segment $[0,1]$. For any function $g(x,k)$ we denote by g' the derivative with respect to x and by g_k the derivative with respect to k.

Let us present some necessary facts from the spectral theory of differential equations.

Let $\{y = \theta(x,k),\, y = \varphi(x,k)\}$ be the fundamental system of solutions of the homogeneous (for $f(x) \equiv 0$) equation (3) such that

$$\begin{cases} \theta(0,k) = 1, & \theta'(0,k) = 0, \\ \varphi(0,k) = 0, & \varphi'(0,k) = 1. \end{cases}$$

It is known [16] that $\theta(x,k)$ and $\varphi(x,k)$ are entire functions in k, real on the real axis, and for $|k|\to\infty$, they have the form

$$\begin{cases} \theta(x,k) = \qquad \cos kx + O(|k|^{-1}e^{|\tau|x}), \\ \varphi(x,k) = \frac{1}{k}\sin kx + O(|k|^{-2}e^{|\tau|x}), \quad \tau = Im\, k, \end{cases}$$

uniformly in $x\in[-b,b]$. These expansions can be differentiated with respect to x and with respect to k.

Let us denote $\theta(k)=\theta(1,k)$, $\theta'(k)=\theta'(1,k)$, $\varphi(k)=\varphi(1,k)$, $\varphi'(k)=\varphi'(1,k)$ and $F(k)\equiv\theta(k)+\varphi'(k)$. The functions $\theta(k)$, $\theta'(k)$, $\varphi(k)$, $\varphi'(k)$ and $F(k)$ are even on the real axis of the complex plane of the variable k.

The Hill operator is the differential operator

$$H_0 := -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x),$$

generated in the Hilbert space $L^2(\mathbb{R})$ by the operation

$$\Lambda_0 y := -(p(x)\,y')' + q(x)\,y,$$

where the functions $p(x)$ and $q(x)$ are periodic with period 1.

The spectrum $\sigma(H_0)$ of the Hill operator H_0 is absolutely continuous and is a finite or infinite sequence of isolated segments (zones) separated by lacunae going to infinity.

Note that the Hill operator has only a continuous spectrum, which lies on the real axis and is left semi-bounded [16].

For a more detailed characterization of the spectrum $\sigma(H_0)$ of the Hill operator H_0, consider the following periodic (anti-periodic) Sturm-Liouville problems.

Let $\hat{v}(x,\lambda_n)$ be an eigenfunction of the periodic Sturm–Liouville problem:

$$\begin{gathered} -(p(x)y')' + q(x)y = \lambda_n y, \quad x\in[0,1], \\ y(0)=y(1), \quad y'(0)=y'(1), \end{gathered} \tag{4}$$

normalized by the condition $||\hat{v};L^2([0,1])||=1$, and $\hat{v}(x,\mu_n)$ is the eigenfunction of the anti-periodic Sturm–Liouville problem:

$$\begin{gathered} -(p(x)y')' + q(x)y = \mu_n y, \quad x\in[0,1], \\ y(0)=-y(1), \quad y'(0)=-y'(1). \end{gathered} \tag{5}$$

normalized by the condition $||\hat{v};L^2([0,1])||=1$, where $\lambda_n=\lambda_n^2$ and $\mu_n=\mu_n^2$, $n=0,1,2,..$, are eigenvalues of the corresponding problems, which are numbered in ascending order, taking into account the multiplicity.

Continuing the function $\hat{v}(x,\lambda_n)$ (or $\hat{v}(x,\mu_n)$) to the entire real axis, in a periodic (or anti-periodic) way, we get a function, which we denote by $v(x,\lambda_n)$ (or $v(x,\mu_n)$).

