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Solution of a renewal equation in population biology and epidemiology

Mario Lefebvre

Abstract. An integral equation for the number of births in unit time in a model for populations with age structure is considered. The equation can sometimes be transformed into a differential-difference equation. It is solved explicitly under various assumptions for the initial age distribution, which can depend on a constant T. The case when T is a random variable is also treated.

Mathematics subject classification: 37N25, 39A06, 45D05, 92D30.
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1 Introduction

In their monograph [2], Brauer and Castillo-Chavez (see also Brauer *et al.*[3]) considered various models for populations with age structure. Other references on this topic include the books by Webb [10] and Perthame [7]. See also Chapter 4 in Metz and Diekmann [6] and Diekmann *et al.*[4].

In this paper, we are interested in the case of linear continuous models. McKendrick (or M'Kendrick) [5] introduced age structure into the dynamics of a one-sex population. His model assumes that the female population can be described by a function $\rho(a,t)$ which denotes the density of individuals of age a at time t, so that $\rho(a,t)\,\Delta a$ is approximately the number of individuals with ages between a and $a+\Delta a$ at time t. Moreover, the total population size is

$$P(t) := \int_0^\infty \rho(a,t)\,\mathrm{d}a. \tag{1}$$

Next, we define the age-dependent death rate $\mu(a)$, known as the *mortality function* or *death modulus*. We can state that, during the time interval $[t, t+\Delta t]$, a fraction $\mu(a)\,\Delta t$ of the members of the population with ages in $[a, a+\Delta a]$ at time t will die. The probability of survival from birth to age a is given by

$$\pi(a) := \exp\left\{-\int_0^a \mu(\alpha)\,\mathrm{d}\alpha\right\}. \tag{2}$$

McKendrick [5] showed that

$$\rho_a(a,t) + \rho_t(a,t) + \mu(a)\,\rho(a,t) = 0. \tag{3}$$

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This equation is now known as the *McKendrick* or the *McKendrick-von Foerster equation*. Von Foerster [9] derived the equation in the context of cellular biology.

Corresponding to the death modulus, we define the *birth modulus* $\beta(a)$. This function, multiplied by Δt, gives the number of offspring produced in the time interval $[t, t+\Delta t]$ by members of the population with ages in $[a, a+\Delta a]$. We have the *renewal condition*

$$B(t) := \rho(0,t) = \int_0^\infty \beta(a)\,\rho(a,t)\,\mathrm{d}a. \tag{4}$$

The function $B(t)$ represents the number of births in unit time t.

Finally, we let $\Psi(t)$ denote the rate of births from members of the population who were present in the population at the initial time zero. We can write that

$$\Psi(t) = \int_0^\infty \beta(t+s)\,\Phi(s)\,\exp\left\{-\int_s^{s+t} \mu(\alpha)\,\mathrm{d}\alpha\right\}\mathrm{d}s, \tag{5}$$

in which the function $\Phi(\cdot)$ is the *initial age distribution*:

$$\Phi(a) := \rho(a,0). \tag{6}$$

It can be shown that the function $B(t)$ satisfies the *renewal equation*

$$B(t) = \Psi(t) + \int_0^t \beta(a)\,\pi(a)\,B(t-a)\,\mathrm{d}a, \tag{7}$$

which is a linear Volterra integral equation of convolution type. Notice that Eq. (7) can be rewritten as follows:

$$B(t) = \Psi(t) + \int_0^t \beta(t-\tau)\,\pi(t-\tau)\,B(\tau)\,\mathrm{d}\tau. \tag{8}$$

In this paper, our aim is to solve the above equation in special cases of interest. Once the function $B(t)$ has been obtained, we can deduce the function $\rho(a,t)$ from the following relation:

$$\rho(a,t) = B(t-a)\,\pi(a) \quad \text{for } t \geq a, \tag{9}$$

while we have

$$\begin{aligned}\rho(a,t) &= \Phi(a-t)\,\exp\left\{-\int_{a-t}^{a} \mu(\alpha)\,\mathrm{d}\alpha\right\} \\ &= \Phi(a-t)\,\frac{\pi(a)}{\pi(a-t)} \quad \text{for } t < a.\end{aligned} \tag{10}$$

2 Explicit solutions

We will consider various special cases for the functions $\Phi(a)$, $\mu(a)$ and $\beta(a)$.