It is known [16] (§ 21.4) that if the Hill operator H_0 is positive, then all eigenvalues of the periodic (anti-periodic) Sturm–Liouville problem are positive. In addition, between the numbers $\lambda_n = \lambda_n^2$ and $\mu_n = \mu_n^2$, $n = 0, 1, 2, ..,$ there is a relation

$$\lambda_0 < \mu_0 \leq \mu_1 < \lambda_1 \leq \lambda_2 < \mu_2 \leq \mu_3 < \lambda_3 \leq \cdots . \tag{6}$$

The set of points $\pm\lambda_n$ coincides with the set of roots of the equation $F(k) = 2$ (correspondingly, $\pm\mu_n$ coincides with the set of roots of the equation $F(k) = -2$), $n = 0, 1, 2, ...$

Gaps in the spectrum, that is, intervals not included in the spectrum,

$$(-\mu_{2n+1}, -\mu_{2n}), (-\lambda_{2n+2}, -\lambda_{2n+1}), (\mu_{2n}, \mu_{2n+1}), (\lambda_{2n+1}, \lambda_{2n+2}), n = 0, 1, 2. \dots ,$$

for which $\mu_{2n} \neq \mu_{2n+1}$ and $\lambda_{2n+1} \neq \lambda_{2n+2}$, are called *lacunae.*

If $\lambda_n = \lambda_n^2$ (or $\mu_n = \mu_n^2$) are ends of a lacunae, then (6) implies that $\pm\lambda_n$ are simple roots of the equation $F(k) = 2$ (or $\pm\mu_n$ are the roots of the equation $F(k) = -2$), $n = 0, 1, 2, ..,$ [17].

Note that each lacuna contains exactly one zero of the function $F_k(k)$, and the functions $\varphi(k)$ and $\theta'(k)$ have one simple zero of the functions in the closure of each lacuna.

If $\lambda = \lambda_{2n} = \lambda_{2n+1}$ (or $\mu = \mu_{2n} = \mu_{2n+1}$), $n \geq 0$, then λ (or μ) is the simple zero of the functions $\varphi(k)$ and $\theta'(k)$ [16].

Let us replace the spectral parameter λ by k^2 so that the spectrum $\sigma(H_0)$ of the operator H_0 on the complex plane of the variable k consists of points for which $H_0 - k^2$ does not have bounded inverse on an everywhere dense set in $L^2(\mathbb{R})$.

Denote by $\mathbb{C}'$ the complex plane of the variable k with cuts along the vertical rays lying in the lower half-plane and starting at the ends of the lacunae.

Let us put

$$m_1(k) = \frac{\varphi'(k) - \theta(k)}{2\varphi(k)} + \frac{\sqrt{(\theta(k) + \varphi'(k))^2 - 4}}{2\varphi(k)}, \quad k \in \mathbb{C}',$$

$$m_2(k) = \frac{\varphi'(k) - \theta(k)}{2\varphi(k)} - \frac{\sqrt{(\theta(k) + \varphi'(k))^2 - 4}}{2\varphi(k)}, \quad k \in \mathbb{C}',$$

where the branch of the root is determined by the condition $\sqrt{F(k)^2 - 4} > 0$ for $k = 0$.

Note that the function $\sqrt{F(k)^2 - 4}$ has branching only at the ends of the lacuna [16], so $m_1(k)$ and $m_2(k)$ are single-valued in $\mathbb{C}'$. Then for any $k, Im\, k > 0$

$$\begin{aligned} \psi_1(x, k) \equiv \quad & \theta(x, k) + m_1(k)\, \varphi(x, k) \in L^2(-\infty, 0), \\ \psi_2(x, k) \equiv \quad & \theta(x, k) + m_2(k)\, \varphi(x, k) \in L^2(0, +\infty). \end{aligned} \tag{7}$$

We define the Green's function of the equation (3) for k from the upper half-plane

$$\Gamma(x, \xi, k) = \begin{cases} \frac{\psi_1(\xi,k)\psi_2(x,k)}{m_2(k)-m_1(k)} & \text{for} \quad \xi < x, \\ \frac{\psi_1(x,k)\psi_2(\xi,k)}{m_2(k)-m_1(k)} & \text{for} \quad \xi > x, \end{cases}$$

and, taking into account the identities (7) and the equality

$$\theta(x,k)\,\varphi'(x,k)-\theta'(x,k)\,\varphi(x,k)=1 \quad \text{for} \quad x\in\mathbb{R},$$

we get

$$-\Gamma(x,\xi,k)=\frac{h(x,\xi,k)}{\sqrt{(\theta(k)+\varphi'(k))^2-4}}+T(x,\xi,k), \tag{8}$$

where

$$\begin{aligned} h(x,\xi,k) &= \varphi(k)\,\theta(x,k)\,\theta(\xi,k)-\theta'(k)\,\varphi(\xi,k)\varphi(x,k)+ \\ &+\tfrac{\varphi'(k)-\theta(k)}{2}\,(\theta(\xi,k)\,\varphi(x,k)+\theta(x,k)\,\varphi(\xi,k)) \end{aligned} \tag{9}$$

and

$$T(x,\xi,k)=\begin{cases} -\frac{1}{2}\,(\theta(\xi,k)\,\varphi(x,k)-\theta(x,k)\,\varphi(\xi,k)) & \text{for} \quad \xi<x, \\ -\frac{1}{2}\,(\theta(x,k)\,\varphi(\xi,k)-\theta(\xi,k)\,\varphi(x,k)) & \text{for} \quad \xi>x. \end{cases} \tag{10}$$

The solution to the problem (1), (2) for $x\in[-b,b]$ and $t>0$ has the form

$$u(x,t)=\frac{1}{2\pi}\int\limits_{Im\,k=a}\frac{(H_0-k^2)^{-1}(f)}{k-\omega}\,e^{-ikt}\,dk,$$

where $a>0$ is some constant, and

$$(H_0-k^2)^{-1}(f)=-\int_0^1\Gamma(x,\xi,k)\,f(\xi)\,d\xi, \quad Im\,k>0. \tag{11}$$

From formulas (8) and (11) it follows that the function $(H_0-k^2)^{-1}$ for each $x,\xi\in[-b,b]$ is analytically continued in $\mathbb{C}'$.

Let us denote by $U(x,\omega)$ the solution to the following equation:

$$(p(x)\,y'(x,\omega))'+(\omega^2-q(x))\,y(x,\omega)=-f(x).$$

Consider the cases:

1) If ω^2 does not belong to the spectrum $\sigma(H_0)$ of the operator H_0 on the complex plane λ, then as a solution $U(x,\omega)$ we take

$$U(x,\omega)=(H_0-\omega^2)^{-1}(f), \tag{12}$$

that is, the solution $U(x,\omega)$ belongs to $L^2(\mathbb{R})$ and it is unique;

2) If $\omega^2\in\sigma(H_0)$ and does not lie on the boundary of the spectrum $\sigma(H_0)$ of the operator H_0 on the complex plane λ, then as a solution $U(x,\omega)$ we take

$$U(x,\omega)=\lim_{k\in\mathbb{C}',k\to\omega}(H_0-k^2)^{-1}(f). \tag{13}$$

Here the limit is understood in terms of a uniform metric on each interval from $\mathbb{R}$. The existence of this limit is an obvious consequence of (9) and (10).

Denote by $L_+ = \{k : Im\, k = a,\, a > 0\}$ and $L_- = \{k : Im\, k = -d,\, d > 0\}$, and by q_l the segment $Re\, k = l\pi + \frac{\pi}{3}$, $-d \le Im\, k \le a$, l is any real number.

From the point $k = p$ lying on the real axis, let us make a vertical cut into the lower half-plane of the variable k.

Denote by l_p the contour going from the point $p - id$ along on the left edge of this cut to the point p, and then from the point p along the right edge of the cut to the point $p - id$, $d > 0$.