Case I. First, assume that $\Phi(a)$ is the Dirac delta function $\delta(a)$. Brauer and Castillo-Chavez [2, p. 279] called this case the *genesis* model, because the members of the population are all at age zero at the initial time zero. Moreover, they chose $\mu(a) \equiv \mu$ and $\beta(a) \equiv \beta$; that is, the death and birth moduli are both (positive) constants. It is then easy to compute

$$\Psi(t) = \beta e^{-\mu t} \quad \text{and} \quad \pi(a) = e^{-\mu a}, \tag{11}$$

so that $B(t)$ satisfies the integral equation

$$B(t) = \beta e^{-\mu t} + \beta e^{-\mu t} \int_0^t e^{\mu\tau} B(\tau)\,\mathrm{d}\tau. \tag{12}$$

Brauer and Castillo-Chavez [2] transformed the above equation into the following simple first-order linear ordinary differential equation (ODE):

$$B'(t) = (\beta - \mu)\,B(t). \tag{13}$$

The solution that satisfies the initial condition $B(0) = \beta$ (which follows from Eq. (12)) is $B(t) = \beta e^{(\beta-\mu)t}$, so that

$$\rho(a,t) = \begin{cases} \beta e^{\beta(t-a)-\mu t} & \text{for } t \geq a, \\ \delta(a-t)\,e^{-\mu t} & \text{for } t < a \end{cases} \tag{14}$$

and

$$P(t) = e^{(\beta-\mu)t}. \tag{15}$$

Actually, it is not necessary to transform Eq. (12) into an ODE. Using the function *intsolve* of the mathematical software program *Maple*, we get at once that $B(t) = \beta e^{(\beta-\mu)t}$.

Case II. Next, suppose that $\mu(a) \equiv \mu$ and

$$\beta(a) = \begin{cases} 0 & \text{for } a \leq T, \\ \beta & \text{for } a > T, \end{cases} \tag{16}$$

where T is a positive constant. For $t \leq T$, Eq. (7) reduces to

$$B(t) = \Psi(t) = \beta e^{-\mu t} \int_{T-t}^{\infty} \Phi(s)\,\mathrm{d}s. \tag{17}$$

We can show (see Brauer and Castillo-Chavez [2, p. 281]) that

$$B(t) = \beta e^{-\mu t} \int_0^{\infty} \Phi(s)\,\mathrm{d}s + \beta e^{-\mu t} \int_0^{t-T} e^{\mu\tau} B(\tau)\,\mathrm{d}\tau \tag{18}$$

for $t > T$, which implies that

$$B'(t) = \beta e^{-\mu T} B(t-T) - \mu B(t). \tag{19}$$

Instead of trying to solve the above differential-difference equation, we will consider the integral equation (18) directly. Assume that the initial age distribution is given by

$$\Phi(a) = \gamma_0 e^{-\gamma a}, \tag{20}$$

where γ_0 and γ are positive constants. Then,

$$\int_0^\infty \Phi(s)\,\mathrm{d}s = \frac{\gamma_0}{\gamma}, \tag{21}$$

so that

$$B(t) = \beta e^{-\mu t} \left\{ \frac{\gamma_0}{\gamma} + \int_0^{t-T} e^{\mu\tau} B(\tau)\,\mathrm{d}\tau \right\} \quad \text{for } t > T. \tag{22}$$

If $T = 0$, the exact solution to Eq. (22) is

$$B(t) = \beta \frac{\gamma_0}{\gamma} e^{(\beta-\mu)t}. \tag{23}$$

Hence, if $\beta = \mu$, the function $B(t)$ is equal to the constant $\beta\gamma_0/\gamma$.