2.2 Auxiliary Statements

Lemma 1. *For any $n = 1, 2, \ldots$ such that $\lambda_{2n} \ne \lambda_{2n-1}$ and for $n = 0$, the expansions*

$$J_{l_{\pm\lambda_{2n}}} = \mp \frac{\pi}{i\sqrt{t}}\, \frac{b_{\pm\lambda_{2n}}}{\lambda_{2n} \mp \omega} a_{\lambda_{2n}} v(x, \lambda_{2n})\, e^{\mp i\lambda_{2n}t - i\frac{\pi}{4}} + R_{\pm\lambda_{2n}}(x, t)$$

hold, where

$$a_{\lambda_{2n}} = \int_0^1 f(x)\, v(x, \lambda_{2n})\, dx,$$

$$b_{\lambda_{2n}} = -\frac{2iC_{\lambda_{2n}}\Gamma(\frac{1}{2})}{\pi} \cdot \lim_{k\to\lambda_{2n}} \frac{\sqrt{k-\lambda_{2n}}}{\sqrt{G(k)}}, \quad b_{-\lambda_{2n}} = -ib_{\lambda_{2n}},$$

and for $R_{\pm\lambda_{2n}}(x, t)$ the following inequalities hold:

$$|R_{\pm\lambda_{2n}}(x, t)| \le C_i t^{-1} ||f; L^2||, \quad x \in [-b, b], \quad t > 0, \quad C_i = \text{const}, \quad i = 1, 2.$$

Remark 1. There exists n_3 such that for $n > n_3$, and $t > 0,\ x \in [-b, b]$, the following estimates

$$|b_{\pm\lambda_{2n}}| \le \frac{C\sqrt{\lambda_{2n} - \lambda_{2n-1}}}{\lambda_{2n-1}}$$

hold, where the constant C depends only on the segment $[-b, b]$.

In the case of $0 \le n \le n_3$ these estimates are replaced by estimates of the form:

$$|b_{\pm\lambda_{2n}}| \le C_{\pm\lambda_{2n}}, \quad 0 \le n \le n_3.$$

Lemma 2. *For any $n = 1, 2, \ldots$ such that $\lambda_{2n-1} \ne \lambda_{2n}$, the expansions*

$$J_{l_{\pm\lambda_{2n-1}}} = \mp \frac{\pi}{i\sqrt{t}} \cdot \frac{b_{\pm\lambda_{2n-1}}}{\lambda_{2n-1} - \omega} \cdot a_{\lambda_{2n-1}} v(x, \lambda_{2n-1}) e^{\mp i\lambda_{2n-1}t + \frac{\pi}{4}i} + R_{\pm\lambda_{2n-1}}(x, t)$$

hold, where

$$a_{\lambda_{2n-1}} = \int_0^1 f(x)\, v(x, \lambda_{2n-1})\, dx,$$

$$b_{\lambda_{2n-1}} = \frac{2iC_{\lambda_{2n-1}}}{\pi} \lim_{k\to\lambda_{2n-1}} \frac{\sqrt{k-\lambda_{2n-1}}}{\sqrt{G(k)}}, \quad b_{-\lambda_{2n-1}} = ib_{\lambda_{2n-1}},$$

and for $R_{\pm\lambda_{2n-1}}(x, t)$ the following inequalities hold:

$$|R_{\pm\lambda_{2n-1}}(x, t)| \le C_i t^{-1} ||f; L^2||, \quad x \in [-b, b], \quad t > 0, \quad C_i = \text{const}, \quad i = 3, 4.$$

Remark 2. There exists n_3 such that for $n > n_3$, and $x \in [-b, b],\ t > 0$, the following estimates

$$|b_{\pm\lambda_{2n-1}}| \leq \frac{C\sqrt{\lambda_{2n} - \lambda_{2n-1}}}{\lambda_{2n-1}}$$

hold, where the constant C depends only on the segment $[-b, b]$.

In the case of $0 \leq n \leq n_3$ these estimates are replaced by estimates of the form:

$$|b_{\pm\lambda_{2n-1}}| \leq C_{\pm\lambda_{2n-1}}, \quad 0 \leq n \leq n_3.$$

3 Main results

Theorem 1. *If the Hill operator* H_0 *is positive,* $p(x) \geq \text{const} > 0$, $q(x) \geq 0$, *and the number* ω^2 *does not lie on the boundary of the spectrum* $\sigma(H_0)$ *of the operator* H_0 *on the complex plane* λ*, then there exist compact operators*

$$M_1, M_3 : L^2[0,1] \longmapsto L^2[0,1],$$
$$M_2, M_4 : L^2[0,1] \longmapsto H^1[0,1]$$

such that for $|x| < b$ *and* $t > 0$*, the solution to the Cauchy problem* (1), (2) *has the form*

$$u(x,t) = -ie^{-i\omega t}U(x,\omega) + \frac{1}{\sqrt{t}}\left(u_1(x,t) + u_2(x,t)\right) + v(x,t), \tag{14}$$

where $U(x,\omega)$ *is defined by* (12) *and* (13), $u_1(x,t)$ *is a periodic solution to the Cauchy problem for which*

$$\hat{u}(x,t)|_{t=0} = M_1 f, \quad \hat{u}_t(x,t)|_{t=0} = M_3 f,$$

$u_2(x,t)$ *is an anti-periodic solution to the Cauchy problem for which*

$$\hat{u}(x,t)|_{t=0} = M_2 f, \quad \hat{u}_t(x,t)|_{t=0} = M_4 f,$$

while the function $v(x,t)$ *for* $|x| < b$, $t > 0$ *satisfies the estimate*

$$|v(x,t)| \leq C(b)\, t^{-1}||f; L^2(\mathbb{R})||; \tag{15}$$

the functions $u_1(x,t)$ *and* $u_2(x,t)$ *have the form*

$$u_1(x,t) = \sum_{n=0}^{\infty} a_{\lambda_n} v(x,\lambda_n) \left(b^{(1)}_{\lambda_n} \cos(\lambda_n t + (-1)^n \frac{\pi}{4}) + b^{(2)}_{\lambda_n} \sin(\lambda_n t + (-1)^n \frac{\pi}{4})\right), \tag{16}$$

$$u_2(x,t) = \sum_{n=0}^{\infty} a_{\mu_n} v(x,\mu_n) \left(b^{(1)}_{\mu_n} \cos(\mu_n t + (-1)^{n+1} \frac{\pi}{4}) + b^{(2)}_{\mu_n} \sin(\mu_n t + (-1)^{n+1} \frac{\pi}{4})\right), \tag{17}$$

where $a_{\lambda_n}(a_{\mu_n})$ *are the coefficients of the Fourier expansion of the function* $f(x)$ *in the system* $\{\hat{v}(x,\lambda_n)\}$ $(\{\hat{v}(x,\mu_n)\})$, *and* $b^{(i)}_{\lambda_n}(b^{(i)}_{\mu_n})$, $i=1,2$, *are of order* $o(n^{-2})$ *for* $n\to\infty$, *and they are given by the formulas*

$$b^{(1)}_{\lambda_j}=\frac{ib_{\lambda_j}\lambda_j}{(\lambda_j-\omega)(\lambda_j+\omega)},\quad b^{(2)}_{\lambda_j}=\frac{\omega b_{\lambda_j}}{(\lambda_j-\omega)(\lambda_j+\omega)}.$$

Similar expressions hold for $b^{(1)}_{\mu_j}$ *and* $b^{(2)}_{\mu_j}$.

Here the summation is carried out only over those n for which λ_n (or μ_n) are simple eigenvalues of the periodic (or anti-periodic) Sturm–Liouville problem.