With the function $\Phi(a)$ defined in Eq. (20), we have

$$\begin{aligned} B(t) &= \beta e^{-\mu t} \int_{T-t}^\infty \gamma_0 e^{-\gamma s}\,\mathrm{d}s \\ &= \frac{\beta\gamma_0}{\gamma} e^{-\mu t + \gamma(t-T)} \quad \text{for } t \in [0, T]. \end{aligned} \tag{24}$$

Let us denote the above function by $B_{0,1}(t)$, and by $B_{k,k+1}(t)$ the value of the function $B(t)$ in the interval $(kT, (k+1)T]$, for $k \in \mathbb{N}$. For $t \in (T, 2T]$, Eq. (22) becomes

$$\begin{aligned} B_{1,2}(t) &= \beta e^{-\mu t} \left\{ \frac{\gamma_0}{\gamma} + \int_0^{t-T} e^{\mu\tau} B_{0,1}(\tau)\,\mathrm{d}\tau \right\} \\ &= \frac{\beta\gamma_0}{\gamma^2} e^{-\mu t} \left\{ \gamma e^{\gamma(t-T)} + \beta e^{-\gamma T} \left[e^{\gamma(t-T)} - 1 \right] \right\}. \end{aligned} \tag{25}$$

In general, for $t \in (kT, (k+1)T]$, with $k \geq 2$, we can write that

$$\begin{aligned} B_{k,k+1}(t) &= \beta e^{-\mu t} \left\{ \frac{\gamma_0}{\gamma} + \int_0^T e^{\mu\tau} B_{0,1}(\tau)\,\mathrm{d}\tau + \dots \right. \\ &\quad \left. + \int_{(k-2)T}^{(k-1)T} e^{\mu\tau} B_{k-2,k-1}(\tau)\,\mathrm{d}\tau + \int_{(k-1)T}^{t-T} e^{\mu\tau} B_{k-1,k}(\tau)\,\mathrm{d}\tau \right\}. \end{aligned} \tag{26}$$

Figure 1 presents the function $B(t)$ in the interval $[0, 3]$ in the particular case when $\gamma_0 = 100$, $\gamma = \mu = T = 1$ and $\beta = 1/2$. We find that $B_{0,1}(t) \equiv 50 e^{-1}$,

$$B_{1,2}(t) = 25 e^{-t} \left(2 - e^{-1} + e^{t-2} \right) \tag{27}$$

and

$$B_{2,3}(t) = \frac{25}{2}\,e^{-3} + \frac{25}{2}\,e^{-t}\left[2 - e^{-1}(1+t) + 2t\right]. \tag{28}$$

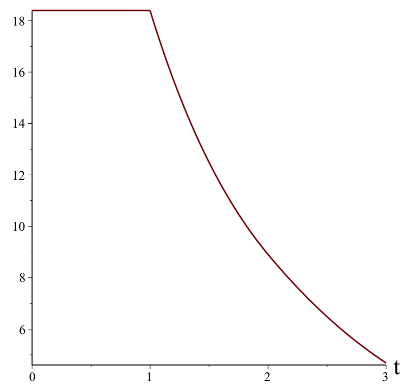

Figure 1. Function $B(t)$ in the interval $[1,3]$ when $\gamma_0 = 100$, $\gamma = \mu = T = 1$ and $\beta = 1/2$.

Remarks. (i) Notice that we do not have to solve any integral or differential equation in order to obtain the function $B(t)$. We merely have to perform simple integrals repeatedly. Of course, if T is small and t is large, this procedure can be rather tedious, but it remains straightforward. Equation (19) is also a simple ODE. However, every time we solve it in an interval of the form $(kT, (k+1)T]$, for $k \in \mathbb{N}$, we must use the solution in the previous interval and the fact that $B(t)$ is a continuous function to obtain a unique solution to the equation in this interval.