Denote by $B(a)$ the circle $B(a)=\{k : |k-\pi a|\le \frac{\pi}{4}\}$. Since the function $\sqrt{(\theta(k)+\varphi'(k))^2-4}$ has a simple zero at the point $k=0$ and in a small neighborhood $B(0)$ of this point has no other zeros, then for $k\in B(0)$ the following equality

$$\sqrt{G(k)}=\sqrt{(\theta(k)+\varphi'(k))^2-4}=k\,G_0(k),$$

holds, and

$$C_2\ge |G_0(k)|\ge C_1 \quad \text{for} \quad k<<1.$$

From this and (8) it follows that the function $(H_0-k^2)^{-1}$ has a simple pole at the point $k=0$ and, for $\omega\ne 0$

$$U_1(x)=\operatorname*{res}_{k=0}(H_0-k^2)^{-1}(f)=b_0 f_0 v(x,0), \tag{18}$$

where the constants f_0 and b_0 are defined as follows

$$f_0=\int_0^1 v(\xi,0)f(\xi)\,d\xi,\quad b_0=\frac{C_0}{G_0(0)},$$

and the function $v(x,0)$ is obtained from the normalized eigenfunction $\hat{v}(x,0)$ of the periodic Sturm–Liouville problem (4) corresponding to the eigenvalue $\lambda_0=\lambda_0^2=0$, if it is continued along the entire axis in a periodic way.

Theorem 2. *If the left end of the spectrum* $\sigma(H_0)$ *of the operator* H_0 *on the complex plane* λ *is equal to zero,* $\omega\ne 0$, *and* ω *does not coincide with any end of the lacunae of the spectrum* $\sigma(H_0)$ *of the operator* H_0 *on the complex plane* k, *then for* $|x|<b$ *and* $t>0$ *the solution to the Cauchy problem* (1), (2) *has the form*

$$u(x,t)=-ie^{-i\omega t}U(x,\omega)+\tfrac{i}{\omega}U_1(x)+\tfrac{1}{\sqrt{t}}\,(u_1(x,t)+u_2(x,t))+v(x,t),$$

where $U(x,\omega)$ *is defined by the formulas* (12) *and* (13), $U_1(x)$ *is given by the formula* (18), *the functions* $u_1(x,t)$ *and* $u_2(x,t)$ *have the same form as in Theorem 1 with the only difference that in the expansion* (16) *for* $u_1(x,t)$ *there is no term for* $n=0$, *while the function* $v(x,t)$ *for* $|x|<b$ *and* $t>0$ *satisfies the following estimate:*

$$|v(x,t)|\le C(b)\,t^{-1}||f;L^2(\mathbb{R})||.$$

As noted above, if the left end of the spectrum $\sigma(H_0)$ of the operator H_0 on the complex plane λ coincides with zero, then the function $(H_0 - k^2)^{-1}$ at the point $k = 0$ has a simple pole. Then

$$(H_0 - k^2)^{-1}(f) = \frac{1}{k} U_1(x) + U_2(x) + O(k) \quad \text{as} \quad |k| \to 0, \tag{19}$$

where the function $U_1(x)$ is defined by the formula (18), and the function $U_2(x)$ can be written explicitly using the formulas (8) and (11).

Theorem 3. *If the left end of the spectrum $\sigma(H_0)$ of the operator H_0 on the complex plane λ coincides with zero and $\omega = 0$, then for $|x| < b$ and $t > 0$, the solution to the Cauchy problem (1), (2), has the form*

$$u(x,t) = -t\, U_1(x) - i\, U_2(x) + \frac{1}{\sqrt{t}} \left(u_1(x,t) + u_2(x,t)\right) + v(x,t),$$

where the functions $U_1(x)$ and $U_2(x)$ are defined by the formulas (18) and (19), the functions $u_1(x,t)$ and $u_2(x,t)$ have the same form as in Theorem 2, and the function $v(x,t)$ for $|x| < b$ and $t > 0$ satisfies the following estimate:

$$|v(x,t)| \le C(b)\, t^{-1} ||f; L^2(\mathbb{R})||.$$

Now let $\omega^2 \neq 0$ and ω^2 lie on the boundary of the spectrum $\sigma(H_0)$ of the operator H_0 on the complex plane λ. Then the operator-function $(H_0 - k^2)^{-1}(f)$ for $k = \omega$ has a branch point of the second order, and for $k \in \mathbb{C}'$, $k \to \omega$ we have

$$(H_0 - k^2)^{-1}(f) = \frac{1}{\sqrt{k-\omega}} H_1(x,\omega) + H_2(x,\omega) + \sqrt{k-\omega}\, H_3(x,\omega) + O(k-\omega). \tag{20}$$