(ii) If, for a different function $\Phi(a)$, we are not able to evaluate the definite integrals analytically, then we can at least try to compute the solution numerically.

Case III. Assume now that $\mu(a) \equiv \mu$ and

$$\beta(a) = \begin{cases} \beta & \text{for } a \leq T, \\ \beta e^{-\mu a} & \text{for } a > T, \end{cases} \tag{29}$$

where T is again a positive constant. This is the case considered by Şterbeţi [8]. He was able to derive the differential equation satisfied by the function $B(t)$ both when $t \leq T$ and $t > T$. For $t \leq T$, we have

$$\begin{aligned} B''(t) - (\beta - 3\mu)\,B'(t) + 2\mu(\mu - \beta)\,B(t) &= -\beta e^{-\mu t}\Big[\mu\,\Phi(T-t) \\ &\quad + \left(1 - e^{-\mu T}\right)\Phi'(T-t)\Big]. \end{aligned} \tag{30}$$

This second-order linear ODE is subject to the boundary conditions

$$B(0) = \beta \int_0^T \Phi(a)\,\mathrm{d}a + \beta \int_T^\infty e^{-\mu a}\,\Phi(a)\,\mathrm{d}a \tag{31}$$

and

$$\begin{aligned} B'(0) &= (\beta-\mu)\,B(0)+\beta\left[\left(e^{-\mu T}-1\right)\Phi(T)-\mu\int_T^\infty e^{-\mu a}\,\Phi(a)\,\mathrm{d}a\right] \\ &= \beta\,(\beta-\mu)\int_0^T \Phi(a)\,\mathrm{d}a+\beta\,(e^{-\mu T}-1)\,\Phi(T) \\ &\quad+\beta\,(\beta-2\,\mu)\int_T^\infty e^{-\mu a}\,\Phi(a)\,\mathrm{d}a. \end{aligned} \tag{32}$$

Remark. The term $\Phi'(T-t)$ in Eq. (30) is interpreted as follows: $\Phi'(T-t) = \frac{\mathrm{d}}{\mathrm{d}s}\Phi(T+s)\times(-1)$, where $s := -t$.

For $t > T$, the differential equation becomes

$$\begin{aligned} B''(t)-(\beta-3\,\mu)\,B'(t)+2\,\mu\,(\mu-\beta)\,B(t) \;=&\; \beta\left(e^{-2\mu T}-e^{-\mu T}\right) \\ &\times\left[B'(t-T)+\mu\,B(t-T)\right]. \end{aligned} \tag{33}$$

We need two boundary conditions to obtain a unique solution to the above ODE. In the interval $(T, 2T]$, one condition is obtained by evaluating $B(T)$ from Eq. (30) and using the fact that $B(t)$ is a continuous function. However, we cannot use the constant $B'(T)$ deduced from Eq. (30) because the function $\beta(a)$ is discontinuous at $a = T$. Instead, we make use of the expression computed by Şterbeţi [8] for $B'(t)$, with $t > T$, to evaluate $B'(T)$:

$$B'(t) = (\beta-2\,\mu)\,B(t)+\beta\left(e^{-2\mu T}-e^{-\mu T}\right)B(t-T)+\mu\beta\int_0^T e^{-\mu a}\,B(t-a)\mathrm{d}a. \tag{34}$$

The same procedure must then be repeated in each interval of the form $(kT, (k+1)\,T]$.

We can proceed as in Case II to obtain an explicit expression for the function $B(t)$. Let us choose again the function $\Phi(a)$ defined in Eq. (20). We compute, for $t \le T$,

$$\begin{aligned} \Psi_{0,T}(t) &:= \beta\,\gamma_0\,e^{-\mu t}\left\{\int_0^{T-t} e^{-\gamma s}\,\mathrm{d}s+e^{-\mu t}\int_{T-t}^\infty e^{-\mu s}\,e^{-\gamma s}\,\mathrm{d}s\right\} \\ &= \beta\,\gamma_0\,e^{-\mu t}\left[\frac{1-e^{-\gamma(T-t)}}{\gamma}+\frac{e^{-\mu T}\,e^{-\gamma(T-t)}}{\gamma+\mu}\right]. \end{aligned} \tag{35}$$

For $t > T$, we have

$$\Psi(t) = \beta\,\gamma_0\,e^{-2\mu t}\int_0^\infty e^{-(\gamma+\mu)s}\,\mathrm{d}s = \frac{\beta\,\gamma_0}{\gamma+\mu}\,e^{-2\mu t}. \tag{36}$$

As in the previous case, we denote the function $B(t)$ in the interval $[0, T]$ by $B_{0,1}(t)$. We have

$$B_{0,1}(t) = \Psi_{0,T}(t)+\beta\,e^{-\mu t}\int_0^t e^{\mu\tau}\,B_{0,1}(\tau)\,\mathrm{d}\tau. \tag{37}$$

The solution of this integral equation is

$$B_{0,1}(t) = \frac{\beta\gamma_0}{(\beta+\mu)(\mu+\gamma)}\, e^{(\beta-\mu)t}\left[\mu e^{-(\beta+\mu)t} + \beta\right]. \tag{38}$$

Next, we have

$$\begin{aligned} B_{1,2}(t) &= \Psi(t) + \beta\int_0^{t-T} e^{-\mu a} B_{1,2}(t-a)\,\mathrm{d}a + \beta\int_{t-T}^{T} e^{-\mu a} B_{0,1}(t-a)\,\mathrm{d}a \\ &\quad + \beta\int_T^t e^{-2\mu a} B_{0,1}(t-a)\,\mathrm{d}a \\ &= \Psi(t) + \beta\int_{t-T}^{T} e^{-\mu a} B_{0,1}(t-a)\,\mathrm{d}a + \beta\int_T^t e^{-2\mu a} B_{0,1}(t-a)\,\mathrm{d}a \\ &\quad + \beta\int_T^t e^{-\mu(t-\tau)} B_{1,2}(\tau)\,\mathrm{d}\tau \end{aligned} \tag{39}$$

for $t \in (T, 2T]$. We can solve explicitly the above integral equation, but the expression obtained is rather involved. Therefore, we will give an example below rather than the general solution.

In the interval $(2T, 3T]$, we can write that

$$\begin{aligned} B_{2,3}(t) &= \Psi(t) + \beta\int_0^{t-2T} e^{-\mu a} B_{2,3}(t-a)\,\mathrm{d}a + \beta\int_{t-2T}^{T} e^{-\mu a} B_{1,2}(t-a)\,\mathrm{d}a \\ &\quad + \beta\int_T^{t-T} e^{-2\mu a} B_{1,2}(t-a)\,\mathrm{d}a + \beta\int_{t-T}^{t} e^{-2\mu a} B_{0,1}(t-a)\,\mathrm{d}a, \end{aligned} \tag{40}$$

etc.

With the same constants as above ($\gamma_0 = 100$, $\gamma = \mu = T = 1$ and $\beta = 1/2$), we obtain that

$$\Psi(t) = 50e^{-t} - 50e^{-1} + 25e^{-2}. \tag{41}$$

We can then compute the function $B_{1,2}(t)$. We find that

$$\begin{aligned} B_{1,2}(t) &\approx 9.0127 + 7.5994e^{-2t-1} - 2.7778e^{-2t-2} \\ &\quad + e^{-t/2}\left(-17.2978t + 16.3406\right). \end{aligned} \tag{42}$$

The function $B(t)$ in the interval $[0, 2]$ is shown in Figure 2. One can check that the same solution is obtained by solving the differential equations derived by Şterbeţi [8], subject to the appropriate boundary conditions.