Theorem 4. *If the Hill operator H_0 is positive, $p(x) \ge \text{const} > 0$, $q(x) \ge 0$, and ω^2 lies on the boundary of the spectrum $\sigma(H_0)$ of the operator H_0 on the complex plane λ, then for $|x| < b$ and $t > 0$ the solution to the Cauchy problem (1), (2) has the form*

$$\begin{aligned} u(x,t) &= h_1 \sqrt{t} e^{-i\omega t} H_1(x,\omega) - i e^{-i\omega t} H_2(x,\omega) + \\ &+ \tfrac{1}{\sqrt{t}} \left(u_1(x,t) + u_2(x,t) + h_2 e^{-i\omega t} H_3(x,\omega)\right) + v(x,t), \end{aligned} \tag{21}$$

where $u_1(x)$ and $u_2(x)$ have the same form as in Theorem 1 with the only difference that expansions (16) and (17) do not include the term corresponding to ω^2, and constants h_1 and h_2 have the form:

$$h_1 = -\frac{2i}{\pi} \cdot e^{-\frac{3}{4}\pi i} \Gamma\left(\frac{1}{2}\right), \quad h_2 = -\frac{1}{\pi} \cdot e^{-\frac{3}{4}\pi i} \Gamma\left(\frac{1}{2}\right),$$

and the function $v(x,t)$ for $|x| < b$ and $t > 0$ satisfies the following estimate:

$$|v(x,t)| \le C(b)\, t^{-1} ||f; L^2(\mathbb{R})||.$$

Taking into account Theorems 3 and 4, it is easy to prove the following theorem.

Theorem 5. *If the left end of the spectrum $\sigma(H_0)$ of the operator H_0 on the complex plane λ coincides with zero, $\omega \neq 0$, and ω^2 lies on the boundary spectrum $\sigma(H_0)$ of the operator H_0 on the complex plane λ, then for $|x| < b$ and $t > 0$ the solution to the Cauchy problem (1), (2), has the form*

$$u(x,t) = -t\,U_1(x) - iU_2(x) + h_1\sqrt{t}e^{-i\omega t}H_1(x,\omega) - ie^{-i\omega t}H_2(x,\omega)$$
$$+\frac{1}{\sqrt{t}}\left(u_1(x,t) + u_2(x,t) + h_2 e^{-i\omega t}H_3(x,\omega)\right) + v(x,t),$$

where the functions $U_1(x)$ and $U_2(x)$, and also the functions $H_1(x,\omega)$, $H_2(x,\omega)$ and $H_3(x,\omega)$ are defined in Theorems 3 and 4, h_1 and h_2 are the same constants defined in Theorem 4, functions $u_1(x,t)$ and $u_2(x,t)$ have the same form as in Theorem 4, and for the function $v(x,t)$ with $|x| < b$ and $t > 0$ the following estimate

$$|v(x,t)| \leq C(b)\,t^{-1}||f; L^2(\mathbb{R})||$$

is valid.

Remark 3. If the left end of the spectrum $\sigma(H_0)$ of the operator H_0 is negative and $\omega > 0$, then, as in Theorem 2 from [11], one can show that the solution to the problem (1), (2) grows exponentially as $t \to \infty$.

Author Contributions: All authors contributed equally to the article. All authors have read and agreed to the published version of the manuscript.

Funding: The research by A.M. Ishkhanyan was supported by the Armenian Science Committe (Grant No.21AG-1C064 and 1-6/23-I/IPR), Armenia.

Conflicts of Interest: The authors declare no conflict of interest.

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Received February 15, 2024

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