Case IV. Finally, let us take $\mu(a) \equiv \mu$, $\beta(a) = \beta e^{-a}$ and $\Phi(a) = \gamma_0 e^{-\gamma a^2}$. We compute

$$\Psi(t) = \frac{\sqrt{\pi}}{2}\frac{\beta\gamma_0}{\sqrt{\gamma}}\left[1 - \mathrm{erf}\left(\frac{1}{2\sqrt{\gamma}}\right)\right] e^{-\mu t}\exp\left(\frac{1}{4\gamma} - t\right), \tag{43}$$

where "erf" is the *error function*, and we must solve the integral equation

$$B(t) = \Psi(t) + \beta\int_0^t e^{-(\mu+1)a} B(t-a)\,\mathrm{d}a. \tag{44}$$

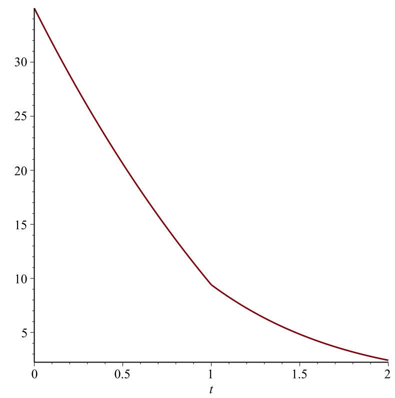

Figure 2. Function $B(t)$ in the interval $[0, 2]$ when $\gamma_0 = 100$, $\gamma = \mu = T = 1$ and $\beta = 1/2$.

In the case when $\mu = \beta = \gamma = 1$ and $\gamma_0 = 100$, we find that

$$B(t) = 50\sqrt{\pi}e^{1/4}\,[1 - \operatorname{erf}(1/2)]\,e^{-t}. \tag{45}$$

Making use of Eqs. (9) and (10), we can obtain the function $\rho(a, t)$ for $a \geq 0$, and then compute the total population size $P(t)$ at time t defined in Eq. (1). We find that

$$\begin{aligned} P(t) &= 50\sqrt{\pi}\left\{e^{1/4}[1 - \operatorname{erf}(1/2)]\,t + 1\right\}e^{-t} \\ &\approx \left(42.4946\,e^{1/4}t + 88.6227\right)e^{-t}. \end{aligned} \tag{46}$$

Now, when we are unable to solve an integral equation exactly, we can at least look for an approximate solution. In *Maple*, the Neumann method (see Arfken [1, p. 879]) has been implemented. This method provides an approximate (or sometimes exact) solution, under certain assumptions, by computing successive approximations. The solution thus obtained is called a *Neumann series solution.* The Neumann series solution $B_N(t)$ of the integral equation (44) provided by *Maple* is given by

$$B_N(t) = e^{-2t}P_N(t), \tag{47}$$

where $P_N(t)$ is a polynomial of degree 6. The functions $B(t)$ and $B_N(t)$ in the interval $[0, 5]$ are presented in Figure 3. We see that they practically coincide in that interval. However, if we look more precisely at the functions in the interval $[4, 5]$, we find that $B_N(t)$ underestimates $B(t)$; see Figure 4.

In this particular case, the Neumann method works well if the constant β is smaller than 1, which ensures the convergence of the series. If we replace $\beta = 1$ by $\beta = 1/2$, the Neumann series solution is indeed more precise, as can be seen in Figure 5.

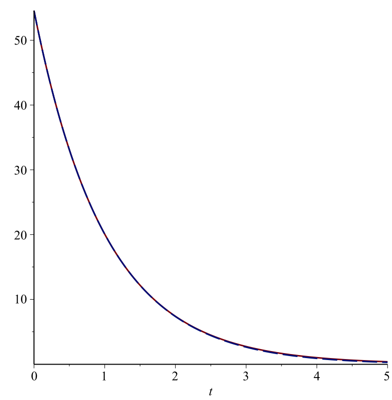

Figure 3. Function $B(t)$ (solid line) and Neumann series solution (dashed line) in the interval $[0, 5]$, when $\gamma_0 = 100$ and $\gamma = \beta = \mu = 1$. The two curves are almost indistinguishable.

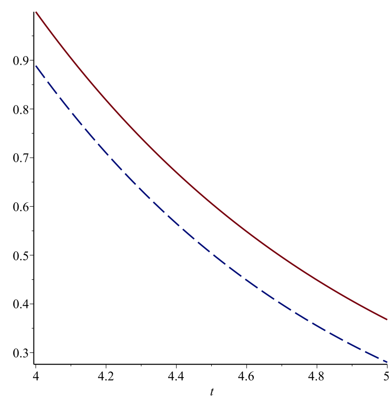

Figure 4. Function $B(t)$ (solid line) and Neumann series solution (dashed line) in the interval $[4, 5]$, when $\gamma_0 = 100$ and $\gamma = \beta = \mu = 1$.

3 The case when T is a random variable

When the constant T in Eq. (16) or in Eq. (29) is actually a (continuous) random variable, one can make use of the *law of total probability* to compute the expected value of the function $B(t)$:

$$E[B(t)] = \int_0^\infty E[B(t) \mid T = s] \, f_T(s) \, \mathrm{d}s, \tag{48}$$

where $f_T(s)$ is the *probability density function* of T.

Remarks. (i) Particularly in epidemiology, the fact that T can be a random time instant is more realistic. Notice that if one assumes that the latent period of a certain disease in a given population is a random variable, this implies that this

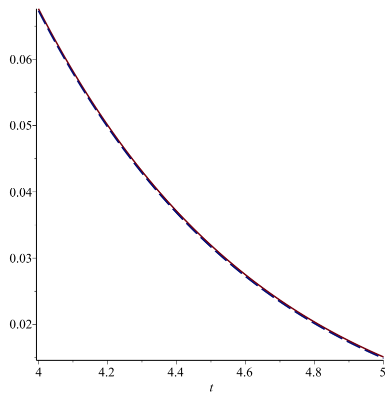

Figure 5. Function $B(t)$ (solid line) and Neumann series solution (dashed line) in the interval $[4,5]$, when $\gamma_0 = 100$, $\gamma = \mu = 1$ and $\beta = 1/2$.

latent period may be different for each member of that population. (ii) If we assume that T is a discrete-type random variable instead, then Eq. (48) can be written as follows:

$$E[B(t)] = \sum_{k=1}^{\infty} E[B(t) \mid T = s_k] p_T(s_k), \tag{49}$$

where $\{s_1, s_2, \ldots\}$ is the set of possible values of T and $p_T(s_k)$ is its *probability mass function*.

We will consider Case II and two particular probability density functions for T. Assume first that T has a uniform distribution on the interval $[1,2]$, so that

$$f_T(s) = 1 \quad \text{for } 1 \le s \le 2. \tag{50}$$

With the constants $\gamma_0 = 100$, $\gamma = \mu = 1$ and $\beta = 1/2$, we have (see Eq. (24) and Eq. (25))

$$B(t) = 50e^{-T} \quad \text{for } t \in [0,T] \tag{51}$$

and

$$B(t) = 25e^{-t}\left(2 + e^{-T} - e^{-t}\right) \quad \text{for } t \in (T, 2T]. \tag{52}$$

Then, for $t \in [0,1]$, we can write that

$$E[B(t)] = \int_1^2 50e^{-s}\,\mathrm{d}s = 50(e^{-1} - e^{-2}). \tag{53}$$

When $t \in (1,2]$, we have

$$\begin{aligned} E[B(t)] &= \int_1^t 25e^{-s}\left(2 + e^{-s} - e^{-t}\right)\mathrm{d}s + \int_t^2 50e^{-s}\,\mathrm{d}s \\ &= 25\left(2te^{-t} - te^{-2t} + e^{-t-1} - 2e^{-2}\right). \end{aligned} \tag{54}$$

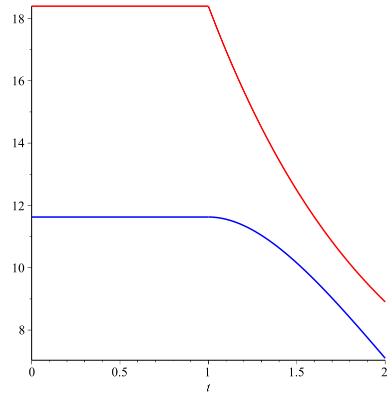

Figure 6. Function $B(t)$ (above) in the interval $[0,2]$, when $\gamma_0 = 100$, $\gamma = \mu = T = 1$ and $\beta = 1/2$, and its expected value (below) if T has a uniform distribution over $[1,2]$.

In Figure 6, we see the difference between the function $B(t)$ in the interval $[0,2]$ when $T = 1$, and its expected value when T is uniformly distributed over $[1,2]$.

For $t \in (k, k+1]$, with $k \in \{2, 3, \ldots\}$, we would have

$$E[B(t)] = \int_1^2 B_{k,k+1}(t,s)\,\mathrm{d}s, \tag{55}$$

where $B_{k,k+1}(t,s)$ is the function defined in Eq. (26), with T replaced by s.

Finally, suppose that

$$f_T(s) = e^{-s+1} \quad \text{for } s \geq 1. \tag{56}$$

That is, T has an exponential distribution shifted one unit to the right. For $t \in [0,1]$, we calculate

$$E[B(t)] = \int_1^\infty 50\,e^{-s}\,e^{-s+1}\,\mathrm{d}s = 25\,e^{-1}, \tag{57}$$

while in the interval $(1,2]$, we have

$$\begin{aligned} E[B(t)] &= \int_1^t 25\,e^{-s}\left(2 + e^{-s} - e^{-t}\right)e^{-s+1}\,\mathrm{d}s + \int_t^\infty 50\,e^{-s}\,e^{-s+1}\,\mathrm{d}s \\ &= \frac{25}{2}\left(e^{-t-1} + 4e^{-t} + e^{1-3t} - 2\,e^{-2t+1} - 2\,e^{-2t}\right). \end{aligned} \tag{58}$$

See Figure 7.

4 Conclusion

In this paper, we considered the integral equation satisfied by the number $B(t)$ of births in unit time t in a model for populations with age structure. Under various

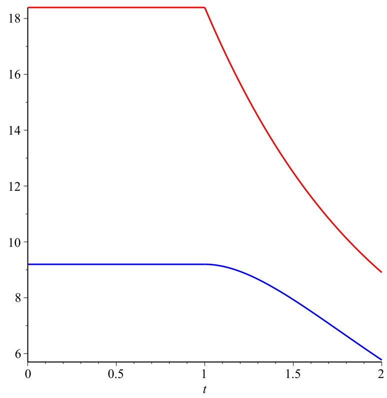

Figure 7. Function $B(t)$ (above) in the interval $[0, 2]$, when $\gamma_0 = 100$, $\gamma = \mu = T = 1$ and $\beta = 1/2$, and its expected value (below) if T has the probability density function defined in Eq. (56).

assumptions, we were able to obtain the exact solution to this integral equation for any $t \geq 0$, or at least in finite intervals.

When it is not possible to solve the integral equation exactly, we could use an approximate method, such as the Neumann method mentioned in Section 2. Alternatively, if it is possible to transform the integral equation into a differential equation, we could at least compute a numerical solution in particular cases.

In Section 2, we treated four particular cases. In Cases II and III, the birth modulus $\beta(a)$ depended on a positive constant T. In Section 3, to make the model more realistic, T was instead a random variable. We saw how to calculate the expected value of $B(t)$ in that case. Similarly, we could have assumed that the parameters μ, β and γ in the model were also random variables.

Finally, we could add a random *noise*, such as a geometric Brownian motion, in the renewal equation (7) to increase the realism of the model.

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Mario Lefebvre
Department of Mathematics and Industrial Engineering,
Polytechnique Montréal, C.P. 6079, Succursale
Centre-ville, Montréal, Canada H3C 3A7
E-mail: *mlefebvre@polymtl.ca*

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